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Three Counterexamples Concerning ω-Chain Continuous Functions and Fixed-point Properties

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The Least Fixed Point Property for ω-Chain Continuous Functions

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ABSTRACT OF THE DISSERTATION

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The basic definitions are given in the first section, including those for ω-chain continuity, ω-chain completeness, and the least fixed point property for ω-chain continuous functions. Some of the relations between completeness and fixed point properties in partially ordered sets are stated and it is briefly shown how the question basic to the dissertation arises.

In the second section, two examples are given showing that a partially ordered set need not be ω-chain complete to have the least fixed point property for ω-chain continuous functions. The first example shows that the least fixed point of an ω-chain continuous function is not in gen-

eral equal to $\sup\{f^n(o):n\in\mathbb{N}\}$. The second example is a partially ordered set which has the least fixed point property for ω-chain continuous functions even though it contains an unbounded countable chain. In this case, the least fixed point is always equal to $\sup\{f^n(o):n\in\mathbb{N}\}$. The free meet-semilattice generated by this second example is also obtained and shown to have the least fixed point property for ω-chain continuous functions.

Retracts are discussed in section 3, where it is seen that they are not sufficient to characterize those partially ordered sets having the least fixed point property for ω-chain continuous functions. A lemma which will be useful later is proved characterizing partially ordered sets which have unbounded countable chains as retracts.

In section 4 the relation between finite width and the least fixed point property for ω-chain continuous functions is explored. Two useful lemmas are proved which will allow us to restrict the cases that need to be checked when proving theorems later. A third lemma is obtained which says that in a partially ordered set of finite width, if A and B are disjoint subsets and B is an upper end, then there are only a finite number of elements of B which are the suprema of countable chains of A. From this lemma, the first theorem is derived. This states that if C is a nonempty unbounded countable chain in a partially ordered set, P, of finite width, then any cofinal subchain of C of order type ω is a retract of P. This leads to the second theorem: a characterization of ω-chain complete partially ordered sets of finite width. This says that a partially ordered set of finite width is ω-chain complete only if it has the least fixed point property for ω-chain continuous functions. It is known that the implication can be reversed.

Section 5 introduces the notion of a layered partially ordered set and discusses some of its problems. The first theorem in this section says that if P is a layered partially ordered set in which every antichain generates a closed lower end, then any chain in P of order type ω is a retract of P. The second theorem says that if P is a partially ordered set in which every antichain generates a closed lower end and every lower end of P which contains a countable chain but none of its upper bounds is layered, then P is ω-chain complete if it has the least fixed point property for ω-chain continuous functions. It is a corollary of a theorem by Edwin Miller that every partially ordered set in which every antichain is finite and every chain contains a countable cofinal subchain is layered. From this and the previous theorem it is easy to see that a partially ordered set in which every antichain is finite and every chain contains a countable cofinal subchain is ω-chain complete if and only if it has the least fixed point property for ω-chain continuous functions.

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I. INTRODUCTION

A partially ordered set is chain complete if each of its chains has a least upper bound. It is ω-chain complete if each of its countable chains has a least upper bound. Since the empty set is considered to be a countable chain, any partially ordered set which is chain or ω-chain complete has a least element, denoted o.

A function, f, mapping a partially ordered set, P, to a partially ordered set, Q, is ω-chain continuous if and only if for every nonempty countable chain, C, of P which has a supremum in P, $f(\sup_P(C))$ equals $\sup_Q(f(C))$. An ω-chain continuous function, then, must preserve order.

A partially ordered set, P, has the (least) fixed point property if and only if every order preserving function from P to itself has a (least) fixed point. It has the (least) fixed point property for ω-chain continuous functions if and only if every ω-chain continuous function from P to itself has a (least) fixed point.

In 1955 A. Tarski ([T], Thm. 1) and A. C. Davis ([Da], Thm. 2) characterized complete lattices as those lattices having the fixed point property. In 1976 G. Markowsky ([M], Thm. 11) characterized chain complete partially ordered sets as those having the least fixed point property. About the same time partially ordered sets and categories were being developed as models for theoretical computer

science ([Sc1],[Sc2],[Sm]). The existence of fixed points is very important for these models, so it is necessary to be sure that the partially ordered set or category in question has the appropriate fixed point property. It was in this context that G. Plotkin asked in 1978 if parially ordered sets with the least fixed point property for ω-chain continuous functions have a characterization similar to that for partially ordered sets with the least fixed point property. That is, is a partially ordered set which has the least fixed point property for ω-chain continuous functions necessarily ω-chain complete? It is known that if a partially ordered set, P, is ω-chain complete, then it has the least fixed point property for ω-chain continuous functions. If $f:P\to P$ is ω-chain continuous, then
$f(\sup\{f^n(o):n\in\mathbb{N}\})=\sup(f(\{f^n(o):n\in\mathbb{N}\}))=\sup\{f^n(o):n\in\mathbb{N}\}$.
A partially ordered set which does not have a least element cannot have the least fixed point property for ω-chain continuous functions, since the identity map is an ω-chain continuous function which, in that case, would not have a least fixed point. Therefore, partially ordered sets will always be assumed to have a least element. The symbol $\mathbb{N}$ will be used to denote the positive integers; $\mathbb{N}_0$ will denote the nonnegative integers; and $\mathbb{N}_\infty$ will denote the positive integers with infinity.

II. SOME EXAMPLES

The following example is a partially ordered set which is not ω-chain complete, yet has the least fixed point property for ω-chain continuous functions.

EXAMPLE 1. Let P_1 be the set $\mathbb{N}_0 \times \{0,1,2\}$. Order P_1 by setting $(n,m) \le (n',m')$ if and only if one of the following conditions holds:

1) $n=m=0$
2) $n \le n'$ and $m=m' \in \{0,1\}$
3) $m=n'=0$
4) $m=0$ and $m'=2$
5) $m=m'=1$ and $n'=0$
6) $0<n \le n'$; $m=1$; and $m'=2$
7) $n=0$ and $m=m'=2$

P_1 can be represented by the diagram below.

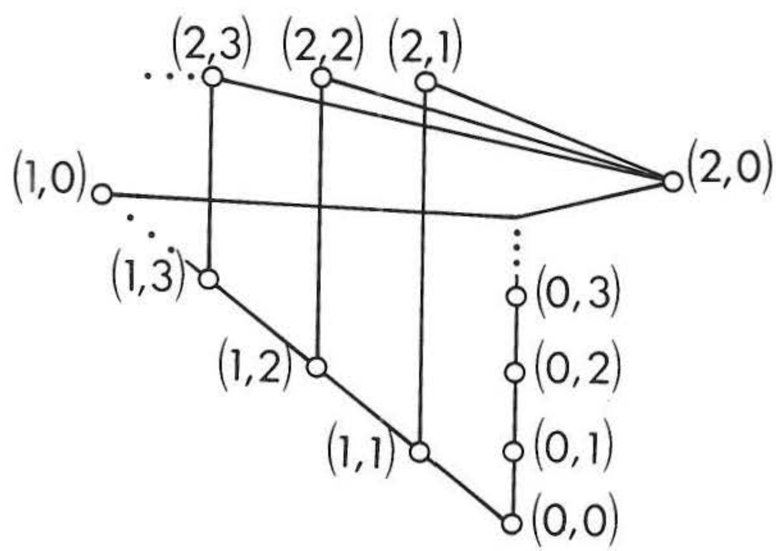

FIG. 1

Let f be an ω-chain continuous function from P_1 to itself such that $f((0,0))$ is not $(0,0)$. If $f((0,0))$ is an element of $P_1-(\mathbb{N}\times\{0\})$, then $f(P_1)$ is contained in $P_1-(\mathbb{N}\times\{0\})$. In that case, f must have a least fixed point since $P_1-(\mathbb{N}\times\{0\})$ is ω-chain complete. Therefore, assume that $f((0,0))$ is an element of $\mathbb{N}\times\{0\}$. Let U denote the set of upper bounds of $\mathbb{N}\times\{0\}$. Then $f(P_1)$ is contained in $(\mathbb{N}\times\{0\})\cup U$. If f has a fixed point in $\mathbb{N}\times\{0\}$, then f has a least fixed point in $\mathbb{N}\times\{0\}$. Assume that f has no fixed points in $\mathbb{N}\times\{0\}$. This forces $f(U)$ to be a subset of U.

Assume that $f((0,2))$ is $(0,1)$. Then for every n in $\mathbb{N}$, $f((n,2))$ equals $(0,1)$, and $f((n,1))\leq(0,1)$. If $f(\mathbb{N}\times\{1\})$ is contained in $\mathbb{N}\times\{0\}$, then $(0,1)$ must also be in $\mathbb{N}\times\{0\}$, contradicting the assumption that $f(U)$ is a subset of U. Thus, there is an N in $\mathbb{N}$ such that, for every $n\geq N$, $f((n,1))$ equals $(0,1)$. Hence $f((0,1))$ equals $(0,1)$.

If $f((0,2))$ equals $(0,2)$ then, for every n in $\mathbb{N}$, $f((n,2))$ must be above $(0,2)$. Since there is no element of P_1 greater than $(0,2)$ and $(0,1)$, $f((n,1))$ cannot equal $(0,1)$ for any n in $\mathbb{N}$. Thus, $f((0,1))$ cannot equal $(0,1)$, and $(0,2)$ is the least fixed point of f.

If $f((0,2))$ equals $(m,2)$ for some m in $\mathbb{N}$, then, for every n in $\mathbb{N}$, $f((n,2))$ equals $(m,2)$. It again follows that $f((0,1))$ cannot equal $(0,1)$; and P_1 has the least fixed point property for ω-chain continuous functions.

As noted earlier, if a partially ordered set is ω-chain complete, then the least fixed point of an ω-chain continuous function, f, from the set to itself is just $\sup\{f^n(o): n\in \mathbb{N}\}$. In P_1, however, this is not always true. Define a function, f, from P_1 to itself by

$$f(x)=\begin{cases}(0,1) & \text{if } x\notin \mathbb{N}\times\{0\}\\ (n+1,0) & \text{if } x=(n,0)\end{cases}$$

This function is clearly ω-chain continuous and has (0,1) as its only fixed point. But $\sup\{f^n((0,0)): n\in\mathbb{N}\}$ does not even exist.

The partially ordered set P_1 has the least fixed point property for ω-chain continuous functions because of the countable chain which sneaks around the antichain of upper bounds of $\mathbb{N}\times\{0\}$. The next example shows that a partially ordered set can have an unbounded countable chain and still have the least fixed point property for ω-chain continuous functions. Also, the least fixed point will be $\sup\{f^n(o): n\in\mathbb{N}\}$.

EXAMPLE 2. Let Q be the set $(\mathbb{N}\times\mathbb{N}_\infty)\cup\{o\}$. Order Q by setting $x\le y$ if and only if one of the following conditions holds.

1) $x=o$

2) $x=(j,k)$; $y=(n,m)$; and either $m=\infty=k$ and $j\le n$, or $j=n$ and $k\le m$

Then Q can be represented be the diagram below.

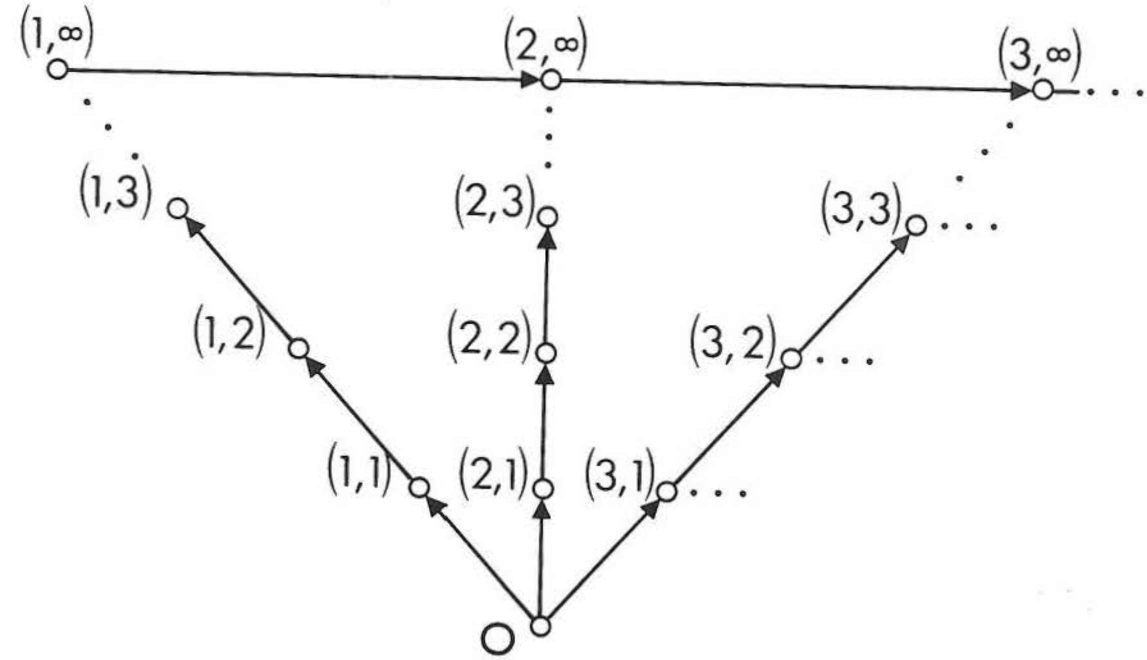

FIG. 2

Let X be the set $\mathbb{N}\times\{\infty\}$. For every n in $\mathbb{N}$, let X_n be the set $\{n\}\times\mathbb{N}$. Let P_2 be the set $Q\cup\mathbb{N}^{\mathbb{N}}$ and order P_2 by setting $x\leq y$ if and only if x and y are elements of Q and $x\leq y$ in Q; or $y=(n_1,n_2,\ldots)$ is an element of $\mathbb{N}^{\mathbb{N}}$, x is an element of Q, and $x\leq(m,n_m)$ for some m in $\mathbb{N}$. Then X is an unbounded countable chain in P_2 and $\mathbb{N}^{\mathbb{N}}$ is an antichain.

Let f be an ω-chain continuous function from P_2 to itself such that f(o) is not o. If $\sup\{f^n(o):n\in\mathbb{N}\}$ exists, then it is the least fixed point of f. It may therefore be assumed that there is no m in $\mathbb{N}$ such that $f^m(o)$ is an element of $\mathbb{N}^{\mathbb{N}}$. If $\{f^n(o):n\in\mathbb{N}\}$ is contained in X_m for some m in $\mathbb{N}$, then $\sup\{f^n(o):n\in\mathbb{N}\}$ exists. Therefore, assume that for some N in $\mathbb{N}$, for every $n\geq N$, $f^n(o)$ is an element of X.

Let M be the smallest positive integer such that $f(o) \le (M,\infty)$. Then $f(\{(n,\infty):n \ge M\})$ is contained in X; and so, for every $n \ge M$, there is a j_n in $\mathbb{N}$ such that, if $k \ge j_n$, $f((k,n))$ equals $f((n,\infty))$. Let x be the element of $\mathbb{N}^{\mathbb{N}}$ whose n^{th} component is j_n for every n in $\mathbb{N}$. Then for every n in $\mathbb{N}$, $f(x) \ge f((j_n,n)) = f((n,\infty))$. It follows that $f(x)$ is an upper bound of $\{f^n(o):n\in\mathbb{N}\}$; that $\{f^n(o):n\in\mathbb{N}\}$ is finite; and that $\sup\{f^n(o):n\in\mathbb{N}\}$ exists.

This example can be used to show another difference between fixed point properties for ω-chain continuous functions and those for order preserving functions. R. E. Smithson ([S2], Thm 1.2) proved the following theorem, which will not be true for the fixed point property for ω-chain continuous functions and ω-chain completeness.

THEOREM 1. Let P be a partially ordered set with a least element such that for every chain, C, of P, the set of upper bounds of C is down directid. If P has the fixed point property, then P is chain complete.

This theorem has the following obvious corollary.

COROLLARY 1. A meet-semilattice has the fixed point property if and only if it is chain complete.

These theorems do not carry over to ω-chain continuous functions and ω-chain completeness, however, since $\mathfrak{F}(P_2)$, the free meet-semilattice generated by P_2, contains an unbounded countable chain, but has the least fixed point property for ω-chain continuous functions. The pairs of elements of P_2 which do not already have infima are those pairs in which both elements are from the antichain $\mathbb{N}^{\mathbb{N}}$, and those in which one element is from $\mathbb{N}^{\mathbb{N}}$ and the other is from X.

EXAMPLE 3. Let P_3 be the set $P_2 \cup (FS'(\mathbb{N}^{\mathbb{N}})) \cup (X \times FS(\mathbb{N}^{\mathbb{N}}))$, where $FS(\mathbb{N}^{\mathbb{N}})$ is the set of all nonempty finite subsets of $\mathbb{N}^{\mathbb{N}}$ and $FS'(\mathbb{N}^{\mathbb{N}})$ is the set of all finite subsets of $\mathbb{N}^{\mathbb{N}}$ which have at least two elements. Define an order on P_3 by setting $x \leq y$ if and only if one of the following conditions holds:

1) $x=A \in FS'(\mathbb{N}^{\mathbb{N}})$; $y \in P_2$; and $y \in A$
2) $x \in P_2$; $y=A \in FS(\mathbb{N}^{\mathbb{N}})$; and $x \leq a$ for every a in A
3) $x=((n,\infty),A) \in X \times FS(\mathbb{N}^{\mathbb{N}})$; $y \in P_2$; and $(n,\infty) \leq y$ or $A \leq y$
4) $x \in P_2$; $y=((n,\infty),A) \in X \times FS(\mathbb{N}^{\mathbb{N}})$; and $x \leq (n,\infty)$ and $x \leq A$
5) $x=A \in FS'(\mathbb{N}^{\mathbb{N}})$; $y=B \in FS'(\mathbb{N}^{\mathbb{N}})$; and $B \subset A$
6) $x=((n,\infty),A) \in X \times FS(\mathbb{N}^{\mathbb{N}})$; $y=B \in FS'(\mathbb{N}^{\mathbb{N}})$; and $B \subset A$
7) $x=((m,\infty),A) \in X \times FS(\mathbb{N}^{\mathbb{N}})$; $y=((n,\infty),B) \in X \times FS(\mathbb{N}^{\mathbb{N}})$; and $m \leq n$ and $B \subset A$

Then P_3 is the free meet-semilattice generated by P_2. Notice that X is still unbounded in P_3. In fact, it is still the only chain that does not have a supremum.

Let $C=\{c_n : n \in \mathbb{N}\}$ be an infinite chain in P_3-X. Clearly, if C is a subset of P_2 then C has a supremum. Therefore, assume that C is a subset of $FS'(\mathbb{N}^{\mathbb{N}}) \cup (X \times FS(\mathbb{N}^{\mathbb{N}}))$. For every k in $\mathbb{N}$, let A_k be an element of $FS(\mathbb{N}^{\mathbb{N}})$ such that either $c_k=((n_k,\infty),A_k)$ or $c_k=A_k$. Then, since every A_k is finite, $A=\bigcup_{k \in \mathbb{N}} A_k$ is in $FS(\mathbb{N}^{\mathbb{N}})$ and $c_k \leq A_k \leq A$ for every k. If there were an m in $\mathbb{N}$ such that, for every k in $\mathbb{N}$, $n_k \leq m$, then C would be finite, since the sequence A_1, A_2,... must be finite. If B is an element of $FS'(\mathbb{N}^{\mathbb{N}})$ and B is an upper bound of C, then B is a subset of A_k for all k and therefore $A \leq B$. If x is an element of $\mathbb{N}^{\mathbb{N}}$ and x is an upper bound of C, then x is an element of A_k for all k and so $A \leq x$. Thus A is the supremum of C. The proof that P_3 has the least fixed point property for ω-chain continuous functions is now the same as that for P_2.

III. RETRACTS AND AN EXAMPLE

A function, f, from a partially ordered set, P, to itself is a retraction if and only if f is ω-chain continuous and is the identity on f(P) . The image of a retraction on P is called a retract of P . Obviously, if P has the least fixed point property for ω-chain continuous functions, then so does every retract of P . Is it possible to make the implication go the other way? If C is a nonempty countable chain in P which does not have a supremum in P ; U is a set of upper bounds of C ; and $C \cup U$ is a retract of P, then if P is to have the least fixed point property, U must have a least element. This is the case in the examples given so far. In [M], Markowsky used such a property for his characterization of chain complete partially ordered sets.

Lemma 1 gives a better idea of when a partially ordered set can have an unbounded countable chain as a retract. Recall that any nonempty countable chain contains a cofinal subchain of order type ω .

DEFINITION. A subset, D, of a partially ordered set, P, is closed if and only if for every nonempty countable chain, C, in D, if C has a supremum in P then sup(C) is in D. A subset of P is open if and only if its complement is closed.

LEMMA 1. An unbounded countable chain $C=\{c_n : n \in \mathbb{N}\}$ of a partially ordered set, P, is a retract of P if and only if there is a collection of disjoint closed subsets of P, $\{D_n : n \in \mathbb{N}\}$, such that

1) $P = \bigcup_{n \in \mathbb{N}} D_n$

2) $\forall n \in \mathbb{N}$, $c_n \in D_n$

3) if $p \in D_m$; $q \in D_n$; and $p \leq q$, then $m \leq n$.

PROOF. Assume that C is a retract of P and let $f: P \to C$ be a retraction. For every n in $\mathbb{N}$, let D_n be the set $f^{-1}(c_n)$. Clearly the D_n's form a collection of disjoint subsets of P; P equals $\bigcup_{n \in \mathbb{N}} D_n$; and, since f preserves order, if p is an element of D_m, q is an element of D_n, and $p \leq q$, then $m \leq n$. Let $X = \{x_n : n \in \mathbb{N}\}$ be a chain in D_m which has a supremum in P. Then $f(\sup(X)) = \sup(f(X)) = c_m$, so that $\sup(X)$ is an element of D_m.

Now let $\{D_n : n \in \mathbb{N}\}$ be a collection of disjoint closed subsets of P having the properties listed in the lemma. Define a function, f, from P to itself by $f(x) = c_n$ if and only if x is an element of D_n. This function is obviously order preserving, so it remains to show that it preserves the supremums of nonempty countable chains. Let $X = \{x_n : n \in \mathbb{N}\}$ be an infinite chain which has a supremum in P. Assume that $\sup(X)$ is an element of D_m. Then X is contained in $\bigcup_{n \leq m} D_n$. Since X is infinte, one of these D_n's must contain an infinite number of elements of X. Then it will contain a cofinal subchain of X and hence sup(X). Thus only D_m contains an infinite number of elements of X. So f(sup(X)) equals sup(f(X)). Example 4 shows that it is not enough to consider retracts as Markowsky was able to do for order preserving functions.

EXAMPLE 4. Let P_4 be the subset of $\mathbb{N}_\infty^{\mathbb{N}}$ consisting of those elements $\bar{x}=(x_1,x_2,\ldots)$ such that x_n equals ∞ for only a finite number of n's, and if x_{n+1} equals ∞ then so does x_n. Order P_4 with the usual product ordering. Then P_4 is a lattice and every nonempty subset of P_4 has an infimum. Thus, every bounded chain in P_4 has a supremum.

If C is a nonempty bounded countable chain in P_4; U is a set of upper bounds of C; and $f:P_4\to C\cup U$ is a retraction, then $\sup_{C\cup U}(C)$ is an element of U since $f(\sup_{P_4}(C))=\sup_{C\cup U}(f(C))=\sup_{C\cup U}(C)$. Therefore U has a least element.

For every n in $\mathbb{N}$ let $\bar{c}_n$ be the element of P_4 having ∞ as its first n components and 1 everywhere else. Then $C=\{\bar{c}_n:n\in\mathbb{N}\}$ is an unbounded chain. Let $\{D_n:n\in\mathbb{N}\}$ be a collection of disjoint closed subsets of P_4 such that, for each n, $\bar{c}_n$ is an element of D_n and if p and q are elements of D_m and D_n respectively, then if $p\le q$, $m\le n$. For every n, let E_n be the set $\bigcup_{m\le n} D_m$.

Set x_1 equal to 1. Assume that x_k has been defined for every $k\le n$. Let X_n be the set of all positive integers, x, such that $(x_1,\ldots,x_n,x,1,1,1,\ldots)$ is an element of E_n. If X_n is empty, set x_{n+1} equal to 1. If not, then it must be finite since E_n is closed and does not contain $\bar{c}_{n+1}$. In this case, set x_{n+1} equal to $(\max(X_n))+1$. Each x_n defined by this process is a positive integer, so $\bar{x}=(x_1,x_2,\ldots)$ is in P_4. But $\bar{x}$ is not contained in any D_n. If $\bar{x}$ were in D_n for some n, then $\bar{y}=(x_1,\ldots,x_{n+1},1,1,1,\ldots)$ would be in D_m for some $m\leq n$, since $\bar{y}\leq\bar{x}$. Thus, $\bar{y}$ would be in E_n, which contradicts the definition of x_{n+1}.

Let $C'=\{\bar{c}'_n:n\in\mathbb{N}\}$ be an unbounded chain in P_4 and let f be an ω-chain continuous map from P_4 to C'. For every n in $\mathbb{N}$, define a sequence whose supremum is $\bar{c}'_n$ as follows. For every $j,k\in\mathbb{N}$, let

$$x_{njk}=\begin{cases} j & \text{if } j\leq c'_{nj} \text{ and } k\leq j \\ c'_{nj} & \text{if } c'_{nj}<j \text{ and } k\leq j \\ 1 & \text{if } j<k \end{cases}$$

where $\bar{c}'_n=(c'_{n1},c'_{n2},\ldots)$, and set $\bar{x}_{nj}=(x_{nj1},x_{nj2},\ldots)$. Let X_n be the set of all the $\bar{x}_{nj}$'s. Notice that for every n in $\mathbb{N}$, every element of X_n is less than an element of C. Therefore, if C' is the image of P_4 under f, f(C) must be a cofinal subchain of C'. Assume that this is so and define a function, g, taking C' onto a cofinal subchain of C by $g(x)=\min\{\bar{c}_n:x\leq f(\bar{c}_n)\}$. Then $g\circ f$ is a retraction from P_4 onto a cofinal subchain of C. But this is impossible, so C' cannot be the image of P_4 under f. Thus, no unbounded chain is a retract of P_4. In fact, the same argument will show that there is no ω-chain continuous function from P_4 onto an unbounded countable chain.

Even though P_4 satisfies this nice property for retracts, it does not have the least fixed point property for ω-chain continuous functions. Denote the point (1,1,...) by $\bar{1}$, and, for every $\bar{x}=(x_1,x_2,\ldots)$ in P_4, let $\bar{x}+\bar{1}$ be the point $(x_1+1,x_2+1,\ldots)$, where $\infty+1$ equals ∞. Define a function, f, taking P_4 into itself by $f(\bar{x})=\bar{x}+\bar{1}$. This function is clearly order preserving. Let $C=\{\bar{c}_n:n\in\mathbb{N}\}$ be a chain which has a supremum in P_4. Then $f(\sup(C))=(\sup(C))+\bar{1}=\sup\{\bar{c}_n+\bar{1}:n\in\mathbb{N}\}=\sup(f(C))$. Thus, f is an ω-chain continuous functions which has no fixed points.

IV. FINITE WIDTH

The partially ordered sets in the first three examples all contained infinite antichains. It is also clear that any chain has the least fixed point property for ω-chain continuous functions if and only if it is ω-chain complete. So it seems reasonable that any partially ordered set in which the cardinalities of all its antichains are finite and bounded would have the least fixed point property for ω-chain continuous functions if and only if it is ω-chain complete. Before this is shown to be true, some notation and a definition are in order.

DEFINITION. A partially ordered set, P, has finite width if there is a positive integer, N, such that, for every antichain, A, of P, the cardinality of A is less than or equal to N.

For every element, x, of P, the lower end generated by x, denoted $\downarrow x$, is the set of all elements of P less than or equal to x. For a subset, X, of P, the lower end generated by X, denoted $\downarrow X$, is $\bigcup_{x \in X} \downarrow x$. The sets $\uparrow x$ and $\uparrow X$ are defined dually.

LEMMA 2. If a partially ordered set, P, contains a countable chain, C, whose set of upper bound, U, contains a chain, D, which is not bounded below by any element of U and if C is a retract of $\bigcap_{d \in D} \downarrow d$, then P does not have the fixed point property for ω-chain continuous functions.

PROOF. Let D' be a subchain of D such that every subset of D' has a largest element and every element of D has an element of D' below it. Denote D' by $\{d_\alpha : \alpha<\sigma\}$ for some ordinal σ. Set E_0 equal to $P-\downarrow d_0$ and, for every $\alpha>0$, let E_α be the set $(\downarrow d_\alpha)-(\downarrow d_{\alpha+1})$. Let f be a retraction of $\bigcap_{d\in D}\downarrow d$ onto C. Define a function, $g:P\to D'\cup C$, by

$$g(x)=\begin{cases} d_\alpha & \text{if } x\in E_\alpha \\ f(x) & \text{if } x\in \bigcap_{d\in D}\downarrow d \end{cases}$$

Let x and y be elements of P such that $x\leq y$. If x and y are elements of $\bigcap_{d\in D}\downarrow d$, then $g(x)=f(x)\leq f(y)=g(y)$. If x is an element of $\bigcap_{d\in D}\downarrow d$ and y is an element of $P-\bigcap_{d\in D}\downarrow d$, then $g(x)$ is in C; $g(y)$ is in D'; and $g(x)\leq g(y)$. If x and y are elements of $P-\bigcap_{d\in D}\downarrow d$, then let $\alpha<\sigma$ such that y is an element of E_α. Then if x is in E_β, β must be at least as large as α. Thus $g(x)=d_\beta\leq d_\alpha=g(y)$. So g preserves order.

Let X be a nonempty countable chain in P which has a supremum in P. If $\sup(X)$ is an element of $\bigcap_{d\in D}\downarrow d$, then X is a subset of $\bigcap_{d\in D}\downarrow d$ and $g(\sup(X))=f(\sup(X))=\sup(f(X))=\sup(g(X))$. If $\sup(X)$ is an element of $P-\bigcap_{d\in D}\downarrow d$, then, because g preserves order and $\bigcap_{d\in D}\downarrow d$ is closed, it may be assumed that X is contained in $P-\bigcup_{d\in D}\downarrow d$. Let $\alpha<\sigma$ such that sup(X) is in E_α. Since $\downarrow E_{\alpha+1}$ equals $\downarrow d_{\alpha+1}$ and is closed, X cannot be contained in $\downarrow E_{\alpha+1}$. Thus, from some point on, every element of X is in E_α. Hence, g(sup(X)) equals sup(g(X)). Therefore, $C\cup D'$ is a retract of P.

Because $C\cup D'$ does not have the fixed point property for ω-chain continuous functions, P does not either.

LEMMA 3. If a partially ordered set, P, contains a countable chain, C , whose set of upper bounds, U , contains two minimal elements, x and y , with a common upper bound, z, and if C is a retract of $(\downarrow x)\cap(\downarrow y)$, then P does not have the least fixed point property for ω-chain continuous functions.

PROOF. Let f be a retraction from $(\downarrow x)\cap(\downarrow y)$ onto C, and define a function, $g:P\to C\cup\{x,y,z\}$, by

$$g(p)=\begin{cases} f(p) & \text{if } p\in(\downarrow x)\cap(\downarrow y)\\ x & \text{if } p\in(\downarrow x)-(\downarrow y)\\ y & \text{if } p\in(\downarrow y)-(\downarrow x)\\ z & \text{if } p\in P-[(\downarrow x)\cup(\downarrow y)]\end{cases}$$

Let p and q be elements of P such that $p\leq q$. The function, g, obviously preserves order in each of the subsets of P given in the definition of g. If p is an element of $(\downarrow x)\cap(\downarrow y)$ and q is not, then g(p) is an element of C and g(q) is in $\{x,y,z\}$. Therefore, g(p) is less than g(q). Assume that p is an element of $(\downarrow x)-(\downarrow y)$. Then q is an element either of $(\downarrow x)-(\downarrow y)$ or of $P-[(\downarrow x)\cup(\downarrow y)]$. In either case, $g(p)$ is less than or equal to $g(q)$. The other case being analogous, g preserves order.

Since each of the subsets given in the definition of g is closed and g is ω-chain continuous on all of them, g is ω-chain continuous on P. Because $C\cup\{x,y,z\}$ does not have the least fixed point property for ω-chain continuous functions, P does not either.

A subset, X, of a partially ordered set, P is an upper end if and only if X equals $\uparrow X$. It is a lower end if and only if X equals $\downarrow X$.

LEMMA 4. If a partially ordered set, P, has finite width and A and B are nonempty disjoint subsets of P such that B is an upper end, then there is only a finite number of points in B which are suprema of countable chains in A.

PROOF. Let L be the set of elements of B which are suprema of countable chains in A. Assume that L is infinite and let M be a maximal antichain of L. Since M is finite, there is some element of M, say p_1, such that p_1 is comparable to an infinite number of elements of L. Either $(\uparrow p_1)\cap L$ is infinite or $(\downarrow p_1)\cap L$ is. If $(\uparrow p_1)\cap L$ is infinite, let K_1 be the set $(\uparrow p_1)\cap L$. Otherwise, set K_1 equal to $(\downarrow p_1)\cap L$.

Let $n\in\mathbb{N}$ and assume that elements $p_1,p_2,\ldots,p_n$ and subsets K_{n-1} and K_n of L have been defined and satisfy the following properties.

1) $\{p_1,\ldots,p_n\}$ is a chain in L

2) K_n is infinite

3) $K_n=(\uparrow p_n)\cap K_{n-1}$ or $K_n=(\downarrow p_n)\cap K_{n-1}$

Let M be a maximal antichain of $K_n-\{p_1,\ldots,p_n\}$. Since M is finite, one of its elements, p_{n+1}, is comparable to an infinite number of elements of K_n. Define K_{n+1} to be $(\uparrow p_{n+1})\cap K_n$ if that set is infinite, and $(\downarrow p_{n+1})\cap K_n$ otherwise.

The chain $\{p_n:n\in\mathbb{N}\}$ thus obtained is infinite and so contains either an increasin or a decreasing chain. Assume that $\{p_n:n\in\mathbb{N}\}$ is increasing, that is, that $p_1<p_2<p_3<\cdots$.

For each n in $\mathbb{N}$, let $X_n=\{x_{nm}:n\in\mathbb{N}\}$ be a chain in A which has p_n as a supremum. Because $\{p_n:n\in\mathbb{N}\}$ is increasing, there is, for every n in $\mathbb{N}$, an N_{nm} in $\mathbb{N}$, where $m<n$, such that if $j\geq N_{nm}$, then for any x_{mk} in X_m, $x_{nj}\not\leq x_{mk}$. If x_{mk} were less than x_{nj} for every k in $\mathbb{N}$, then p_m would be less than or equal to x_{nj}. This is not possible since B is an upper end and A and B are disjoint. Thus, for every $j\geq N_{nm}$, there is an M_{nmj} in $\mathbb{N}$ such that if $k\geq M_{nmj}$, then x_{mk} and x_{nj} are incomparable. Let $n\in\mathbb{N}$ and, for each $0\leq j\leq n-1$, set k_j equal to $\max(\{N_{n-j,i}:1\leq i<n-j\}\cup\{M_{n-i,n-j,k_i}:0\leq i\leq j-1\})$. Then $\{x_{n-j,k_j}:0\leq j\leq n-1\}$ is an antichain. Therefore, P contains arbitrarily large finite antichains, contradicting the assumption that P has finite width.

By reversing the order of P , the same argument works when $\{p_n:n\in\mathbb{N}\}$ is a decreasing chain. Thus, L is finite.

THEOREM 2. If P is a parially ordered set of finite width, then any unbounded chain in P of order type ω is a retract of P.

PROOF. Let $C=\{c_n : n\in\mathbb{N}\}$ be an unbounded chain of P. Set E_1 equal to $P-(\uparrow c_2)$ and, for every $n\geq 2$, let E_n be the set $(\uparrow c_n)-(\uparrow c_{n+1})$. For every n in $\mathbb{N}$, let L_n' be the set of all the elements of $P-E_n$ which are suprema of nonempty countable chains in E_n. Let L_n equal $L_n' - \bigcup_{k<n} L_k'$. By Lemma 4 each L_n is finite. For every $m,n\in\mathbb{N}$, let K_{nm} be the set $L_n \cap E_n$. Notice that K_{nm} is empty if $n\geq m$.

Let p be an element of K_{nm} and let V_p'' be an open upper end containing p. Because it is an upper end, V_p'' is also closed. Let V_p' be the intersection of V_p'' with $\uparrow c_n$. There may be elements of V_p' which are the suprema of nonempty countable chains in $P-(\uparrow c_n)$. Let A be the collection of all such elements of V_p'. By Lemma 4, A is finite, so $V_p = V_p' - \bigcup_{a\in A}(\downarrow a)$ is an open upper end. Now let U_p be the set $(V_p - \bigcup_{n\leq k\leq m}(\downarrow c_k)) \cap (\bigcup_{n\leq k\leq m} E_k)$. This is an open set contained in $\bigcup_{n\leq k\leq m} E_k$ and is an upper end in that set. Hence it is open and closed with respect to $\bigcup_{n\leq k\leq m} E_k$. Set D_n equal to

$$[E_n \cup (\bigcup_{k=1}^{n} \bigcup_{p\in K_{kn}} U_p)] - \bigcup_{k=1}^{n} \bigcup_{n<j} \bigcup_{q\in K_{kj}} U_q .$$

The following properties will be shown to be true.

1) $P=\bigcup_{n\in\mathbb{N}} D_n$

2) if $n \neq m$, then $D_n \cap D_m = \phi$

3) for all n, $c_n \in D_n$

4) if $x,y \in P$; $x \leq y$; $x \in D_m$; and $y \in D_n$, then $m \leq n$

5) each D_n is closed

1) Let x be an element of P. Since P equals $\bigcup_{n \in \mathbb{N}} E_n$, x must be in E_n for some n in $\mathbb{N}$. If x is not in D_n, then there is some $1 \leq a \leq n$ and some $b > n$ and some $p \in K_{ab}$ such that x is an element of U_p. Let M be the set $\{j > n : \exists 1 \leq k \leq n, \exists q \in K_{kj} \ni x \in U_q\}$ and set m equal to max(M). Then x is an element of $E_m \cup (\bigcup_{k=1}^{n} \bigcup_{p \in K_{km}} U_p)$. Let $1 \leq a \leq m$; let $b > m$; and let p be an element of K_{kj}. Then U_q is a subset of $\bigcup_{a \leq i < b} E_i$. So if x is an element of U_q, then $a \leq n$. But if $k \leq n$, then b is an element of M, contradicting the choice of m. Thus x is not an element of $\bigcup_{k=1}^{m} \bigcup_{j > m} \bigcup_{q \in K_{km}} U_q$ and x is in D_m.

2) Let m and n be positive integers such that $m < n$. Then the intersection of E_m with E_n is empty. Let $1 \leq a \leq m$ and let p be an element of K_{am}. Then U_p is a subset of $\bigcup_{a \leq i < m} E_i$ which has an empty intersection with E_n. Therefore the intersection of $\bigcup_{k=1}^{m} \bigcup_{p \in K_{km}} U_p$ with E_n is empty. Assume that $E_m \cap (\bigcup_{k=1}^{m} \bigcup_{p \in K_{kn}} U_p)$ is not empty and let $1 \leq a \leq n$ be an element of K_{an} such that $E_m \cap U_x$ is not empty. Now U_x is a subset of $\bigcup_{a \leq i < n} E_i$, so $a \leq m$. But then U_x is a subset of $\bigcup_{k=1}^{m} \bigcup_{j > m} \bigcup_{q \in K_{kj}} U_q$;

so $(E_m - (\bigcup_{k=1}^{m} \bigcup_{j>m} \bigcup_{q\in K_{kj}} U_q)) \cap (\bigcup_{k=1}^{n} \bigcup_{p\in K_{kn}} U_p)$ is empty. Now assume that $(\bigcup_{k=1}^{m} \bigcup_{p\in K_{km}} U_p) \cap (\bigcup_{j=1}^{n} \bigcup_{q\in K_{jn}} U_q)$ is not empty. Then there exist $1\le a\le m$, $1\le b\le n$, $x\in K_{am}$, and $y\in K_{bn}$ such that $U_x \cap U_y$ is not empty. If $b\le m$, then U_y is a subset of $\bigcup_{k=1}^{m} \bigcup_{j>m} \bigcup_{q\in K_{kj}} U_q$ and $D_m \cap U_p$ is empty. Because U_y is a subset of $\bigcup_{b\le i\le n} E_i$ and U_x is a subset of $\bigcup_{a\le i\le m} E_i$, b is less than or equal to m. Therefore, $D_m \cap U_p$ is empty. Thus, $D_m \cap D_n$ is empty.

3) For every n in $\mathbb{N}$, c_n is an element of E_n. By the way they were defined, no U_p contains a c_n. So c_n is an element of D_n.

4) Let x and y be elements of P such that $x\le y$, x is an element of D_m and y is an element of D_n. Let x be an element of E_a and y an element of E_b. Since x is less than y, a must be less than or equal to b. Obviously a is less than or equal to m and b is less than or equal to n. If m is equal to a, then $m\le b\le n$. If m is greater than a, then there exists $1\le k\le a$ and $j>a$ and an element, p, of K_{kj} such that x is an element of U_p. If there are no j's greater than b, then $m\le b\le n$. If $j>b$, then U_p is an upper end of $\bigcup_{k\le i<j} E_i$, so y is an element of U_p. Therefore, m is less than or equal to n.

5) Let n be an element of $\mathbb{N}$ and let X be a nonempty countable chain in D_n. If X is a subset of E_n, then either sup(X)

is an element of E_n or $\sup(X)$ is an element of K_{kj} for some $1 \le k \le n$ and $j > n$. If $\sup(X)$ is an element of K_{kj}, then the intersection of X with $U_{\sup(X)}$ is not empty and X is not a subset of D_n. For the same reason, $\sup(X)$ cannot belong to U_p for any element, p, of K_{kj}, where $1 \le k \le n$ and $j > n$. Thus, $\sup(X)$ is an element of D_n. If X is a subset of $\bigcup_{k=1}^{n} \bigcup_{p \in K_{kn}} U_p$, then it may be assumed that X is a subset of U_p for some element, p, of K_{kn} where $1 \le k \le n$. Assume that $\sup(X)$ is not in U_p and that $\sup(X)$ is not in E_n. Let $\sup(X)$ be an element of E_m. Since U_p is a subset of $\bigcup_{k \le i < n} E_i$ and U_p is closed in this set, m must be greater than n. Again it may be assumed that X is a subset of E_j for some $k \le j \le n$. But then $\sup(X)$ is an element of K_{im} for some $1 \le i \le n$ and the intersection of X with $\bigcup_{k=1}^{n} \bigcup_{j>n} \bigcup_{q \in K_{kj}} U_q$ is not empty, contradicting the assumption that X is a subset of D_n. Therefore, $\sup(X)$ is an element of D_n.

It follows from Lemma 1 that C is a retract of P. Therefore, P does not have the fixed point property for ω-chain continuous functions.

THEOREM 3. If a partially ordered set with finite width has the least fixed point property for ω-chain continuous functions, then it is ω-chain complete.

PROOF. By Theorem 2, it may be assumed that every countable chain in a partially ordered set, P , is bounded. Let

C be a nonempty countable chain in P which does not have a supremum. Let U be the set of upper bounds of C.

By Lemmas 2 and 3 it may be assumed that U contains a set of minimal elements, M, that every element of U is above some element of M, and that no two elements of M have a common upper bound. Denote the elements of M by $m_1, m_2, \ldots, m_r$. Let D be the set of all elements, x, of P such that $\uparrow x$ is not contained in $\downarrow(\uparrow m_i) - \bigcup_{j \neq i} \downarrow(\uparrow m_j)$ for any $1 \leq i \leq r$.

Let $X = \{x_n : n \in \mathbb{N}\}$ be a chain in D which has a supremum in P. Assume that for every n in $\mathbb{N}$, there are $i \geq n$, $1 \leq j_i \neq k_i \leq r$, and $y_{j_i} \geq m_{j_i}$, and $y_{k_i} \geq m_{k_i}$ such that x_i is less than both y_{j_i} and y_{k_i}. Then there are an infinite number of pairs, (j_i, k_i), and one of the pairs, (j,k), must be repeated an infinite number of times. Let Y_j be the set $\{y_{j_i} : j_i = j\}$ and Y_k the set $\{y_{k_i} : k_i = k\}$. If Y_j is finite, then it contains an element which is an upper bound of X. If it is infinite, it must contain an infinite chain which, by assumption, is bounded. Any upper bound of this chain is, of course, an upper bound of X. In either case there is a $y_j \geq m_j$ which is an upper bound of X. Similarly, there is a $y_k \geq m_k$ which is an upper bound of X. Thus, sup(X) is less than both y_j and y_k and sup(X) is an element of D.

Assume that for every n in $\mathbb{N}$, there is a y_n in $P - \bigcup_{i=1}^{r} \downarrow(\uparrow m_i)$ such that $x_n \leq y_n$. Let Y be the set of all these

y_n's. If Y is finite, one of its elements must be an upper bound of X. If Y is infinite, it must contain an infinite chain which, by assumption, is bounded. Any upper bound of this chain will be an upper bound of X. In either case there is an element, y, of $P-\bigcup_{i=1}^{r}\downarrow(\uparrow m_i)$ which is an upper bound of X. Then sup(X) is less than or equal to y and sup(X) is an element of D. Therefore D is closed.

Since C is unbounded in D, there is, by Theorem 2, a retraction, f, of D onto a cofinal subchain, C′ of C. Define a function, $g:P\to C'\cup M$ by

$$g(x)=\begin{cases} m_i & \text{if } x\in\downarrow(\uparrow m_i)-D \quad 1\le i\le r \\ f(x) & \text{if } x\in D \end{cases}$$

Let x and y be elements of P with $x\le y$. If x is in D, then $g(x)\le g(y)$. If x is not in D, then there is some $1\le i\le r$ such that $\uparrow x$ is a subset of $\downarrow(\uparrow m_i)-\bigcup_{j\ne i}\downarrow(\uparrow m_j)$. But then y is an element of $\downarrow(\uparrow m_i)-D$, so $g(x)=m_i=g(y)$. Therefore, g preserves order.

Clearly, g is ω-chain continuous on D and on $\downarrow(\uparrow m_i)-D$ for each $1\le i\le r$. Thus g is ω-chain continuous on P; $C'\cup M$ is a retract of P; and P cannot have the least fixed point property for ω-chain continuous function.

COROLLARY 2. A partially ordered set, P, having finite width is ω-chain complete if and only if it has the least fixed point property for ω-chain continuous functions.

V. LAYERED PARTIALLY ORDERED SETS

The proof of Theorem 2 relied heavily on Lemma 4, which is no longer true if partially ordered sets not of finite width are considered. Another way must be found if Theorem 3 is to be generalized.

Recall that in the first three examples, the partially ordered sets all contained a countable chain which sneaked around an antichain, that is, they all contained an antichain which generates a nonclosed lower end. Notice also that the lower end generated by a finite antichain will always be closed. Thus no partially ordered set of finite width could have an antichain which generates a lower end that is not closed. If a partially ordered set contains an unbounded chain $C=\{c_n : n\in\mathbb{N}\}$ and there is a collection, $\{A_n : n\in\mathbb{N}\}$, of antichains such that c_n is an element of $\downarrow A_n$; P, the partially ordered set, equals $\bigcup_{n\in\mathbb{N}} \downarrow A_n$; and for each n in $\mathbb{N}$, $\downarrow A_n$ is closed, then C is arretract of P. This is the approach which will be taken next.

The first question which arises concerning this method is, Can a partially ordered set actually be partitioned in such a manner? The answer is that not all can be. For example, if a partially ordered set contains a maximal chain which has no countable cofinal subchain, then it cannot be the union of a countable number of lower ends generated by antichains. Such chains must therefore be avoided. Even so, it is not certain that such partitions occur in general. A Souslin tree is a partially ordered set which cannot be partitioned in the desired way even though all of its chains and antichains are countable. The existence of a Souslin tree is independant of the usual axioms of set theory (see [D]), so it may be that the existence of such partitions is also independant.

DEFINITION. A partially ordered set, P, is layered if and only if there is a collection, $\{A_n : n\in\mathbb{N}\}$, of antichains of P such that P is equal to $\bigcup_{n\in\mathbb{N}} {\downarrow}A_n$.

THEOREM 4. If every antichain of a layered partially ordered set, P, generates a closed lower end, then any unbounded chain in P of order type ω is a retract of P.

PROOF. Let $\{A_n : n\in\mathbb{N}\}$ be a collection of antichains of P such that P equals $\bigcup_{n\in\mathbb{N}} {\downarrow}A_n$. First, assume that for every n in $\mathbb{N}$, ${\downarrow}A_n$ is a subset of ${\downarrow}A_{n+1}$. Let $C=\{c_n : n\in\mathbb{N}\}$ be an unbounded chain in P and assume that C is not contained in ${\downarrow}A_n$ for any n. Set d_n equal to the largest element of C contained in ${\downarrow}A_n$, if there are any elements of C in ${\downarrow}A_n$, and c_1 if there are not. Define a function, $f:P\to C$, by

$$f(x)=\begin{cases} c_1 \text{ if } x\in{\downarrow}c_1 \\ c_n \text{ if } x\in({\downarrow}c_n)-({\downarrow}c_{n-1}) \text{ and } n>1 \\ d_1 \text{ if } x\in(P-{\downarrow}C)\cap({\downarrow}A_1) \\ d_n \text{ if } x\in(P-{\downarrow}C)\cap(({\downarrow}A_n)-({\downarrow}A_{n-1})) \text{ and } n>1 \end{cases}$$

Let x and y be elements of P such that $x\le y$. If y is an element of $\downarrow C$, then x is an element of $\downarrow C$ and $f(x) \leq f(y)$. If y is an element of $(P - \downarrow C) \cap ((\downarrow A_n) - (\downarrow A_{n-1}))$, where $\downarrow A_{n-1}$ is empty if n is 1, then x is an element of $\downarrow A_n$ and $f(x) \leq d_n = f(y)$.

Let X be a nonempty countable chain in P which has a supremum in P. If sup(X) is an element of $\downarrow c_n$ for some n, then X is a subset of $\downarrow c_n$. Furthermore, from some point on, all the elements of X must be in $(\downarrow c_n) - (\downarrow c_{n-1})$. Thus, $f(\sup(X)) = c_n = \sup(f(X))$. Assume that sup(X) is an element of $(P - \downarrow C) \cap ((\downarrow A_n) - (\downarrow A_{n-1}))$, where $\downarrow A_{n-1}$ is again empty if n is 1. Then, since $\downarrow A_{n-1}$ is closed, X may be assumed to be a subset of $(\downarrow A_n) - (\downarrow A_{n-1})$. In fact, it may be assumed that X is a subset of $(P - \downarrow C) \cap ((\downarrow A_n) - (\downarrow A_{n-1}))$, since if it were not it would be a subset of $\downarrow d_n$, which is closed. Therefore, $f(\sup(X)) = d_n = \sup(f(X))$. Hence C is a retract of P.

If there is an n in $\mathbb{N}$ such that C is a subset of $\downarrow A_n$, then it may be assumed that C is a subset of $\downarrow A_n$ for every n in $\mathbb{N}$. Set D_1 equal to $\bigcup_{n \in \mathbb{N}} \downarrow (A_n - (\uparrow c_2))$. For all m>1, set D_m equal to $(\bigcup_{n \in \mathbb{N}} \downarrow ((A_n - (\uparrow c_{m+1})) \cap (\uparrow c_m))) - \bigcup_{j<m} D_j$. Let x be an element of P, n an element of $\mathbb{N}$, and a an element of A_n such that x is less than a. Now a is not an upper bound of C, so either there is an m in $\mathbb{N}$ such that a is an element of $(A_n - (\uparrow c_{m+1})) \cap (\uparrow c_m)$ or it is an element of $A_n - (\uparrow c_2)$. Thus, either x is in D_m or x is in D_1. Hence P equals $\bigcup_{n \in \mathbb{N}} D_n$.

Let X be a nonempty countable chain in D_m which has a supremum in P. Since P equals $\bigcup_{n\in\mathbb{N}} {\downarrow}A_n$, there is a k in $\mathbb{N}$ such that X is contained in ${\downarrow}A_k$. Then X is contained in $({\downarrow}A_k)\cap D_m$ which is equal to ${\downarrow}((A_k-({\uparrow}c_{n+1}))\cap({\uparrow}c_n))$ for some $n>1$ or ${\downarrow}(A_k-({\uparrow}c_2))$. In either case, X is a subset of a lower end generated by an antichain. But such a lower end is closed, so sup(X) is an element of D_m. Thus, D_m is closed. Let x and y be elements of P such that x is less than y. Assume that x is an element of D_m and that y is an element of D_n. Clearly, if y is an element of ${\downarrow}((A_k-({\uparrow}c_{n+1}))\cap({\uparrow}c_n))$, then x must be also. Therefore, m is less than or equal to n. By Lemma 1, C is a retract of P.

The requirement that every antichain of P generate a closed lower end is not necessary for the first part of the previous proff. There it is only needed that each of the A_n's generate a closed lower end. The requirement is necessary in the second part, however, as is shown by the following example.

EXAMPLE 5. Let $\mathfrak{A}$ be the collection of infinite countable subsets of ω_1 which don't have a largest element. Let P_5 be the set $\omega_1\cup(\bigcup_{\alpha\in\omega_1-\omega}\ \bigcup_{A\in\mathfrak{A}} (\alpha\times A\times\{A\}))$. Let $\{X_\alpha:\alpha\in\omega\}$ be a partition of $\omega_1-\omega$ such that, for every α, X_α is infinite. If x and y are elements of P_5, set $x\le y$ if and only if one of the following conditions holds.

1) $x\in\omega$; $y\in X_\alpha$ for some $\alpha\in\omega$; and $x\le\alpha$ in ω

2) $x=(\alpha,\beta,A)\in\alpha\times A\times\{A\}$; $y=(\alpha,\delta,A)\in\alpha\times A\times\{A\}$; and $\beta\le\delta$

3) $x=(\alpha,\beta,A)\in\alpha\times A\times\{A\}$ and $y=\alpha$ or $y=\beta$

4) $x,y\in\omega$ and $x\leq y$ in ω

5) $x=0$ (the least element of ω)

Clearly, ω is an unbounded chain in P; $\omega_1-\omega$ is an antichain; and $\downarrow(\omega_1-\omega)$ is all of P_5 and is closed. Let f be an ω-chain continuous function from P_5 to itself and assume that $f(P_5)$ is a subset of ω. Then there is an element, m, of ω such that $f^{-1}(m)$ is infinite. Let A be an element of $\mathcal{G}$ which is contained in $f^{-1}(m)$. Then for every α in $\omega_1-\omega$, $f(\alpha\times A\times\{A\})$ is a subset of $\{1,2,\ldots,m\}$. Since α is the supremum of $\alpha\times A\times\{A\}$, it must be less than or equal to m. Hence for every $n\geq m$, f(n) is less than or equal to m. Thus, no cofinal subchain of ω is a retract of P_5.

In fact, P_5 has the least fixed point property for ω-chain continuous functions. Assume that f(0) is not 0. Since $P-\omega$ is ω-chain complete, it may be assumed that $\{f^n(0):n\in\mathbb{N}\}$ is contained in ω. By the argument above, it may also be assumed that tere is an α in $\omega_1-\omega$ such that $f(\alpha)$ is an element of $\omega_1-\omega$. Let $f(\alpha)$ be an element of X_m. For every $n\geq m$, let A_n be the element of $\mathcal{G}$ contained in X_n. Since α is the supremum of $\alpha\times A_n\times\{A_n\}$, there is an element, β, of A_n such that $f((\alpha,\beta,A))$ equals $f(\alpha)$. Therefore, $f(\beta)$ equals $f(\alpha)$ and f(n) is less than or equal to m. It follows that f has a least fixed point.

THEOREM 5. If every antichain of a partially ordered set, P, generates a closed lower end; every lower end of P which contains a countable chain but none of its upper bounds is layered; and P has the least fixed point property for ω-chain continuous functions, then P is ω-chain complete.

PROOF. Assume that P contains a nonempty countable chain, C, which does not have a supremum. By Theorem 4 and Lemmas 2 and 3, it may be assumed that every countable chain in P is bounded: that the set, U, of upper bounds of C has a set, M, of minimal elements; that every element of U is above an element of M; and that no two elements of M have a common upper bound.

Let D be the set of all the elements, x, of P such that $\uparrow x$ is not contained in $(\downarrow(\uparrow m)) - \bigcup_{n\in M-\{m\}} \downarrow(\uparrow n)$ for any m in M. Let $X=\{x_n : n\in \mathbb{N}\}$ be a chain in D which has a supremum in P. Assume that for every k in $\mathbb{N}$, there is a $j\geq k$, elements $m_{r_j} \neq m_{s_j}$ of M, and y_{r_j} of $\uparrow m_{r_j}$ and y_{s_j} of $\uparrow m_{s_j}$ such that x_j is less than both y_{r_j} and y_{s_j}. Let Y_r be the set of all the y_{r_j}'s and Y_s the set of all the y_{s_j}'s.

Assume that both Y_r and Y_s are infinite and that Y_r contains an infinite antichain, A_r. Since the lower end generated by an antichain is closed, sup(X) is below all but a finite number of the elements of A_r. Let Z_s be those elements of Y_s which are paired with the elements of A_r. If Z_s has an infinite antichain, A_s, then sup(X) is below all but a finite number of the elements of A_s. Thus, there is a j in $\mathbb{N}$ such that sup(X) is less than y_{r_j} and y_{s_j}. Then sup(X) is an element of D.

If every antichain of Z_s is finite, then it contains an infinite chain, C_s, which, by assumption, is bounded. Let y_s be an upper bound for this chain. Then sup(X) is less than or equal to y_s. Let B_r be the set of elements of A_r which are paired with the elements of C_s. Since B_r is an antichain and X is contained in $\downarrow B_r$, sup(X) is in $\downarrow B_r$. Let y_r be an element of B_r which is above sup(X). Then y_r and y_s are elements of U which are above distinct elements of M and are both greater than sup(X). Thus sup(X) is in D.

Assume that all antichains in both Y_r and Y_s are finite. Let C_r be an infinite chain in Y_r; let y_r be an upper bound of C_r; and let m_r be the element of M less than or equal to y_r. Let Z_s be the set of elements of Y_s which are paired with the elements of C_r. Then no element of Z_s is greater than m_r. Since every antichain of Z_s is finite, it must contain an infinite chain, C_s. Let y_s be an upper bound of C_s and let m_s be the element of M less than or equal to y_s. Then m_r does not equal m_s and sup(X) is less than both y_r and y_s. Therefore, sup(X) is in D.

Assume that for every k in $\mathbb{N}$, there is a $y_k \geq x_k$ such that y_k is not contained in $\downarrow(\uparrow m)$ for any m in M. Let Y be the set of all the y_k's. Assume that Y is infinite. If Y contains an infinite antichain, then sup(X) is in the lower end it generates and is therefore in D. If Y does not contain an infinite antichain, then it contains an infinite chain which is bounded. Let y be an upper bound for this chain. Then y is above sup(X) and is not in $\downarrow(\uparrow m)$ for any m in M. Therefore, D is closed.

Since C is unbounded in D, there is, by Theorem 4, a retraction, f, of D onto C′, a cofinal subchain of C of order type ω. Let m_1 and m_2 be distinct elements of M. Define a function $g:P\to C'\cup\{m_1,m_2\}$, by

$$g(x)=\begin{cases} m_1 & \text{if } \uparrow x\subset(\downarrow(\uparrow m_1))-D \\ m_2 & \text{if } \uparrow x\subset(\downarrow(\uparrow m))-D \text{ and } m\neq m_1 \\ f(x) & \text{if } x\in D \end{cases}$$

Then f preserves order. Since the sets $(\downarrow(\uparrow m_1))-D$ and $\bigcup_{m\in M-\{m_1\}}((\downarrow(\uparrow m))-D)$ are closed and g is ω-chain continuous on them and on D, it is ω-chain continuous on P. Thus, $C'\cup\{m_1,m_2\}$ is a retract of P and P cannot have the least fixed point property for ω-chain continuous functions.

COROLLARY 3. If every antichain of a partially ordered set, P, generates a closed lower end and every lower end of P which contains a countable chain but none of its upper bounds is layered, then P is ω-chain complete if and only if it has the least fixed point property for ω-chain continuous functions.

The following theorem by Edwin Miller appears as Theorem

B in [Mi].

THEOREM 6. If every antichain of an uncountable partially ordered set, P, is finite, then P contains an uncountable chain.

COROLLARY 4. If every antichain of a partially ordered set, P, is finite and every chain contains a countable cofinal subchain, then P is layered.

PROOF. Assume that there is no countable collection, $\mathcal{C}$, of antichains of P such that P equals $\bigcup_{A\in\mathcal{C}} A$. Let A_1 be a maximal antichain of P. If β is less than ω_1 and, for every α less than β, A_α has been defined, then let A_β be a maximal antichain of $P-\bigcup_{\alpha<\beta} A_\alpha$. The set $A=\bigcup_{\alpha<\omega_1} A_\alpha$ is an uncountable partially ordered set and, by Theorem 6, contains an uncountable chain, C. But C must intersect an uncountable number of A_α's and so cannot contain a countable cofinal subchain.

COROLLARY 5. If every antichain of a partially ordered set, P, is finite and every chain contains a countable cofinal subchain, then P is ω-chain complete if and only if it has the least fixed point property for ω-chain continuous functions.

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