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Abstract In 2002 Coecke and Martin ([3]) created a model for the finite classical and quantum states in physics. This model is based on a type of ordered set which is standard in the study of information systems. It allows the information content of its elements to be compared and measured. Their work is extended to a model for the infinite classical states. These are the states which result when an observable is applied to a quantum system. When this extended order is restricted to a finite number of coordinates, the model of Coecke and Martin is obtained. The infinite model retains many desirable aspects of the finite model, such as pure states as maximal elements and expected behavior of thermodynamic entropy. But it looses some of the important domain theoretic aspects, such as having a least element and exactness. Shannon entropy is no longer defined over the entire model and both it and thermodynamic entropy cease to be a measurements in the sense of Martin.

1 Introduction

In 1970 Dana Scott ([8]) introduced the concept of using ordered sets as models for systems of information. The ordered sets he used, called domains, have proved useful in many settings that can be expressed in terms of some type of knowledge or information. The role of information has become important in the interpretation of quantum physics. See, for example, Clifton, Bub, and Halvorson [2] or Fuchs [5]. Recently Coecke and Martin ([3]) used a modified version of these domains to create a model, Δ^n, for finite classical states and a model, Ω^n for finite quantum states. The classical states are the states which result when an observable is applied to a quantum system. One can also think of choosing a frame of reference for the

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quantum system. Then there are a finite number of possible outcomes, each with its own probability of actually occurring. The model Δ^n is therefore based on the set of n-tuples with nonnegative coordinates whose sum equals one. The idea is that each coordinate represents the probability that the system under consideration has the particular outcome represented by that coordinate. The order defined on this set, called the Bayesian order, captures the notion of increasing certainty. If $x, y \in \Delta^n$ and $x < y$ then y represents a movement from the uncertain state x towards a particular pure state, in which one coordinate has value 1 and the others have value 0. These pure states form the maximal elements of the model. The model contains a minimum element (the completely mixed state each of whose coordinates has value $1/n$). It also satisfies the Mixing Law and Degeneracy. In this model, entropy decreases as certainty increases.

The Bayesian order gives an order-theoretic structure to the physical information contained in the classical states. The content of the information contained in a state can be measured and the information contained in different states can be compared. The model can predict which states contain essential information for other states.

In this paper, we will extend the model for finite classical states to one for infinite classical states. We will make use of symmetries, similar to those used by Coecke and Martin, to define an order on functions defined on ω, the set of nonnegative integers. These functions take the place of the n-tuples and behave like ω-tuples with nonnegative coordinates. This model retains some of the nice properties of Δ^n such as pure states, Degeneration, the Mixing Law, and thermodynamic entropy. The order also represents increasing certainty. But some properties are lost. There is no longer a minimal element, and an important relation defined by Coecke and Martin to represent approximation does not work in this model. This means that the idea of essential information is lost. While the notion of entropy associated with thermodynamics is retained Shannon entropy, more commonly associated with quantum information theory, is not defined over the entire infinite model. Nevertheless, Shannon entropy still provides a picture of what is happening in the system. The standard functions which measure the information content of elements of the model are no longer able to do so in the infinite case.

In the next section we will give the background and basic definitions and notation that are needed in the following sections. In Section 3, the order is defined and its basic properties are established. In Section 4 the structure of the model is developed. We will see that the Mixing Law applies, that the model has an infinite number of maximal elements, that it is directed complete, that it can be divided into an infinite number of order isomorphic pieces, and that it has no minimal elements. In Section 5 it is shown that the notion of approximation used in [3] does not help with approximation in this model because no element is weakly way below any other element. In Section 6 it is shown that while the natural retraction still retains the nice behavior exhibited in Δ^n, as does the function $v(f) = 1 - f^+$ and entropy $s(f) = -\ln f^+$, Shannon entropy is not defined everywhere. We will also see that these functions are no longer measurements of the model.

2 Background

An order ($<$) on a set X is a relation that is asymmetric and transitive. A partial ordering ($\leq$) is a relation that is reflexive, antisymmetric and transitive. We can clearly switch back and forth easily between the two kinds of relations. If $a \in X$ then $\uparrow a = \{b \in X : a < b\}$, $\downarrow a = \{b \in X : b < a\}$, $\underline{\uparrow}\, a = \{b \in X : a \leq b\}$, and $\overline{\downarrow}\, a = \{b \in X : b \leq a\}$. We say that a subset A of X is *increasing* if $A = \underline{\uparrow}\, A = \bigcup_{a \in A} \underline{\uparrow}\, a$.

Definition 1 A subset D of an ordered set X is *directed* if and only if for every $a, b \in D$ there is $c \in D$ such that $a \leq c$ and $b \leq c$.

In terms of information, to say that D is directed means that for every pair of states contained in D there is another state in D which extends the information contained in both states in the pair. This means that the information contained in states in D is consistent. If a (possibly infinite) number of observers reach conclusions on the state of a physical system based on their individual results and these various states are collected into a set D, then to say that D is directed means when states for observers A and B are compared there is a state belonging to another observer which is consistent with which the first two states. That is, while the results of A and B differ, neither is surprised by the results of C.

Definition 2 An ordered set X is said to be *directed complete* if and only if every nonempty directed subset of X has a supremum in X.

If X is directed complete then we will say that it is a directed complete partial order, or a *dcpo*. As the name implies, a dcpo represents an information system that is complete in the sense that consistent collections of information lead somewhere. There is a larger piece of information that is obtained by combining all the smaller bits.

Definition 3 A set U of an ordered set X is *Scott open* if and only if it is increasing and, for every directed $D \subseteq X$, if $\sup D \in U$ then $D \cap U \neq \emptyset$.

The collection of Scott open subsets forms the Scott topology. A subset of X, or collection of states, will be Scott open when the subset contains all the states that give more information about the system than a state known to be in the collection, and when it is impossible to reach the information contained within a state from the collection without first obtaining information from a state already known to be in the collection. Every ordered set admits a Scott topology. A relation that is intimately connected to the Scott topology is the *way below* relation. Let X be an ordered set and $a, b \in X$. Then a is way below b, or $a \ll b$, if and only if, for every directed $D \subseteq X$, if $\sup D \geq b$ then $D \cap \underline{\uparrow}\, a \neq \emptyset$. When a state a is way below a state b it means that a contains information that is essential if one is to obtain the information contained in state b. Every path to b could use a as its starting point. Let $\twoheaduparrow a = \{b \in X : a \ll b\}$ and $\twoheaddownarrow a = \{b \in X : b \ll a\}$. The ordered set X is *continuous* if, for every $a \in X$, $\twoheaddownarrow a$ is directed and $\sup \twoheaddownarrow a = a$. When X is continuous, every element of X can be approximated by the elements that are way below it. Coecke and Martin relate this to the notion of *partiality*, in which the approximating elements are said to be partial, and the approximated elements are total. See [4] for more information on partiality in physics. The continuity of X also gives us a good way of constructing the Scott topology, for when X is continuous then $\{\twoheaduparrow a : a \in X\}$ is a basis for the Scott topology. For more information on the way below relation and the Scott topology see [1] and [8].

Unfortunately, Δ^n is not continuous. The set $\{\twoheaduparrow x : x \in \Delta^n\}$ still generates a topology, but not the Scott topology. Many elements of Δ^n are not way below any other element. Coecke and Martin were able to preserve a notion of approximation by slightly changing the definition of the way below relation. We will say that a is *weakly way below* b, or $a \ll_w b$, if and only if, for every directed $D \subseteq X$, if $\sup D = b$ then $D \cap \uparrow a \neq \emptyset$. Let $\twoheaduparrow_w a = \{b \in X : a \ll_w b\}$ and $\twoheaddownarrow_w a = \{b \in X : b \ll_w a\}$. X is said to be *exact* if, for every $a \in X$, $\twoheaddownarrow_w a$ is directed and $\sup \twoheaddownarrow_w a = a$. Exactness, like continuity, provides an idea of approximation. Δ^n is exact. The weakly way below relation provides a notion of essential information just as the way below relation does. If $a \ll_w b$ then we cannot build a knowledge of b without knowing a. The difference in the relations is that if $a \ll b$ then we cannot build a knowledge of a state which contains even more information than b without knowing a. In the weakly way below relation we can build up such knowledge without knowing a. For more information on the weakly way below relation see [7]. In [6] Martin modifies the order on Δ^n to obtain a continuous model for the classical states. But this order does not extend to quantum states, whereas the Bayesian order does.

In the following sections ω is the set of natural numbers, or nonnegative integers. For every $n \in \omega$, $n = \{m \in \omega : m < n\} = \{0, 1, \ldots, n-1\}$. Saying that $m \in n$ is the same as saying that $m < n$. We will use the standard notation of $\operatorname{dom} f$ as the domain of the function f, and $\operatorname{ran} f$ as its range. We will use f^+ to represent $\max \operatorname{ran} f$.

3 Definition of the Model

The model we will construct should reflect the situation when an observable is applied to an infinite dimensional quantum system. Then the system is in a state in which an infinite number of outcomes are possible and each outcome has a certain probability of actually occurring. This situation can be represented by a function with an infinite number of possible values. Since the values are supposed to be probabilities, they must all be nonnegative and add up to one. Let Δ^ω be the set of functions from ω into $[0,1]$ such that $\sum_{n\in\omega} f(n) = 1$. We can again think of f as giving us probabilities for being in a particular state. Here $f(n)$ is the probability that our system has the outcome represented by n. Note that if $f, g \in \Delta^\omega$ and there is $m \in \omega$ such that $f(m) < g(m)$ then there must be $n \in \omega$ such that $f(n) > g(n)$. This means that if $f(m) \leq g(m)$ for all $m \in \omega$ then $f = g$.

Some of our results require a more general setting, so let C be the set of functions from ω into $[0,1]$ such that $\sum_{n\in\omega} f(n)$ converges.

Let $f \in C$ and think of f as an infinite string of numbers whose sum is finite. We can reorder f so that the numbers are decreasing (in the sense of nonincreasing). We will use such reorderings to define the order on Δ^ω, and we would like all of them to "look alike". But we will have trouble getting this because of zeroes. If $f(n) > 0$ for a finite number of values of n then a reordering of f will result in a sequence of order type ω. But if $f(n) > 0$ for an infinite number of values of n and there are still some values of n at which $f(n) = 0$ then a reordering of f results in a sequence of order type larger than ω. We will have an infinite sequence of nonzero numbers, followed by a sequence of zeros. When this happens we will essentially throw the extra zeros away. Since the outcomes represented by the coordinates giving us a value of 0 cannot occur, we do not lose any physical significance by eliminating them.

Definition 4 For every $f \in C$ let $R(f)$ be the set of one-to-one functions σ from ω into ω such that $f^{-1}[(0,1]] \subseteq \operatorname{ran}\sigma$ and $f \circ \sigma$ is decreasing.

For every $f \in C$ the set $R(f)$ represents the number of ways that f can be rearranged into a decreasing sequence.

The following fairly obvious lemma will be very useful in the development of the model.

Lemma 5 *Let $f \in C$. If $\sigma, \tau \in R(f)$ then $f \circ \sigma = f \circ \tau$.*

Before defining the order on Δ^ω we must consider what it is that the order should tell us. The relation $f < g$ should occur when g gives us more certain information about the state of our system and is consistent with what we know from the earlier observation. This means that outcomes with lower probabilities in f should become even lower in g, and those with higher probabilities in f should become even higher in g. The relative differences in the values we get from g should become larger than those we get from f. Coecke and Martin were able to codify this in the following way. Let $f, g \in C$. To simplify notation, we will assume that f and g are already decreasing, and that they never take on the value 0. Then g is more certain than f when $f(n)/f(n+1) \leq g(n)/g(n+1)$ for all $n \in \omega$. If we write this inequality as $f(n)g(n+1) \leq f(n+1)g(n)$ then we can drop the assumption that f and g are never zero. This leads us to the following two definitions.

Definition 6 For every $f \in C$ set $T(f,g)$ equal to the set of all $\sigma \in R(f) \cap R(g)$ such that

$$(f \circ \sigma)(n)(g \circ \sigma)(n+1) \leq (f \circ \sigma)(n+1)(g \circ \sigma)(n) \qquad (*)$$

for every $n \in \omega$.

Definition 7 For every $f, g \in \Delta^\omega$ set $f \leq g$ if and only if $T(f,g) \neq \emptyset$.

The following lemmas will establish some basic behavior of this relation and allow us to show that it is indeed an order. First, if $f \leq g$ and $f(n) = 0$ for some n then, since g should not contain any surprises, based on the information from f, we should have $g(n) = 0$.

Lemma 8 *Let $f, g \in C$. If $T(f,g) \neq \emptyset$ and there is $n \in \omega$ such that $f(n) = 0$ then $g(n) = 0$. If $\{m \in \omega : g(m) > 0\}$ is infinite and there is $n \in \omega$ such that $g(n) = 0$ then $f(n) = 0$.*

Proof Let $\sigma \in T(f,g)$. If $n \notin \operatorname{dom}\sigma$ then $g(n) = 0$. So we may assume that $n \in \operatorname{dom}\sigma$. Let $m \in \omega$ such that $\sigma(m) = n$. We may also assume that $m = \min\{k \in \omega : (f \circ \sigma)(k) = 0\}$. Obviously $m > 0$, so

$$(f \circ \sigma)(m-1)(g \circ \sigma)(m) \leq (f \circ \sigma)(m)(g \circ \sigma)(m-1) = 0$$

and therefore $g(n) = (g \circ \sigma)(m) = 0$.

Now assume that $\{m \in \omega : g(m) > 0\}$ is infinite and let $n \in \omega$ such that $g(n) = 0$. Let $\sigma \in T(f,g)$. If $f(n) > 0$ then $n \in \operatorname{ran}\sigma$. Let $j \in \omega$ such that $\sigma(j) = n$. There must be $m \in \omega$ such that $g(m) > 0$ and $m = \sigma(k)$ for some $k > j$. But this is impossible because $g(\sigma(k)) \leq g(\sigma(j)) = 0$. Thus $f(n) = 0$. □

A consequence of this lemma is that if $T(f,g) \neq \emptyset$ and $\{n \in \omega : g(n) > 0\}$ is infinite then $\{n \in \omega : f(n) > 0\} = \{n \in \omega : g(n) > 0\}$. So if $f \in \Delta^\omega$ and f is positive on an infinite number of coordinates, then we cannot create a larger element of Δ^ω by changing a finite number of these coordinates to 0 and adjusting the rest. If we want to change one of these coordinates to 0, then we must change all but a finite number of them to 0. If we determine that a finite number of outcomes are impossible and remain consistent with the information represented by f, then we must in fact determine that all but a finite number of outcomes are impossible.

Lemma 9 *If $f, g \in C$ and $\sigma \in T(f,g)$ then*

$$(f \circ \sigma)(m)(g \circ \sigma)(n) \leq (f \circ \sigma)(n)(g \circ \sigma)(m)$$

for all $m, n \in \omega$ with $m < n$.

Proof Let $m, n \in \omega$ with $m < n$ and assume that the following inequality holds.

$$(f \circ \sigma)(m)(g \circ \sigma)(n) \leq (f \circ \sigma)(n)(g \circ \sigma)(m)$$

We want to show that $(f \circ \sigma)(m)(g \circ \sigma)(n+1) \leq (f \circ \sigma)(n+1)(g \circ \sigma)(m)$. This is clearly true if $(g \circ \sigma)(n+1) = 0$, so we may assume that $(g \circ \sigma)(n+1) \neq 0$. This means that $(g \circ \sigma)(n) \neq 0$ and, by Lemma 8, $(f \circ \sigma)(n) \neq 0$ and $(f \circ \sigma)(n+1) \neq 0$ as well. Our assumption tells us that

$$\frac{(f \circ \sigma)(m)}{(f \circ \sigma)(n)} \leq \frac{(g \circ \sigma)(m)}{(g \circ \sigma)(n)}$$

and the fact that $\sigma \in T(f,g)$ gives us

$$\frac{(f \circ \sigma)(n)}{(f \circ \sigma)(n+1)} \leq \frac{(g \circ \sigma)(n)}{(g \circ \sigma)(n+1)}$$

Therefore

$$\begin{aligned}\frac{(f \circ \sigma)(m)}{(f \circ \sigma)(n+1)} &= \frac{(f \circ \sigma)(m)}{(f \circ \sigma)(n)} \cdot \frac{(f \circ \sigma)(n)}{(f \circ \sigma)(n+1)} \\ &\leq \frac{(g \circ \sigma)(m)}{(g \circ \sigma)(n)} \cdot \frac{(g \circ \sigma)(n)}{(g \circ \sigma)(n+1)} = \frac{(g \circ \sigma)(m)}{(g \circ \sigma)(n+1)}\end{aligned}$$

The result now follows by induction. □

The next lemma establishes that Δ^ω satisfies Degeneration.

Lemma 10 (Degeneration) *Let $f, g \in C$ with $T(f,g) \neq \emptyset$. If there are $m, n \in \omega$ with $g(m) = g(n) > 0$ then $f(m) = f(n)$.*

Proof Assume that $m \neq n$. Let $\sigma \in T(f,g)$ and let $j, k \in \omega$ such that $\sigma(j) = m$ and $\sigma(k) = n$. We may assume that $j < k$. Then $(f \circ \sigma)(j)(g \circ \sigma)(k) \leq (f \circ \sigma)(k)(g \circ \sigma)(j)$ or $(f \circ \sigma)(j) \leq (f \circ \sigma)(k)$. But $f \circ \sigma$ is decreasing, so $f(m) = (f \circ \sigma)(j) = (f \circ \sigma)(k) = f(n)$. □

The proceeding lemma shows that the relation follows our intuition on more certain states. If g is more certain than f and f tells us that outcome m is less likely than outcome n then it would be a surprise for g to say that they were equally likely, unless they both went to zero.

Lemma 11 *Let $f, g \in C$ and let $\sigma \in T(f,g)$. If $(g \circ \sigma)(m) < (f \circ \sigma)(m)$ for some $m \in \omega$ then $(g \circ \sigma)(n) < (f \circ \sigma)(n)$ for all $n \in \omega$ with $m < n$ and $(g \circ \sigma)(n) > 0$.*

Proof Let $n \in \omega$ with $m \leq n$ and assume that $(g \circ \sigma)(n+1) > 0$. Then none of $(f \circ \sigma)(n)$, $(f \circ \sigma)(n+1)$, and $(g \circ \sigma)(n)$ can be zero. Further assume that $(g \circ \sigma)(n) < (f \circ \sigma)(n)$. From $\sigma \in T(f,g)$ we get that

$$\frac{(g \circ \sigma)(n+1)}{(f \circ \sigma)(n+1)} \leq \frac{(g \circ \sigma)(n)}{(f \circ \sigma)(n)} < 1$$

and therefore $(g \circ \sigma)(n+1) < (f \circ \sigma)(n+1)$. □

Lemma 12 *Let $f, g \in \Delta^\omega$ with $T(f,g) \neq \emptyset$. If $f^+ \geq g^+$ then $f = g$.*

Proof Let $\sigma \in T(f,g)$. Then $(f \circ \sigma)(0) = f^+ = g^+ = (g \circ \sigma)(0)$. Let $n \in \omega$ and assume that $(g \circ \sigma)(n) \leq (f \circ \sigma)(n)$. If $(f \circ \sigma)(n) = 0$ then $(g \circ \sigma)(n) = 0$ and $(g \circ \sigma)(n+1) = 0 = (f \circ \sigma)(n+1)$. We may therefore assume that $(f \circ \sigma)(n) > 0$. Then

$$(g \circ \sigma)(n+1) \leq \frac{(g \circ \sigma)(n)}{(f \circ \sigma)(n)}(f \circ \sigma)(n+1) \leq (f \circ \sigma)(n+1)$$

and it follows that $g(n) \leq f(n)$ for all $n \in \omega$ and that $f = g$. □

This means that if $f \leq g$ then $f^+ \leq g^+$.

Theorem 13 *The relation $\leq$ is an order.*

Proof It is obvious that $f \leq f$ for all $f \in \Delta^\omega$. Let $f, g \in \Delta^\omega$ with $f \leq g$ and $g \leq f$. Let $\sigma \in T(f,g)$ and $\tau \in T(g,f)$. If $f^+ < g^+$ then $(f \circ \tau)(0) < (g \circ \tau)(0)$ so $f(n) < g(n)$ for all $n \in \omega$ with $f(n) > 0$ by Lemma 11, which is a contradiction. If $g^+ < f^+$ then $(g \circ \sigma)(0) < (f \circ \sigma)(0)$ so $g(n) < f(n)$ for all $n \in \omega$ with $g(n) > 0$ by Lemma 11, which is a contradiction. Therefore $f^+ = g^+$ and $f = g$ by Lemma 12.

Let $f, g, h \in \Delta^\omega$ with $f \leq g$ and $g \leq h$. Let $\rho \in T(f,g)$ and $\sigma \in T(g,h)$. We will define a new function τ and show that $\tau \in T(f,h)$. First assume that

$g(n) > 0$ for all $n \in \omega$. Then $\operatorname{ran}\sigma = \omega$ so automatically $f^{-1}[(0,1]] \subseteq \operatorname{ran}\sigma$ and $h^{-1}[(0,1]] \subseteq \operatorname{ran}\sigma$.

Let $n \in \omega$. If $\rho(n) = \sigma(n)$ then $(f \circ \rho)(n) = (f \circ \sigma)(n)$. If $\rho(n) \neq \sigma(n)$ then, since $(g \circ \rho)(n) = (g \circ \sigma)(n)$ we have $(f \circ \rho)(n) = (f \circ \sigma)(n)$ by Lemma 10. Therefore $f \circ \rho = f \circ \sigma$ and $\sigma \in R(f)$. Since $\rho \in T(f,g)$ we know that

$$\frac{(f \circ \rho)(n)}{(f \circ \rho)(n+1)} \leq \frac{(g \circ \rho)(n)}{(g \circ \rho)(n+1)} \text{ or}$$
$$\frac{(f \circ \sigma)(n)}{(f \circ \sigma)(n+1)} \leq \frac{(g \circ \sigma)(n)}{(g \circ \sigma)(n+1)}$$

for all $n \in \omega$.

If $(h \circ \sigma)(n) = 0$ then $(h \circ \sigma)(n+1) = 0$ and $(f \circ \sigma)(n)(h \circ \sigma)(n+1) = (f \circ \sigma)(n+1)(h \circ \sigma)(n)$. Assume that $(h \circ \sigma)(n) > 0$. If $(h \circ \sigma)(n+1) = 0$ then $(f \circ \sigma)(n)(h \circ \sigma)(n+1) = 0 \leq (f \circ \sigma)(n+1)(h \circ \sigma)(n)$. If $(h \circ \sigma)(n+1) > 0$ then

$$\frac{(f \circ \sigma)(n)}{(f \circ \sigma)(n+1)} \leq \frac{(g \circ \sigma)(n)}{(g \circ \sigma)(n+1)} \leq \frac{(h \circ \sigma)(n)}{(h \circ \sigma)(n+1)}$$

In each of these cases, $\sigma \in T(f,g)$.

Now assume that there is $k \in \omega$ such that $g(k) = 0$. Let m be the minimum element of $\{k \in \omega : (g \circ \rho)(k) = 0\}$. Note that if $n \geq m$ then $(g \circ \sigma)(n) = (g \circ \rho)(n) = 0$ so $(h \circ \rho)(n) = 0$. For every $n \in \omega$ set

$$\tau(n) = \begin{cases} \sigma(n), & \text{if } n < m \\ \rho(n), & \text{if } m \leq n \end{cases}$$

We need to ensure that $f^{-1}[(0,1]] \subseteq \operatorname{ran}\tau$ and $h^{-1}[(0,1]] \subseteq \operatorname{ran}\tau$. Let $j \in \omega$ such that $f(j) > 0$. If $g(j) > 0$ then there is $n \in \omega$ such that $\sigma(n) = j$. Now $n < m$ so $\tau(n) = \sigma(n) = j$. Assume that $g(j) = 0$. Since $f(j) > 0$ there is $n \in \omega$ such that $\rho(n) = j$. Now $m \leq n$ so $\tau(n) = \rho(n) = j$. Therefore $f^{-1}[(0,1]] \subseteq \operatorname{ran}\tau$. Let $h(j) > 0$. Then $g(j) > 0$ so there is $n \in \omega$ such that $\sigma(n) = j$. Now $n < m$ so $\tau(n) = \sigma(n) = j$. Therefore $h^{-1}[(0,1]] \subseteq \operatorname{ran}\tau$.

If $n < m$ then $(f \circ \tau)(n) = (f \circ \sigma)(n) = (f \circ \rho)(n)$. Thus, for every $n \in \omega$, $(f \circ \tau)(n) = (f \circ \rho)(n) \geq (f \circ \rho)(n+1) = (f \circ \tau)(n+1)$. If $n+1 < m$ then $(h \circ \tau)(n) = (h \circ \sigma)(n) \geq (h \circ \sigma)(n+1) = (h \circ \tau)(n+1)$. If $n+1 = m$ then $(h \circ \tau)(n) = (h \circ \sigma)(n) \geq 0 = (h \circ \tau)(n+1)$. And if $n+1 > m$ then $(h \circ \tau)(n) = 0 = (h \circ \tau)(n+1)$. Therefore $\tau \in R(f) \cap R(h)$.

The proof that $(f \circ \tau)(n)(h \circ \tau)(n+1) \leq (f \circ \tau)(n+1)(h \circ \tau)(n)$ is the same as the previous case when $n+1 < m$. If $n+1 \geq m$ then $(h \circ \tau)(n+1) = 0$ so $(f \circ \tau)(n)(h \circ \tau)(n+1) = 0 \leq (f \circ \tau)(n+1)(h \circ \tau)(n)$. Therefore $\tau \in T(f,h)$ and $f \leq h$. □

4 Structure of Δ^ω

We have already seen that Δ^ω satisfies Degeneracy. If our model is to be a good one, it should also satisfy the Mixing Law and have maximal elements that represent pure states where our knowledge or information is absolute. It is an immediate consequence of the definition of Δ^ω that this model satisfies the Mixing Law.

Theorem 14 (The Mixing Law) *Let $f, g \in \Delta^\omega$. If $f \leq g$ and $t \in (0,1)$ then $f \leq (1-t)f + tg \leq g$.*

Proof Let $h = (1-t)f + tg$ and let $\sigma \in T(f,g)$. Since $1-t > 0$ and $t > 0$ we know that $\sigma \in R(h)$. Let $n \in \omega$.

$$\begin{aligned}
(f \circ \sigma)(n)(h \circ \sigma)(n+1) &= (f \circ \sigma)(n)[(1-t)(f \circ \sigma)(n+1) + t(g \circ \sigma)(n+1)] \\
&= (1-t)(f \circ \sigma)(n)(f \circ \sigma)(n+1) + t(f \circ \sigma)(n)(g \circ \sigma)(n+1) \\
&\leq (1-t)(f \circ \sigma)(n+1)(f \circ \sigma)(n) + t(f \circ \sigma)(n+1)(g \circ \sigma)(n) \\
&= (f \circ \sigma)(n+1)[(1-t)(f \circ \sigma)(n) + t(g \circ \sigma)(n)] \\
&= (f \circ \sigma)(n+1)(h \circ \sigma)(n)
\end{aligned}$$

Therefore $\sigma \in T(f,g)$ and $f \leq h$. The proof that $h \leq g$ is similar. □

Just as in Δ^n, the pure states will be the maximal elements of Δ^ω.

Definition 15 For every $m \in \omega$ let e_m be the element of Δ^ω given by

$$e_m(n) = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$$

Let $\max \Delta^\omega$ denote the set of maximal elements of Δ^ω.

Theorem 16 *For every $f \in \Delta^\omega$ and $m \in \omega$, $f \leq e_m$ if and only if $f(m) = f^+$.*

Proof Let $f \in \Delta^\omega$ and let $m \in \omega$ such that $f(m) = f^+$. Let $\sigma \in R(f)$ such that $\sigma(0) = m$. Then $\sigma \in R(e_m)$ and $(f \circ \sigma)(n)(e_m \circ \sigma)(n+1) = 0 \leq (f \circ \sigma)(n+1)(e_m \circ \sigma)(n)$ for every $n \in \omega$. Therefore $f \leq e_m$.

If $f(m) \neq f^+$ and $\sigma \in R(f)$ then $\sigma(0) \neq m$. Thus $R(f) \cap R(e_m) = \emptyset$ and f and e_m are incomparable. □

Since e_m and e_n are incomparable when $m \neq n$ the preceding proof shows that $\max \Delta^\omega = \{e_m : m \in \omega\}$ and $\overline{\downarrow} \max \Delta^\omega = \Delta^\omega$.

We will see later that Δ^ω does not have any minimal elements. Another property that our model should have is completeness. We want to know that if we follow consistent states that are becoming more certain that we will be led to a state with at least as much certainty or information. The next lemma and theorem will allow us to show that Δ^ω has this property by being directed complete.

Lemma 17 *Let γ be an ordinal and let $\langle f_\alpha : \alpha \in \gamma \rangle$ be a sequence in C such that for every $\alpha, \beta \in \gamma$, if $\alpha < \beta$ then $T(f_\alpha, f_\beta) \neq \emptyset$. Then there is $m \in \omega$ such that $f_\alpha(m) = f_\alpha^+$ for every $\alpha \in \gamma$.*

Proof For every $\alpha \in \gamma$ let $M_\alpha = \{m \in \omega : f_\alpha(m) = f_\alpha^+\}$. Let $\alpha, \beta \in \gamma$ with $\alpha < \beta$ and let $\sigma \in T(f_a, f_b)$. Then $f_\alpha(\sigma(0)) = f_\alpha^+$ and $f_\beta(\sigma(0)) = f_\beta^+$. If $m \in M_\beta$ then $f_\beta(m) = f_\beta(\sigma(0))$ so $f_\alpha(m) = f_\alpha(\sigma(0))$ by Lemma 10. Therefore $m \in M_\alpha$ and $M_\beta \subseteq M_\alpha$. If $\gamma = \beta + 1$ then $M_\beta \subseteq \bigcap_{\alpha \in \gamma} M_\alpha$ so $\bigcap_{\alpha \in \gamma} M_\alpha = M_\beta$. If γ is a limit ordinal then $\bigcap_{\alpha \in \gamma} M_\alpha \neq \emptyset$ because each M_α is finite. In either case, if $m \in \bigcap_{\alpha \in \gamma} M_\alpha$ then $f_\alpha(m) = f_\alpha^+$ for all $\alpha \in \gamma$. □

Theorem 18 *Let γ be an ordinal. If $\langle f_\alpha : \alpha \in \gamma \rangle$ is an increasing sequence in Δ^ω then there is a one-to-one function $\sigma : \omega \to \omega$ such that $\sigma \in T(f_\alpha, f_\beta)$ for all $\alpha, \beta \in \gamma$ with $\alpha < \beta$.*

Proof We will define σ recursively. The idea is simple enough, but the details are technical. We will use Lemma 17 to find a coordinate m on which all of the functions in the sequence reach their maximum value. We will then eliminate this coordinate, and repeat the process to find the next coordinate. Set $M_0 = \emptyset$. Let $i \in \omega$ and assume that M_i is a finite subset of ω. Let λ_i be a strictly increasing function from ω onto $\omega - M_i$ and set $f_{i\alpha} = f_\alpha \circ \lambda_i$ for every $\alpha \in \gamma$. Then $\langle f_{i\alpha} : \alpha \in \gamma \rangle$ is a sequence in C.

Let $\alpha, \beta \in \gamma$ with $\alpha < \beta$. We want to show that $T(f_{i\alpha}, f_{i\beta}) \neq \emptyset$. Let $\rho_{\alpha\beta} \in T(f_\alpha, f_\beta)$ and let $\mu_{i\alpha\beta}$ be a strictly increasing function from ω onto $\omega - \rho_{\alpha\beta}^{-1}[M_i]$. Set $\tau_{i\alpha\beta} = \lambda_i^{-1} \circ \rho_{\alpha\beta} \circ \mu_{i\alpha\beta}$. Then $\operatorname{dom} \tau_{i\alpha\beta} = \omega$ and $\operatorname{ran} \tau_{i\alpha\beta} \subseteq \omega$. If $k \in \omega$ and $f_{i\alpha}(k) > 0$ then $f_\alpha(\lambda_i(k)) > 0$ so $\lambda_i(k) \in \operatorname{ran} \rho_{\alpha\beta}$. Let $m \in \omega$ such that $\rho_{\alpha\beta}(m) = \lambda_i(k)$ or $(\lambda_i^{-1} \circ \rho_{\alpha\beta})(m) = k$. Now $\lambda_i(k) \in \omega - M_i$ so $m \in \omega - \rho_{\alpha\beta}^{-1}[M_i]$. Let $n \in \omega$ such that $\mu_{i\alpha\beta}(n) = m$. Then $\tau_{i\alpha\beta}(n) = (\lambda_i^{-1} \circ \rho_{\alpha\beta} \circ \mu_{i\alpha\beta})(n) = k$ and $k \in \operatorname{ran} \tau_{i\alpha\beta}$. The same argument shows that $f_{i\beta}^{-1}[(0,1]] \subseteq \operatorname{ran} \tau_{i\alpha\beta}$.

We must show that $f_{i\alpha} \circ \tau_{i\alpha\beta}$ and $f_{i\beta} \circ \tau_{i\alpha\beta}$ are decreasing. Let $m, n \in \omega$ with $m < n$. Then $\mu_{i\alpha\beta}(m) < \mu_{i\alpha\beta}(n)$ and therefore

$$\begin{aligned}(f_{i\alpha} \circ \tau_{i\alpha\beta})(m) &= (f_\alpha \circ \lambda_i \circ \lambda_i^{-1} \circ \rho_{\alpha\beta} \circ \mu_{i\alpha\beta})(m) \\ &= (f_\alpha \circ \rho_{\alpha\beta})(\mu_{i\alpha\beta}(m)) \\ &\geq (f_\alpha \circ \rho_{\alpha\beta})(\mu_{i\alpha\beta}(n)) \\ &= (f_{i\alpha} \circ \tau_{i\alpha\beta})(n)\end{aligned}$$

The same argument shows that $f_{i\beta} \circ \tau_{i\alpha\beta}$ is decreasing.

Still with $m < n$ we get the following inequalities.

$$\begin{aligned}(f_{i\alpha} \circ \tau_{i\alpha\beta})(m)(f_{ib} \circ \tau_{i\alpha\beta})(n) &= (f_\alpha \circ \rho_{\alpha\beta})(\mu_{i\alpha\beta}(m))(f_\beta \circ \rho_{\alpha\beta})(\mu_{i\alpha\beta}(n)) \\ &\leq (f_\alpha \circ \rho_{\alpha\beta})(\mu_{i\alpha\beta}(n))(f_\beta \circ \rho_{\alpha\beta})(\mu_{i\alpha\beta}(m)) \\ &= (f_{i\alpha} \circ \tau_{i\alpha\beta})(n)(f_{ib} \circ \tau_{i\alpha\beta})(m)\end{aligned}$$

Therefore $\tau_{i\alpha\beta} \in T(f_{i\alpha}, f_{i\beta})$.

By Lemma 17 there is $b_i \in \omega$ such that $f_{i\alpha}(b_i) = f_{i\alpha}^+$ for every $\alpha \in \gamma$. Let $\sigma(i) = \lambda_i^{-1}(b_i)$ and set $M_{i+1} = M_i \cup \{\sigma(i)\}$.

This process recursively defines a function $\sigma : \omega \to \omega$. For every $i \in \omega$, $\sigma(i) \in \omega - M_i$ and $\sigma(i) \in M_{i+1}$, so σ is one-to-one. Let $\alpha \in \gamma$. If $f_\alpha(m) > 0$ and $f_\alpha(m) \notin \operatorname{ran} \sigma$ then the set $\{n \in \omega : f_\alpha(n) \geq f_\alpha(m)\}$ is infinite, contradicting $\sum_{n\in\omega} f_\alpha(n) = 1$. Also, $(f_\alpha \circ \sigma)(n) = \max\{f_\alpha(m) : m \notin M_i\}$ so $(f_a \circ \sigma)(i) \geq (f_\alpha \circ \sigma)(i+1)$.

Finally, let $m, n \in \omega$ and $\alpha, \beta \in \gamma$ with $m < n$ and $\alpha < \beta$. Let $\rho \in T(f_\alpha, f_\beta)$. Then

$$\begin{aligned}(f_\alpha \circ \sigma)(m)(f_\beta \circ \sigma)(n) &= (f_\alpha \circ \rho)(m)(f_\beta \circ \rho)(n) \\ &\leq (f_\alpha \circ \rho)(n)(f_\beta \circ \rho)(m) \\ &= (f_\alpha \circ \sigma)(n)(f_\beta \circ \sigma)(m) \qquad \square\end{aligned}$$

Theorem 19 *If $\langle f_n : n \in \omega \rangle$ is an increasing sequence in Δ^ω then $\langle f_n : n \in \omega \rangle$ has a supremum $g = \sup_{n\in\omega} f_n$ in Δ^ω and $g(k) = \lim_{n\to\infty} f_n(k)$ for all $k \in \omega$.*

Proof We will first show that $\langle f_n(k) : n \in \omega \rangle$ converges for every $k \in \omega$. By Theorem 18 there is a function σ so that $\sigma \in T(f_m, f_n)$ for all $m, n \in \omega$ with $m < n$. Then $(f_n \circ \sigma)(0) = f_n^+$ for all $n \in \omega$ and $\langle (f_n \circ \sigma)(0) : n \in \omega \rangle$ is an increasing sequence in $[0, 1]$ and must converge. Let $k \in \omega$ and assume that $\langle (f_n \circ \omega)(k) : n \in \omega \rangle$ converges. If $\lim_{n\to\infty}(f_n \circ \sigma)(k) = 0$ then $\langle (f_n \circ \sigma)(k + 1) : n \in \omega \rangle$ also converges to 0 since $(f_n \circ \sigma)(k + 1) \leq (f_n \circ \sigma)(k)$ for all $n \in \omega$. We may therefore assume that $\lim_{n\to\infty}(f_n \circ \sigma)(k) > 0$. This means that $(f \circ \sigma)(k) > 0$ for all $n \in \omega$ by Lemma 8. The same lemma allows us to assume that $(f_n \circ \sigma)(k + 1) > 0$ for all $n \in \omega$. Because $\langle (f_n \circ \sigma)(k) : n \in \omega \rangle$ converges, for every $\epsilon > 0$ there is $j \in \omega$ such that for every $m, n \in \omega$, if $j < m < n$ then $(f_n \circ \sigma)(k)/(f_m \circ \sigma)(k) < 1+\epsilon$. If $\langle (f_n \circ \sigma)(k+1) : n \in \omega \rangle$ does not converge then there is $\epsilon > 0$ such that for every $j \in \omega$ there are $m, n \in \omega$ such that $j < m < n$ and $(f_n \circ \sigma)(k + 1)/(f_m \circ \sigma)(k + 1) > 1 + \epsilon$. Using this last ϵ we can choose $m, n \in \omega$ such that $m < n$ and

$$\frac{(f_n \circ \sigma)(k)}{(f_m \circ \sigma)(k)} < \frac{(f_n \circ \sigma)(k + 1)}{(f_m \circ \sigma)(k + 1)}$$

a contradiction.

Now that we have coordinate-wise convergence, we can define g by $g(k) = \lim_{n\to\infty} f_n(k)$ for all $k \in \omega$. Obviously g is a function from ω into $[0, 1]$ and

$$\sum_{k\in\omega} g(k) = \sum_{k\in\omega} \lim_{n\to\infty} f_n(k) = \lim_{n\to\infty} \sum_{k\in\omega} f_n(k) = 1$$

We also know that $g\circ\sigma$ is decreasing because $f_n\circ\sigma$ is decreasing for every $n \in \omega$. If $g(m) > 0$ then $f_n(m) > 0$ for all $n \in \omega$. Thus $m \in \operatorname{ran} \sigma$. Therefore $\sigma \in R(g)$.

If $k, m, n \in \omega$ with $m < n$ then

$$(f_m \circ \sigma)(k)(f_n \circ \sigma)(k + 1) \leq (f_m \circ \sigma)(k + 1)(f_n \circ \sigma)(k)$$

So $\lim_{n\to\infty}(f_m \circ \sigma)(k)(f_n \circ \sigma)(k + 1) \leq \lim_{n\to\infty}(f_m \circ \sigma)(k + 1)(f_n \circ \sigma)(k)$ or $(f_m \circ \sigma)(k)(g \circ \sigma)(k + 1) \leq (f_m \circ \sigma)(k + 1)(g \circ \sigma)(k)$. Therefore $\sigma \in T(f_m, g)$ and $f_m \leq g$.

Now let h be an upper bound of $\langle f_n : n \in \omega \rangle$ in Δ^ω. By Theorem 18 there is a one-to-one function $\tau : \omega \to \omega$ such that $\tau \in T(f_n, h)$ for every $n \in \omega$. Let $k \in \omega$. For every $n \in \omega$,

$$(f_n \circ \sigma)(k)(h \circ \tau)(k + 1) \leq (f_n \circ \sigma)(k + 1)(h \circ \tau)(k)$$

So

$$\lim_{n\to\infty}(f_n \circ \sigma)(k)(h \circ \tau)(k + 1) \leq \lim_{n\to\infty}(f_n \circ \sigma)(k + 1)(h \circ \tau)(k)$$

or

$$(g \circ \tau)(k)(h \circ \tau)(k + 1) \leq (g \circ \tau)(k + 1)(h \circ \tau)(k)$$

As happened with σ we must have $g^{-1}[(0, 1]] \subseteq \operatorname{ran} \tau$ and $\gamma \circ \tau$ is decreasing. Therefore $\tau \in T(g, h)$ and $g \leq h$. □

The next corollary is proved the same way that Prop 2.16(ii) is proved in [3] and it greatly simplifies our study of directed sets, or sets of consistent information, by allowing us to consider only sequences. This means that the information is being built steadily in a linear fashion by adding new or more complete information to what we already knew.

Corollary 20 *Δ^ω is a dcpo and every directed subset D of Δ^ω contains and increasing sequence $\langle f_n : n \in \omega\rangle$ with $\sup_{n\in\omega} f_n = \sup D$.*

When Coecke and Martin defined their order on Δ^n they based it on the proposition that the order on Δ^{n+1} should be determined by the order on Δ^n in the sense that for every $f, g \in \Delta^{n+1}$, f should be set less than or equal to g if and only if the projection of f into Δ^n is less than or equal to the projection of g into Δ^n. The next theorem shows that Δ^ω also displays this kind of behavior. First we need to define what we mean be a projection.

Definition 21 For every $f \in \Delta^\omega$ let $\mathcal{P}(f) = \{A \subseteq \omega : 0 < \sum_{k\in A} f(k)\}$. If $A \in \mathcal{P}(f)$ we define the *projection of f onto A*, $p_A(f)$, by setting

$$p_A(f)(n) = \begin{cases} \dfrac{f(n)}{\sum\limits_{m\in A} f(m)}, & n \in A \\ 0, & n \notin A \end{cases}$$

for all $n \in \omega$.

Note that $p_A(f) \in \Delta^\omega$. Under $p_A(f)$ we have examined all the outcomes not listed in A and determined them to be impossible. We have no further information on the outcomes listed in A relative to one another, so their probabilities remain comparatively the same. We only need to adjust to ensure that the sum is still 1.

Theorem 22 *For every $f, g \in \Delta^\omega$, $f \leq g$ if and only if $p_A(f) \leq p_A(g)$ for all $A \in \mathcal{P}(f) \cap \mathcal{P}(g)$.*

Proof Assume that $f \leq g$ and let $\sigma \in T(f, g)$. Let $A \in \mathcal{P}(f) \cap \mathcal{P}(g)$. We will consider two cases. First assume that $|A| = \omega$. Let $\rho : \omega \to \sigma^{-1}[A]$ be an order isomorphism (strictly increasing and onto). For easier notation set $v = p_A(f)$ and $w = p_A(g)$. Set $\tau = \sigma \circ \rho$. Then τ is a one-to-one function from ω into ω. Let $n \in \omega$. If $v(n) > 0$ then $n \in A$ and $f(n) > 0$. There is $m \in \omega$ such that $\sigma(m) = n$, since $\sigma \in R(f)$, and $m \in \sigma^{-1}[A]$. So there is $k \in \omega$ such that $\rho(k) = m$. Then $n = (\sigma \circ \rho)(k) = \tau(k)$. Since ρ is strictly increasing and $f \circ \sigma$ is decreasing, $v \circ \tau$ must also be decreasing. Thus $\tau \in R(v)$. It can be shown in the same way that $\tau \in R(w)$. Property $(*)$ of Definition 6 must also be satisfied, again because ρ is strictly increasing. Therefore $\tau \in T(f, g)$ and $p_A(f) \leq p_A(g)$.

For the second case, assume that $|A| < \omega$. Let $n = |A|$ and let $\rho_0 : n \to \sigma^{-1}[A]$ be strictly increasing and $\rho_1 : \omega - n \to \omega - \sigma^{-1}[A]$ one-to-one. Set $\tau = \sigma \circ (\rho_0 \cup \rho_1)$. Then τ is a one-to-one function from ω into ω. Note that if $m \geq n$ then $\rho_1(m) \notin \sigma^{-1}[A]$ so $\tau(m) = (\sigma \circ \rho_1)(m) \notin A$. Therefore $(v \circ \tau)(m) = (w \circ \tau)(m) = 0$. The same argument as that used in the first case shows that $v^{-1}[(0, 1]] \subseteq \operatorname{ran} \tau$ and $w^{-1}[(0, 1]] \subseteq \operatorname{ran} \tau$. That $v \circ \tau$ and $w \circ \tau$ are decreasing follows from the facts that ρ_1 is strictly increasing on n and that

$(v \circ \tau)(m) = (w \circ \tau)(m) = 0$ when $m \geq n$. It remains to show that Property ($*$) of Definition 6 is satisfied.

Let $m \in \omega$. If $m + 1 < n$ then Property ($*$) of Definition 6 is satisfied because ρ_0 is strictly increasing. If $m + 1 \geq n$ then $(w \circ \tau)(m + 1) = 0$ so Property ($*$) is satisfied. Therefore $p_A(f) \leq p_A(g)$.

For the other direction, if $p_A(f) \leq p_A(g)$ for all $A \in \mathcal{P}(f) \cap \mathcal{P}(g)$ then $f = p_\omega(f) \leq p_w(g) = g$. □

Lemma 23 *If $\sigma : \omega \to \omega$ is one-to-one and onto then the function $r_\sigma : \Delta^\omega \to \Delta^\omega$ given by $r_\sigma(f) = f \circ \sigma$ is an order isomorphism.*

Proof If $f \in \Delta^\omega$ then $f \circ \sigma \in \Delta^\omega$ and $f \circ \sigma^{-1} \in \Delta^\omega$. Therefore $\operatorname{ran} r_\sigma = \Delta^\omega$. If $f \neq g$ then there is $n \in \omega$ such that $f(n) \neq g(n)$. Let $m \in \omega$ such that $\sigma(m) = n$. Then $r_\sigma(f)(m) \neq r_\sigma(g)(m)$ and $r_s(f) \neq r_\sigma(g)$. Thus r_σ is one-to-one. If $f, g \in \Delta^\omega$ and $\tau \in T(f, g)$ then $\sigma^{-1} \circ \tau \in T(f \circ \sigma, g \circ \sigma)$ so $r_\sigma(f) \leq r_\sigma(g)$. If $\tau \in T(f \circ \sigma, g \circ \sigma)$ then $\sigma \circ \tau \in T(f, g)$. Therefore r_σ is an order isomorphism. □

This proof shows that if $\sigma : \omega \to \omega$ is one-to-one, $\sigma \in R(f) \cap R(g)$, and $f < g$ then $r_\sigma(f) < r_\sigma(g)$. If σ is not onto then there will be an $f \in \Delta^\omega$ such that $r_\sigma(f) \notin \Delta^\omega$.

Coecke and Martin showed that the model Δ^{n+1} contains $n + 1$ copies of the model Δ^n. These are obtained be taking those elements of Δ^{n+1} whose first coordinates are zero, then those whose second coordinate is zero, and so forth. We can do a similar thing with Δ^ω.

Definition 24 $\Delta^\omega_+ = \{f \in \Delta^\omega : \forall m \in \omega(f(m) > 0)\}$. For every $A \subseteq \omega$, $\Delta^\omega(A) = \{f \in \Delta^\omega : (\forall m \in \omega - A)(f(m) = 0)\}$ and $\Delta^\omega_+(A) = \{f \in \Delta^\omega : \forall m \in \omega(f(m) > 0 \iff m \in A)\}$.

In the following theorem we will use $\simeq$ to mean order isomorphic.

Theorem 25 *Let $n \in \omega$ and $A \subseteq \omega$.*

1. *If $|A| = n$ then $\Delta^\omega(A) \simeq \Delta^n$.*
2. *If $|A| = \omega$ then $\Delta^\omega(A) \simeq \Delta^\omega$.*
3. *If $|A| = \omega$ then $\Delta^\omega_+(A) \simeq \Delta^\omega_+$.*

Proof Assume that $|A| = n$ and let $\rho : n \to A$ be strictly increasing. Define $\phi : \Delta^\omega(A) \to \Delta^n$ by setting $\phi(f) = f \circ \rho$ for all $f \in \Delta^\omega(A)$. This function is one-to-one and if $g \in \Delta^n$ then $\phi(f) = g$ when

$$f(m) = \begin{cases} (g \circ \rho^{-1})(m), & \text{if } m \in A \\ 0, & \text{if } m \notin A \end{cases}$$

Therefore $\operatorname{ran} \phi = \Delta^n$.

Let $f, g \in \Delta^\omega(A)$ such that $f \leq g$ and let $\sigma \in T(f, g)$. We would like to use $\rho^{-1} \circ \sigma$ to show that $\phi(f) \leq \phi(g)$, however A may not be a subset of $\operatorname{ran} \sigma$, in which case n is not a subset of $\operatorname{ran}(\rho^{-1} \circ \sigma)$. If $m \in A$ and $m \notin \operatorname{ran} \sigma$ then $f(m) = g(m) = 0$. Let $k = \min\{m \in \omega : (f \circ \sigma)(m) = 0\}$. Note that $(g \circ \sigma)(m) = 0$ for all $m \geq k$. We can enumerate $A - \operatorname{ran}\sigma$ as $\{m_j : j = k, \ldots, n-1\}$. For every $m \in \omega$ set

$$\sigma_1(j) = \begin{cases} \sigma(j), & \text{if } j < k \\ m_j, & \text{if } k \leq j < n \\ \sigma(j-n+k), & \text{if } n \leq j \end{cases}$$

Let $i, j \in \omega$ with $i < j$. If $j < k$ then $\sigma_1 = \sigma(i) \neq \sigma(j) = \sigma_1(j)$. If $n \leq i$ then $\sigma_1(i) = \sigma(i-n+k) \neq \sigma(j-n+k) = \sigma_1(j)$. If $i < k$ and $n \leq j$ then $i < j-n+k$ so $\sigma_1(i) = \sigma(i) \neq \sigma(j-n+k) = \sigma_1(j)$. Thus $\sigma_1 : \omega \to \omega$ is one-to-one.

Let $m \in \omega$ with $f(m) > 0$. There is $j \in \omega$ such that $\sigma(j) = m$. Now $j < k$, so $\sigma_1(j) = \sigma(j) = m$. Similarly, if $g(m) > 0$ then $m \in \operatorname{ran}\sigma_1$.

Again let $i, j \in \omega$ with $i < j$. If $j < k$ then

$$(f \circ \sigma_1)(i) = (f \circ \sigma)(i) \geq (f \circ \sigma)(j) = (f \circ \sigma_1)(j) \text{ and}$$
$$(g \circ \sigma_1)(i) = (g \circ \sigma)(i) \geq (g \circ \sigma)(j) = (g \circ \sigma_1)(j)$$

so $f \circ \sigma_1$ and $g \circ \sigma_1$ are decreasing. We also have the following property.

$$\begin{aligned}(f \circ \sigma_1)(i)(g \circ \sigma_1)(j) &= (f \circ \sigma)(i)(g \circ \sigma)(j) \\ &\leq (f \circ \sigma)(j)(g \circ \sigma)(i) \\ &= (f \circ \sigma_1)(j)(g \circ \sigma_1)(i)\end{aligned}$$

If $k \leq j$ then $(f \circ \sigma_1)(j) = (g \circ \sigma_1)(j) = 0$ so $(f \circ \sigma_1)(i) \geq (f \circ \sigma_1)(j)$, $(g \circ \sigma_1)(i) \geq (g \circ \sigma_1)(j)$, and $(f \circ \sigma_1)(i)(g \circ \sigma_1)(j) \leq (f \circ \sigma_1)(j)(g \circ \sigma_1)(i)$. Therefore $\sigma_1 \in T(f, g)$. Now set $\tau = \rho^{-1} \circ \sigma_1$. Then $\tau \restriction n$ is a permutation on n and will show that $\phi(f) \leq \phi(g)$ in Δ^n.

To show that ϕ^{-1} also preserves order, let $f, g \in \Delta^\omega(A)$ such that $\phi(f) \leq \phi(g)$ in Δ^n. There is a permutation σ on n such that $(f \circ \rho) \circ \sigma$ and $(g \circ \rho) \circ \sigma$ are decreasing, and Property $(*)$ of Definition 6 is satisfied by $\phi(f)$, $\phi(g)$, and σ on n.

Define $\sigma_1 : \omega \to \omega$ by setting

$$\sigma_1(m) = \begin{cases} \sigma(m), & \text{if } m < n \\ m, & \text{if } n \leq m \end{cases}$$

Let $\rho_0 : \omega - n \to \omega - A$ be one-to-one and set $\rho_1 = \rho \cup \rho_0$. Set $\tau = \rho_1 \circ \sigma_1$. Then $\tau : \omega \to \omega$ is one-to-one. Let $i, j \in \omega$ with $i < j$. If $j < n$ then

$$(f \circ \tau)(i) = ((f \circ \rho) \circ \sigma)(i) \geq ((f \circ \rho) \circ \sigma)(j) = (f \circ \tau)(j)$$
$$(g \circ \tau)(i) = ((g \circ \rho) \circ \sigma)(i) \geq ((g \circ \rho) \circ \sigma)(j) = (g \circ \tau)(j)$$

so $f \circ \tau$ and $\gamma \circ \tau$ are decreasing. Also,

$$\begin{aligned}(f \circ \tau)(i)(g \circ \tau)(j) &= ((f \circ \rho) \circ \sigma)(i)((g \circ \rho) \circ \sigma)(j) \\ &\leq ((f \circ \rho) \circ \sigma)(j)((g \circ \rho) \circ \sigma)(i) \\ &= (f \circ \tau)(j)(g \circ \tau)(i)\end{aligned}$$

If $n \leq j$ then $\sigma_1(j) = j$ and $\rho_1(j) \in A$, so $(f \circ \tau)(j) = (g \circ \tau)(j) = 0$. Thus $(f \circ \tau)(i) \geq (f \circ \tau)(j)$, $(g \circ \tau)(i) \geq (g \circ \tau)(j)$, and $(f \circ \tau)(i)(g \circ \tau)(j) = 0 \leq (f \circ \tau)(j)(g \circ \tau)(i)$.

If $m \in \omega$ and $f(m) > 0$ or $g(m) > 0$ then $m \in A$ and there is $j \in n$ such that $\rho(j) = m$. Let $i = \sigma^{-1}(j)$. Then $\tau(i) = (\rho_1 \circ \sigma_1)(i) = m$. Therefore $\tau \in T(f,g)$ and $f \leq g$. So ϕ is an order isomorphism.

For the proofs of parts 2 and 3, let $\rho : \omega \to A$ be strictly increasing and onto. Set $\phi(f) = f \circ \rho$. The proof that $\phi : \Delta^\omega(A) \to \Delta^\omega$ or $\phi : \Delta^\omega_+(A) \to \Delta^\omega_+$ is an order isomorphism is similar to the proof of part 1. The only part in which we must be careful is in showing that ϕ preserves order. Let $f, g \in \Delta^\omega(A)$ with $f \leq g$ and let $\sigma \in T(f,g)$. We would like to proceed as before in using σ to create a one-to-one function $\sigma_1 : \omega \to \omega$ such that $\sigma_1 \in T(f,g)$, then using $\tau = \rho^{-1} \circ \sigma_1$. The problem arises from the fact that $\rho^{-1} \circ \sigma$ need not be defined on all of ω, so we must make some adjustments to σ in order to create σ_1. If $\operatorname{ran} \sigma \subseteq A$ then we can just set $\sigma_1 = \sigma$.

Assume that $\operatorname{ran} \sigma \not\subseteq A$. If $n \in (\operatorname{ran} \sigma) - A$ and $m \in \omega$ such that $\sigma(m) = n$ then $(f \circ \sigma)(m) = 0$. But then $\{n \in \omega : f(n) > 0\}$ is finite, so $\{n \in A : f(n) = 0\}$ is infinite. Also, $\{n \in \omega : (f \circ \sigma)(n) = 0\}$ is infinite, so we can define a one-to-one function $\sigma_1 : \omega \to \omega$ such that $\sigma_1(n) = \sigma(n)$ if $(f \circ \sigma)(n) > 0$ and $\sigma_1(n) \in A$ with $f(\sigma_1(n)) = 0$ if $(f \circ \sigma)(n) = 0$. Then $\sigma_1 \in T(f,g)$ and $\rho^{-1} \circ \sigma_1$ is a one-to-one function from ω into ω as desired. □

The model Δ^n can be nicely broken into sections that depend upon how the elements within the section can be written in decreasing order. Each permutation σ on n defines a section, which consists of those functions f with the property that $f \circ \sigma$ is decreasing. The section based on the identity permutation is the section of functions that are already decreasing. The model Δ^ω is also sectioned in the same way.

Definition 26 For every one-to-one function $\sigma : \omega \to \omega$ let $\Delta^\omega_\sigma = \{f \in \Delta^\omega : \sigma \in R(f)\}$. If id is the identity function then $\Lambda^\omega = \Delta^\omega_{id} = \{f \in \Delta^\omega : f \text{ is decreasing}\}$. Set $\Lambda^\omega_+ = \{f \in \Delta^\omega : \forall n \in \omega (f(n) > 0)\}$.

Theorem 27 *For every one-to-one function $\sigma : \omega \to \omega$, $\Delta^\omega_\sigma \simeq \Lambda^\omega$.*

Proof For the order isomorphism use the restriction s of the function r_σ defined in Lemma 23 to the set Δ^ω_σ. The proof of Lemma 23 shows that s is one-to-one and that both s and s^{-1} are strictly increasing. That $\operatorname{ran} s = \Lambda^\omega$ follows from the fact that $\operatorname{dom} s = \Delta^\omega_s$. □

Theorem 28 *For every one-to-one function $\sigma : \omega \to \omega$, Δ^ω_σ is closed under suprema of directed subsets.*

Proof If $\langle f_n : n \in \omega \rangle$ is an increasing sequence in Λ^ω then it has a supremum f in Δ^ω by Corollary 20. Now

$$f(m) = \lim_{n \in \omega}(f_n(m)) \geq \lim_{n \in \omega}(f_n(m+1)) = f(m+1)$$

for all $m \in \omega$ and therefore $f \in \Lambda^\omega$. That Λ^ω is closed under the suprema of directed sets follows again from Corollary 20. That every Δ^ω_σ is follows from Theorem 27. □

Lemma 29 *Let $f, g \in \Delta^\omega$. If $g \in \Lambda^\omega_+$ and $f \leq g$ then $f \in \Lambda^\omega_+$.*

Proof If follows from Lemma 8 that $f(n) > 0$ for all $n \in \omega$. Assume that there is $m \in \omega$ such that $f(m) \leq f(m+1)$. Let $\sigma \in T(f, g)$. There are $j, k \in \omega$ such that $\sigma(j) = m+1$, $\sigma(k) = m$, and $j < k$. Then $g(m) = (g \circ \sigma)(k) \leq (g \circ \sigma)(j) = g(m+1)$. But $g(m_1) \leq g(m)$, so $g(m) = g(m+1)$. Therefore $f(m) = f(m+1)$, and f is decreasing. □

Theorem 30 *Δ^ω contains no minimal elements.*

Proof Let $f \in \Lambda^\omega$. First assume that $\{n \in \omega : f(n) > 0\}$ is finite. Choose $m \in \omega$ such that if $n \in \omega$ and $f(n) > 0$ then $n < m$ and $1/m < f(n)$. Define g as follows.

$$g(n) = \begin{cases} \dfrac{1}{m}, & \text{if } n < m \\ 0, & \text{if } m \leq n \end{cases}$$

Then $g \in \Lambda^\omega$. If $n + 1 < m$ then

$$g(n)f(n+1) = \frac{1}{m}f(n+1) \leq \frac{1}{m}f(n) = g(n+1)f(n)$$

If $m \leq n+1$ then $f(n+1) = 0$ and $g(n)f(n+1) \leq g(n+1)f(n)$. Therefore $g < f$.

Now assume that $\{n \in \omega : f(n) > 0\}$ is infinite. Since $f \in \Delta^\omega$ this means that $f(n) > 0$ for all $n \in \omega$. We can think creating a function g from f by multiplying each $f(n)$ by a number a_n. Of course, we want $g \in \Lambda^\omega$ so we need $\sum_{n\in\omega} a_n f(n) = 1$. We also need for g to be decreasing. Therefore we want $a_{n+1}f(n+1) \leq a_n f(n)$ or $a_{n+1} \leq a_n \dfrac{f(n)}{f(n+1)}$. To get $g < f$, the maximum value of g, which is $g(0)$, must be less than the maximum value of f, which is $f(0)$. Finally, we must have $g(n)f(n+1) \leq g(n+1)f(n)$ for all n. This is equivalent to $a_n f(n)f(n+1) \leq a_{n+1}f(n+1)f(n)$, so we need $a_n \leq a_{n+1}$. If we generate a sequence $\langle a_n \rangle$ of positive real numbers that satisfy the following conditions, then $g(n) = a_n f(n)$ will give us the desired function.

1. $a_0 < 1$
2. $\sum_{n\in\omega} a_n f(n) = 1$
3. $\forall n \in \omega \left(a_n \leq a_{n+1} \leq a_n \dfrac{f(n)}{f(n+1)} \right)$

We will generate such a sequence as follows. Pick an index m such that $f(m) > f(m+1)$. At or below m, we will use a single number a for a_n, and above m we will use another number b for a_n. First use property 2 to see how b must be related to a.

$$\begin{aligned}1 &= \sum_{n\in\omega} a_n f(n)\\ &= a\sum_{n=0}^{m} f(n) + b\sum_{n=m+1}^{\infty} f(n)\\ &= a\sum_{n=0}^{m} f(n) + b\left(1-\sum_{n=0}^{m} f(n)\right)\\ b\left(1-\sum_{n=0}^{m} f(n)\right) &= 1 - a\sum_{n=0}^{m} f(n)\\ b &= \frac{1-a\sum_{n=0}^{m} f(n)}{1-\sum_{n=0}^{m} f(n)}\end{aligned}$$

Whatever we use for a, this definition of b will ensure that property 2 is satisfied. To simplify the notation, we will use $c = \sum_{n=0}^{m} f(n)$. Note that $0 < c < 1$. Now consider property 3. If $n+1 \le m$ or $m < n$ then $a_n = a_{n+1}$ and property 3 holds. We want $a = a_m \le a_{m+1} = b$ and

$$b = a_{m+1} \le a_m \frac{f(m)}{f(m+1)} = a\frac{f(m)}{f(m+1)}$$

Make this inequality an equality and solve for a.

$$\begin{aligned}\frac{1-ac}{1-c} &= a\frac{f(m)}{f(m+1)}\\ f(m+1)(1-ac) &= af(m)(1-c)\\ f(m+1) - af(m+1)c &= af(m)(1-c)\\ a[f(m+1)c + f(m)(1-c)] &= f(m+1)\\ a &= \frac{f(m+1)}{f(m+1)c + f(m)(1-c)}\end{aligned}$$

Now $f(m+1) < f(m)$. Therefore

$$\begin{aligned}f(m+1)(1-c) &< f(m)(1-c)\\ f(m+1) - f(m+1)c &< f(m)(1-c)\\ f(m+1) &< f(m+1)c + f(m)(1-c)\end{aligned}$$

So $0 < a < 1$.

The definition of a and b guarantees that $b = a\dfrac{f(m)}{f(m+1)}$, so property 3 is satisfied. Therefore $g < f$.

The result now holds for all $f \in \Delta^\omega$ because each Δ^ω_σ is order isomorphic to Λ^ω. □

5 Approximation

In this section we will investigate the notion of approximation in $\varDelta^\omega$. The ability to tell when the information contained in one state is essential to the information contained in another is a powerful property of $\varDelta^n$. An important tool in the study of this property in [3] is the path from the least element of $\varDelta^n$ to another given element. We will lose the ability to approximate in the sense of [3] because of the fact that $\varDelta^\omega$ does not have a minimum element. These paths will play an significant role in our investigation.

Definition 31 Let $f, g \in \varDelta^\omega$. The *path* from f to g is the function $\pi_{fg} : [0,1] \to \varDelta^\omega$ given by $\pi_{fg}(t) = (1-t)f + tg$ for every $t \in [0,1]$.

The Mixing Law tells us that if $f \leq g$ then the range of $\pi_{f,g}$ is a chain. In $\varDelta^n$ these paths are used to determine when one element is weakly way below another. There $f \ll_w g$ if and only if there is $t \in [0,1)$ such that $f \leq \pi_{fg}(t)$. In $\varDelta^\omega$, we will use these paths to show that, in fact, no element of $\varDelta^\omega$ is weakly way below another.

Theorem 32 *For every $f \in \varDelta^\omega$, $\twoheaduparrow_w f = \emptyset$.*

Proof Let $f, h \in \varLambda^\omega$ with $f < h$. First assume that $\{n \in \omega : h(n) > 0\}$ and $\{n \in \omega : f(n) > 0\}$ are both finite. Pick $m \in \omega$ such that $m \geq 1$, $f(n) = 0$ for all $n \geq m$, and $1/(m+1) < \min\{h(n) : h(n) > 0\}$. Define $g : \omega \to [0,1]$ as follows.

$$g(n) = \begin{cases} \frac{1}{m+1} & 0 \leq n \leq m \\ 0 & m < n \end{cases}$$

Then $g \in \varLambda^\omega$. If $n < m$ then

$$g(n)h(n) = \frac{1}{m+1}h(n+1) \leq \frac{1}{m+1}h(n) = g(n+1)h(n)$$

If $m \leq n$ then $h(n) = 0$ so $g(n)h(n+1) = g(n+1)h(n)$. Therefore $g < h$. Then $C = \pi_{gh}[[0,1)]$ is a chain in $\varLambda^\omega$ whose supremum is h. Let $t \in [0,1)$ and let $k < m$ such that $f(k) > f(k+1) = 0$. Then $h(k+1) = 0$ and

$$\begin{aligned} f(k)\pi_{gh}(t)(k+1) &= f(k)\left[\frac{1-t}{m+1} + th(k+1)\right] \\ &= \frac{f(k)(1-t)}{m+1} \\ &> 0 = f(k+1)\pi_{gh}(t)(k) \end{aligned}$$

Therefore $C \cap \updownarrow f = \emptyset$ and $f \not\ll_w h$.

Next assume that $\{n \in \omega : h(n) > 0\}$ is finite and $\{n \in \omega : f(n) > 0\}$ is infinite. Since $f \in \varLambda^\omega$ this means that $f(n) > 0$ for all $n \in \omega$. Choose $m \in \omega$ such that $m \geq 1$, $h(n) = 0$ for all $n \geq m$, and $f(m+1) < f(m)$. Define $g : \omega \to [0,1]$ as follows.

$$g(n) = \begin{cases} \frac{1}{m+2} & n \leq m+1 \\ 0 & m+1 < n \end{cases}$$

Then $g \in \Lambda^\omega$. If $n < m$ then

$$g(n)h(n+1) = \frac{1}{m+2}h(n+1) \leq \frac{1}{m+2}h(n) = g(n+1)h(n)$$

If $m \leq n$ then $h(n) = 0$ so $g(n)h(n+1) = g(n+1)h(n)$. Therefore $g < h$. Again, $C = \pi_{gh}[[0,1)]$ is a chain in Λ^ω whose supremum is h. Let $t \in [0,1)$.

$$f(m)\pi_{gh}(t)(m+1) = f(m)\frac{1-t}{m+2} > f(m+1)\frac{1-t}{m+2} = f(m+1)\pi_{gh}(t)(m)$$

Therefore $C \cap \updownarrow f = \emptyset$ and $f \not\ll_w h$.

Finally, assume that $\{n \in \omega : h(n) > 0\}$ is infinite. Then $h(n) > 0$ for all $n \in \omega$, as is $f(n)$. Let $m \in \omega$ such that $m \geq 1$, there is $j < m$ such that $f(j+1) < f(j)$, and there is $j < m$ such that $h(j+1) < h(j)$. Set

$$b = \frac{1}{(m+1)h(m) + \sum_{n=m+1}^{\infty} h(n)}$$

and $a = bh(m)$. Define $g : \omega \to [0,1]$ as follows.

$$g(n) = \begin{cases} a & 0 \leq n \leq m \\ bh(n) & m < n \end{cases}$$

Then $g(n) > 0$ and $g(n) \geq g(n+1)$ for all $n \in \omega$.

$$\begin{aligned}\sum_{n \in \omega} g(n) &= a(m+1) + b\sum_{n=m+1}^{\infty} h(n) \\ &= b\left[(m+1)h(m) + \sum_{n=m+1}^{\infty} h(n)\right] \\ &= 1\end{aligned}$$

Therefore $g \in \Lambda^\omega$. If $n+1 < m$ then

$$g(n)h(n+1) = ah(n+1) \leq ah(n) = g(n+1)g(n)$$

If $m < n$ then $g(n)h(n+1) = bh(n)h(n+1) = g(n+1)h(n)$. Finally,

$$g(m)h(m+1) = ah(m+1) = bh(m)h(m+1) = g(m+1)h(m)$$

Therefore $g < h$ and $C = \pi_{gh}[[0,1)]$ is a chain in Λ^ω whose supremum is h. Let $k < m$ such that $f(k+1) < f(k)$ and let $t \in [0,1)$. If $f(k)\pi_{gh}(t)(k) \leq f(k+1)\pi_{gh}(t)(k+1)$ then

$$\begin{aligned} f(k)[(1-t)a + th(k+1)] &\leq f(k+1)[(1-t)a + th(k)] \\ (1-t)af(k) + tf(k)h(k+1) &\leq (1-t)af(k+1) + tf(k+1)h(k) \\ 0 < (1-t)a[f(k) - f(k+1)] &\leq t[f(k+1)h(k) - f(k)h(k+1)] \leq 0 \end{aligned}$$

Therefore $C \cap \updownarrow f = \emptyset$ and $f \not\ll_w h$. The result follows because Δ_σ^ω is order isomorphic to Λ^ω for all σ. $\square$

It is easy to see that in Δ^n the sets $\downarrow f$ are not directed as long as $f \neq \min \Delta^n$. For example, $\langle .5, .5, 0\rangle$ and $\langle .5, 0, .5\rangle$ are both smaller than $\langle 1, 0, 0\rangle$, but they have no common successor other than $\langle 1, 0, 0\rangle$. It is conceivable that, due to the breadth of Δ^ω, if $f \in \Delta^\omega$ and $\{n \in \omega : f(n) > 0\}$ is infinite, then $\downarrow f$ is directed. But this does not happen.

Theorem 33 *For every $h \in \Delta^\omega$, $\downarrow h$ is not directed.*

Proof The argument that $\downarrow e_0$ is not directed is essentially the same as the example given above for Δ^n. Just put an infinite string of zeros at the end of each 3-tuple. We will prove our result for $h \in \Lambda^\omega$. The general result then follows. Assume that $h \in \Lambda^\omega$ with $h \neq e_0$ and that $\{n \in \omega : h(n) > 0\}$ is finite. Let $m = \max\{n \in \omega : h(n) > 0\}$ and set $a = 1/(1 + h(m))$ and $b = h(m)/(1 + h(m))$. Note that $0 < a < 1$, $0 < b < 1$, and $b = ah(m)$. Define f and g as follows.

$$f(n) = \begin{cases} ah(n) & n \leq m \\ 2b/3 & n = m+1 \\ b/3 & n = m+2 \\ 0 & m+2 < n \end{cases} \qquad g(n) = \begin{cases} ah(n) & n \leq m \\ b/3 & n = m+1 \\ 2b/3 & n = m+2 \\ 0 & m+2 < n \end{cases}$$

Then

$$\sum_{n\in\omega} f(n) = \sum_{n=0}^{m} ah(n) + b = a + b = 1$$

and $\sum_{n\in\omega} g(n) = 1$ by the same argument. So $f, g \in \Delta^\omega$. If $n < m$ then $f(n+1) = ah(n+1) \leq ah(n) = f(n)$. Also, $f(m+1) = 2b/3 < b = ah(m) = f(m)$ and $f(m+2) = b/3 < 2b/3 = f(m+1)$. If $n \geq m+2$ then $f(n+1) = 0 \leq f(n)$. Therefore $f \in \Lambda^\omega$. If $n < m$ then $f(n)h(n+1) = ah(n)h(n+1) = f(n+1)f(n)$. If $m \leq n$ then $f(n)h(n+1) = 0 \leq f(n+1)h(n)$. Thus $f \leq h$.

Let ρ be the function from ω onto ω which sends $m+1$ to $m+2$ and $m+2$ to $m+1$, and is the identity everywhere else. If $n < m$ then $(h \circ \rho)(n+1) = h(n+1) \leq h(n) = (h \circ \rho)(n)$. If $n \geq m$ then $\rho(n+1) \geq m+1$ so $(h \circ \rho)(n+1) = 0 \leq (h \circ \rho)(n)$. Thus $\rho \in R(h)$.

If $n < m$ then

$$(g \circ \rho)(n+1) = g(n+1) = ah(n+1) \leq ah(n) = g(n) = (g \circ \rho)(n)$$

Now $\rho(m) = m$ and $\rho(m+1) = m+2$ so

$$(g \circ \rho)(m+1) = g(m+2) = \frac{2b}{3} < b = ah(m) = g(m) = (g \circ \rho)(m)$$

Also, $\rho(m+2) = m+1$ so

$$(g \circ \rho)(m+2) = g(m+1) = \frac{b}{3} < \frac{2b}{3} = g(m+2) = (g \circ \rho)(m+1)$$

Finally, if $n \geq m+2$ then $\rho(n+1) = n+1 > m+2$ so $(g \circ \rho)(n+1) = 0 \leq (g \circ \rho)(n)$. Therefore $\rho \in R(g)$. We will next show that $\rho \in T(g, h)$.

If $n < m$ then

$$(g \circ \rho)(n)(h \circ \rho)(n+1) = g(n)h(n+1) = ah(n)h(n+1)$$
$$= g(n+1)h(n) = (g \circ \rho)(n+1)(h \circ \rho)(n)$$

If $m \leq n$ then $n+1 \geq m+1$ so $\rho(n+1) \geq m+1$. Therefore $(g \circ \rho)(n)(h \circ \rho)(n+1) = 0 \leq (g \circ \rho)(n+1)(h \circ \rho)(n)$, and $\rho \in T(g,h)$.

Now let $p \in \downarrow h$ such that $f \leq p$ and $g \leq p$. If $n > m+2$ then $f(n) = g(n) = 0$ so $p(n) = 0$. Let $\sigma \in T(f,p)$ and $\tau \in T(f,p)$. Since $\sigma \in R(f)$ we know that $\sigma(m+1) = m+1$ and $\sigma(m+2) = m+2$. Since $\tau \in R(g)$ we know that $\tau(m+1) = m+2$ and $\tau(m+2) = m+1$. Therefore

$$p(m+1) = (p \circ \sigma)(m+1) = (p \circ \tau)(m+1) = p(m+2)$$

By Degeneration it must be true that $p(m+1) = p(m+2) = 0$.

Because $p < h$, $p^+ < h^+$ by Lemma 12, so $p(0) < h(0)$. There must be some $k < m$ such that $p(k) < h(k)$ and $h(k+1) < p(k+1)$. It is obvious from the second inequality that $0 < p(k+1)$, but it is also true that $0 < p(k)$ because $p < h$ and $h(k) > 0$. Therefore there are $i, j \in \omega$ such that $\sigma(i) = k$ and $\sigma(j) = k+1$. We also know that $i < j$ because h is decreasing. But

$$(f \circ \sigma)(i)(p \circ \sigma)(j) = f(k)p(k+1) = ah(k)p(k+1)$$
$$> ah(k+1)p(k) = f(k+1)p(k) = (f \circ \sigma)(j)(p \circ \sigma)(i)$$

a contradiction.

Now assume that $h(n) > 0$ for all $n \in \omega$. Let $k_0, k_1 \in \omega$ such that $k_0 < k_1$, $h(k_0+1) < h(k_0)$, and $h(k_1+1) < h(k_1)$. Define f and g as in Theorem 30 using k_0 in the definition of f and k_1 in the definition of g. Let a_1 and b_1 be the constants used in the definition of f, and let a_2 and b_2 be the constants used in the definition of g. Let $p \in \overline{\downarrow} h$ such that $f, g \leq p$. Note that $f, g, p \in \Lambda^\omega_+$ by Lemma 29. If $n < k_0$ then

$$\frac{h(n)}{h(n+1)} = \frac{a_1 h(n)}{a_1 h(n+1)} = \frac{f(n)}{f(n+1)} \leq \frac{p(n)}{p(n+1)}$$

If $k_0 \leq n < k_1$ then

$$\frac{h(n)}{h(n+1)} = \frac{a_2 h(n)}{a_2 h(n+1)} = \frac{g(n)}{g(n+1)} \leq \frac{p(n)}{p(n+1)}$$

If $k_1 \leq n$ then

$$\frac{h(n)}{h(n+1)} = \frac{b_1 h(n)}{b_1 h(n+1)} = \frac{f(n)}{f(n+1)} \leq \frac{p(n)}{p(n+1)}$$

Therefore $h \leq p$. □

This means that for every $g \in \Delta^\omega$ there is no $f < g$ such that f is a natural starting point for a path to g. We can always bypass f to get to g.

6 Continuous Functions and Measurements

The next theorem corresponds to Corollary 2.18 in [3] and the following corollary to Corollary 2.19 in [3]. They are proved in same way.

Theorem 34 *Let X be a dcpo. An increasing function $\phi : \Delta^\omega \to X$ is Scott continuous if and only if $\phi(\sup_{n\in\omega} f_n) = \sup_{n\in\omega} \phi(f_n)$ for every increasing sequence $\langle f_n : n \in \omega\rangle$ in Δ^ω.*

Let $[0,\infty)^*$ represent the set $[0,\infty)$ with its reverse order.

Corollary 35 *The function $s : \Delta^\omega \to [0,\infty)^*$ given by $s(f) = -\ln(f^+)$ for all $f \in \Delta^\omega$ is Scott continuous and has the following properties.*

1. *For all $f, g \in \Delta^\omega$, if $f \le g$ and $s(f) = s(g)$ then $f = g$. That is, if $f < g$ then $s(f) < s(g)$.*
2. *For all $f \in \Delta^\omega$, $s(f) = 0$ if and only if $f \in \max \Delta^\omega$.*

Definition 36 The *natural retraction* of Δ^ω is the function $r : \Delta^\omega \to \Lambda^\omega$ given by $r(f) = f \circ \sigma$ for all $f \in \Delta^\omega$ and $\sigma \in R(f)$.

That r is a function follows from the fact the $f \circ \sigma = f \circ \tau$ for all $\sigma, \tau \in R(f)$.

Theorem 37 *The natural retraction is a Scott continuous retraction from Δ^ω onto Λ^ω.*

Proof It is obvious from the definition of r that r is the identity on Λ^ω and that $\operatorname{ran} r = \Lambda^\omega$. If $f, g \in \Delta^\omega$ and $f < g$ then there is $\sigma \in R(f) \cap R(g)$. By the comments after Lemma 23, $r(f) = r_\sigma(f) < r_\sigma(g) = r(g)$. If $f, g \in \Delta^\omega_\sigma$ and $f \circ \sigma < g \circ \sigma$ then σ also shows that $f < g$. Thus r is strictly increasing and $r \restriction \Delta^\omega_\sigma$ is an order isomorphism. It follows from Theorem 34 that r is Scott continuous. □

The natural retraction is also an open mapping under the Scott topology. We will use the following lemma to prove this.

Lemma 38 *Let $g \in \Lambda^\omega$ and let $\sigma : \omega \to \omega$ be one-to-one. If*

$$f(n) = \begin{cases} g(\sigma^{-1}(n)) & n \in \operatorname{ran} \sigma \\ 0 & n \notin \operatorname{ran} \sigma \end{cases}$$

for all $n \in \omega$ then $f \in \Delta^\omega$ and $\sigma \in R(f)$.

Proof Obviously f is a function from ω into $[0,1]$. If $f(n) > 0$ then $n \in \operatorname{ran} \sigma$. Also, $(f \circ \sigma)(n) = f(\sigma(n)) = g(\sigma^{-1}(\sigma(n))) = g(n)$ for every $n \in \omega$. Therefore

$$\sum_{n\in\omega} f(n) = \sum_{m\in\operatorname{ran}\sigma} f(n) = \sum_{n\in\omega} f(\sigma(n)) = \sum_{n\in\omega} g(n) = 1$$

Thus $f \in \Delta^\omega$ and, since $f \circ \sigma = g \in \Lambda^\omega$, we also have $\sigma \in R(f)$. □

Theorem 39 *If U is a Scott-open subset of Δ^ω and r is the natural retraction then $r[U]$ is a Scott-open subset of Λ^ω.*

Proof Let $s \in r[U]$ and let $t \in \Lambda^\omega$ with $s < t$. Let $f \in U$ such that $s = r(f)$. There is $\sigma \in R(f)$ such that $r(f) = f \circ \sigma$. Set

$$g(n) = \begin{cases} t(\sigma^{-1}(n)) & n \in \operatorname{ran}\sigma \\ 0 & n \notin \operatorname{ran}\sigma \end{cases}$$

for all $n \in \omega$. Then $g \in \Delta^\omega$ and $\sigma \in R(g)$. If $n \in \omega$ then

$$(f\circ\sigma)(n)(g\circ\sigma)(n+1) = s(n)t(n+1) \leq s(n+1)t(n) = (f\circ\sigma)(n+1)(g\circ\sigma)(n)$$

so $f \leq g$. But U is increasing, so $g \in U$. Therefore $t = r(g) \in r[U]$.

Let $\langle t_k : k \in \omega\rangle$ be an increasing sequence in Λ^ω such that $\sup_{n\in\omega} t_k = s \in r[U]$. Let $f \in U$ and let $\sigma \in R(f)$ such that $f \circ \sigma = r(f) = s$. For every $k \in \omega$ and every $n \in \omega$ define $f_k(n)$ by the following equation.

$$f_k(n) = \begin{cases} t_k(\sigma^{-1}(n)) & n \in \operatorname{ran}\sigma \\ 0 & n \notin \operatorname{ran}\sigma \end{cases}$$

Then $\langle f_k : k \in \omega\rangle$ is a sequence in Δ^ω and $\sigma \in R(f_k)$ for every $k \in \omega$. If $k, n \in \omega$ then

$$\begin{aligned}(f_k\circ\sigma)(n)(f_{k+1}\circ\sigma)(n+1) &= t_k(n)t_{k+1}(n+1) \\ &\leq t_k(n+1)t_{k+1}(n) = (f_k\circ\sigma)(n+1)(f_{k+1}\circ\sigma)(n)\end{aligned}$$

so $f_k \leq f_{k+1}$.

If $n \in \operatorname{ran}\sigma$ then

$$\lim_{k\to\infty} f_k(n) = \lim_{k\to\infty} t_k(\sigma^{-1}(n)) = s(\sigma^{-1}(n)) = f(n)$$

so $f = \sup_{k\to\infty} f_k$ by Theorem 19. But $f \in U$ and U is Scott-open, so there is $k \in \omega$ such that $f_k \in U$. Then $t_k = r(f_k) \in r[U]$ and $r[U]$ is Scott-open. □

The following definition is from [3].

Definition 40 Let X be a set. A function $\phi : \Delta^\omega \to X$ is *symmetric* if and only if $\phi(f \circ \sigma) = \phi(f)$ for every one-to-one function σ from ω onto ω.

The next theorem corresponds to Lemma 2.29 in [3] and is proved the same way.

Theorem 41 *If X is a set and $\phi : \Lambda^\omega \to X$ then there is a unique symmetric extension $\overline{\phi} : \Delta^\omega \to X$ given by $\overline{\phi} = \phi \circ r$. If X is an ordered set and ϕ is increasing, strictly increasing, or Scott continuous, then so is $\overline{\phi}$.*

The examples of symmetric functions given in [3] apply here, except for Shannon entropy. Shannon entropy for Δ^n is defined by

$$S(f) = -\sum_{k=0}^{n-1} f(k)\ln(f(k))$$

where the summand is assumed to be 0 when $f(k) = 0$. There is no problem with the existence of this function for finite sequences. However, there are some elements of Δ^ω for which the infinite version of the series diverges. I am indebted to my University of Dayton colleague, Bob Gorton, for the following example. Let $f(n) = \dfrac{1}{(n+2)\ln(n+2)}$ and $g(n) = \dfrac{1}{(n+2)(\ln(n+2))^2}$ for $n \in \omega$. A simple integral test shows that $\sum_{n\in\omega} f(n)$ diverges and $\sum_{n\in\omega} g(n)$ converges. Also

$$f(n) = \frac{\ln(n+2)}{(n+2)(\ln(n+2))^2} \leq \frac{\ln[(n+2)(\ln(n+2))^2]}{(n+2)(\ln(n+2))^2} = |g(n+2)\ln(g(n+2))|$$

for all $n \geq 1$. Therefore $\sum_{n\in\omega} g(n)\ln(g(n))$ diverges. Of course, $g \notin \Delta^\omega$, but we can rectify that by multiplying g by a constant. The resulting element of Δ^ω has the same convergence properties as g. We will say that the states for which the Shannon entropy diverges have infinite Shannon entropy. One can think of these states as having probabilities that are so close to one another that the entropy measurement sees all the outcomes as essentially the same and therefore gives an infinite value to the state. Or one can think of these states as having so much noise that any underlying structure is completely lost. As we will see the relation between the states with finite Shannon entropy and those with infinite Shannon entropy is rather interesting. One would expect that if state f has finite Shannon entropy and state g is larger than f and therefore carries more information than f, then g would not only have finite Shannon entropy but would have a lower entropy value than f. This is precisely what happens. But we can also find a sequence of states, each with infinite Shannon entropy, which converge to a pure state, which has Shannon entropy 0. This would be like having an infinite number of completely unintelligible messages suddenly resolving themselves into one in which the message is certain.

It is not hard to use the properties of the function $-x\ln x$ to show that if $f < g$ and f has finite Shannon entropy, then g has finite Shannon entropy. It is not so easy to show that the value of the entropy is actually decreasing. We will follow the approach used in [3].

Definition 42 For every $f, g \in \Delta^\omega$ the relative Shannon entropy of g given f is

$$S(g\|f) = \sum_{n\in\omega} g(n)\ln\left[\frac{g(n)}{f(n)}\right]$$

if the series converges.

If $f(m) = 0$ and $g(m) \neq 0$ for some m then we may assign the value ∞ to $g(m)\ln[g(m)/f(m)]$ and therefore to $\sum g(n)\ln[g(n)/f(n)]$. If $g(m) = 0$ then $g(m)\ln[g(m)/f(m)]$ is assumed to be 0 regardless of the value of $f(m)$.

Lemma 43 $S(g\|f) \geq 0$ *for all* $f, g \in \Delta^\omega$ *and* $S(g\|f) = 0$ *if and only if* $f = g$.

Proof Fix $g \in \Delta^\omega$. Then $\sum g(n) = 1$. Let $U(f) = \sum g(n) \ln[g(n)/f(n)]$. The maximum value of $U(f)$ is obviously ∞. We will use the method of Lagrange multipliers to show that the minimum value of $U(f)$ subject to $\sum f(n) = 1$ is 0 and it occurs when $f = g$. We need consider only those functions f which are 0 only when g is also 0.

For every $n \in \omega$ when we take a partial derivative of $U(f)$ with respect to the variable $f(n)$ we get $-g(n)/f(n) = \lambda$ or $-g(n) = \lambda f(n)$. Therefore $-1 = -\sum g(n) = \lambda \sum f(n) = \lambda$. It follows that U reaches its minimum if and only if $f = g$. In that case, $U(f) = \sum g(n) \ln[g(n)/f(n)] = 0$. □

Theorem 44 *If $f, g \in \Delta^\omega$ and f has finite Shannon entropy then g has finite Shannon entropy and $S(f) > S(g)$.*

Proof Assume that $f, g \in \Lambda^\omega$. There is $m \in \omega$ such that if $n \leq m$ then $f(n) < g(n)$ and if $n > m$ then $f(n) \geq g(n)$. Note that if $f(m) = 0$ and $f(k) = 0$ then $g(k) = 0$ and we may assign the value 0 to $[g(k) - f(k)] \ln[f(k)/f(m)]$. Furthermore, if $f(m) = 0$ and $f(k) \neq 0$ then $k < m$ so $f(k) < g(k)$ and we may assign the value ∞ to $[g(k) - f(k)] \ln[f(k)/f(m)]$. If follows that for every $n > m$

$$\begin{aligned}
&\sum_{k=0}^{n} [g(k) - f(k)] \ln f(k) \\
&= \sum_{k=0}^{m-1} \ln \left[\frac{f(k)}{f(m)} \right] + \sum_{k=m+1}^{n} [g(k) - f(k)] \ln \left[\frac{f(k)}{f(m)} \right] \geq 0
\end{aligned}$$

and therefore

$$\sum_{k=0}^{n} g(k) \ln f(k) - \sum_{k=0}^{n} f(k) \ln f(k) \geq 0$$

or

$$-\sum_{k=0}^{n} f(k) \ln f(k) \geq -\sum_{k=0}^{n} g(k) \ln f(k).$$

By Lemma 43 we have $\sum_{k=0}^{n} g(k) \ln[g(k)/f(k)] \geq 0$, so

$$-\sum_{k=0}^{n} g(k) \ln f(k) \geq -\sum_{k=0}^{n} \ln g(k)$$

and

$$-\sum_{k=0}^{n} f(k) \ln f(k) \geq -\sum_{k=0}^{n} g(k) \ln g(k).$$

Thus $\lim_{n\to\infty} \sum_{k=0}^{n} g(k) \ln g(k)$ converges, g has finite entropy, and $S(f) \geq S(g)$.

If $S(f) = S(g)$ then

$$-\sum_{k=0}^{\infty} g(k) \ln f(k) = -\sum_{k=0}^{\infty} g(k) \ln g(k)$$

and $S(g \| f) = 0$, which means that $f = g$. The result for general f and g follows from symmetry. □

As a consequence, S is increasing as a function into $[0,\infty)^*$. Also, if $f \le g$ and g has infinite entropy then so does f.

We will use the following lemma in an example of a sequence of states with infinite entropy which converge to a state with finite entropy.

Lemma 45 *Let $f \in \Delta^\omega$. If f has infinite Shannon entropy and $0 < a$ then af has infinite Shannon entropy.*

Proof If the series $S(af) = \sum_{n\in\omega} af(n)\ln[af(n)]$ converges, then so does the series $\sum_{n\in\omega}[f(n)\ln[af(n)] - f(n)\ln a] = \sum_{n\in\omega} f(n)\ln f(n) = S(f)$. □

Example 46 There is an increasing sequence $\langle f_n\rangle$ in Δ^ω such that each f_n has infinite Shannon entropy and $\lim_{n\to\infty} f_n = e_0$.

Let $f \in \Lambda^\omega$ with infinite Shannon entropy. Let a be a number between 0 and 1, and set $b = \dfrac{a[f(0)-1]+1}{f(0)}$. Then

$$\begin{aligned} a &< 1 \\ a[f(0)-1] &> f(0)-1 \\ b = \frac{a[f(0)-1]+1}{f(0)} &> 1 \end{aligned}$$

If we set $g(0) = bf(0)$ and $g(n) = af(n)$ for all $n > 0$ then

$$\begin{aligned} \sum_{n\in\omega} g(n) &= bf(0) + a\sum_{n\in\omega} f(n) \\ &= bf(0) + a[1-f(0)] \\ &= 1 \end{aligned}$$

So $g \in \Lambda^\omega$. Furthermore,

$$f(0)g(1) = af(0)f(1) \le bf(0)f(1) = f(1)g(0)$$

and, for $n > 0$,

$$f(n)g(n+1) = af(n)f(n+1) = f(n+1)g(n)$$

and therefore $f < g$.

Set $f_0 = f$ and, for every $n \in \omega$, set $f_{n+1}(0) = bf_n(0)$ and $f_{n+1}(m) = af_n(m)$ for $m > 0$. Then $\langle f_n\rangle$ is an increasing sequence of states in Λ^ω each of which has infinite Shannon entropy by Lemma 45. But for every $m > 0$ we have $\lim_{n\to\infty} f_n(m) = \lim_{n\to\infty} a^n f(m) = 0$. Thus $\lim_{n\to\infty} f_n(0) = 1$ and $\lim_{n\to\infty} f_n = e_0$.

The examples of symmetric functions in [3] play a very important role in the model Δ^n. They are *measurements*. They measure the information content of an element of Δ^n and can tell us how close to a pure state a particular element of Δ^n lies. But Δ^ω offers too many ways to approach a maximal element. These symmetric functions cannot keep track of all the elements as well in this setting. This means, in particular, that entropy is no longer a measurement.

Definition 47 Let X be an ordered set and $b \in X$. A Scott continuous map $\mu : X \to [0, \infty)^*$ *measures the content* of b if and only if for every Scott neighborhood U of b there is a Scott neighborhood V of $\mu(b)$ such that $\mu^{-1}[V] \cap \downarrow b \subseteq U$.

We say that μ measures $A \subseteq X$ when μ measures the content of every element of A.

Definition 48 Let X be an ordered set. A Scott continuous function $\mu : X \to [0, \infty)^*$ is a *measurement* of X if and only if μ measures $\ker \mu = \{a \in X : \mu(a) = 0\}$.

These definitions can be generalized by replacing $[0, \infty)^*$ by a dcpo.

Theorem 49 *If* $\mu : \Delta^\omega \to [0, \infty)^*$ *is symmetric and* $\ker \mu = \max \Delta^\omega$ *then* μ *is not a measurement of* Δ^ω.

Proof In order to be a measurement, μ must be Scott continuous, so we may assume that μ is Scott continuous. For every $k \in \omega$ let f_k be the element of Δ^ω such that $f_k(0) = 1 - 2^{-k-1}$, $f_k(1) = 2^{-k-1}$, and $f_k(n) = 0$ for all $n > 1$. Then $\langle f_k : k \in \omega \rangle$ is an increasing sequence in Λ^ω and $\sup_{k \in \omega} f_k = e_0$. Define a new sequence by setting $g_k(0) = 1 - 2^{-k-1}$, $g_k(k+1) = 2^{-k-1}$, and $g_k(n) = 0$ for all other values of n. We will show that $K = \bigcup_{k \in \omega} \overline{\downarrow} g_k$ is Scott-closed. It is clearly decreasing.

Assume that there is $f \in \Delta^\omega$ such that $B = \{k \in \omega : f \leq g_k\}$ is infinite. We can write $B = \{k_j : j \in \omega\}$ in such a way that $k_i < k_j$ when $i < j$. Let $j \in \omega$ with $j > 0$, and let $\sigma \in R(f) \cap R(g_{k_0})$ and $\tau \in R(f) \cap R(g_{k_j})$. Now $\sigma(0) = 0$ and $\sigma(1) = k_0 + 1$ by the definition of g_{k_0}. Therefore $f(k_0 + 1) \geq f(k_j + 1)$. But $\tau(0) = 0$ and $\tau(1) = k_j + 1$ by the definition of g_{k_j}, so $f(k_j + 1) \geq f(k_0 + 1)$. Thus $f(k_j + 1) = f(k_0 + 1)$ for all $j \in \omega$. But then $f(k_j + 1) = 0$ for all $j \in \omega$, which is impossible because $g_{k_j}(k_j + 1) > 0$. Therefore if $f \in K$ then $\{k \in \omega : f \leq g_k\}$ is finite.

Let $\langle h_n : n \in \omega \rangle$ be an increasing sequence in K. There is a finite $F \subseteq \omega$ such that $\{h_n : n \in \omega\} \subseteq \bigcup_{k \in F} \overline{\downarrow} g_k$. This means that there is $k \in \omega$ such that $\{h_n : n \in \omega\} \subseteq \overline{\downarrow} g_k$. Then $\sup_{n \in \omega} h_n \in \overline{\downarrow} g_k \subseteq K$. It follows that K is closed under the suprema of directed subsets and that K is Scott-closed.

Set $U = \Delta^\omega - K$. U is a Scott neighborhood of e_0. Let V be a Scott neighborhood of $\mu(e_0) = 0$. There is $k \in \omega$ such that $\mu(f_k) \in V$. But $\mu(g_k) = \mu(f_k)$ because μ is symmetric, so $g_k \in \mu^{-1}[V] \cap \downarrow e_0$ and $\mu^{-1}[V] \cap \downarrow e_0 \nsubseteq U$. □

It is hard to imagine a meaningful measurement which is not symmetric, that is, which can tell the difference between various outcomes within a state and will assign a different measure of information to the state when the same probabilities are rearranged among the outcomes.

7 Conclusion

We have seen that the model of Coecke and Martin can be extended to the infinite dimensional classical states with mixed results. It does provide a picture of increasing certainty or increasing information about the system and it also satisfies some basic principles such as Degeneracy. But the important ability to approximate total or pure states by partial states completely vanishes as does, for the most part, the ability to measure content of a state in the sense of Martin. Thermodynamic entropy is still defined on the model and behaves as it should, but Shannon entropy is not defined over the entire model and displays some surprising behavior. It is not unusual for problems to arise when passing from a finite dimensional to an infinite dimensional model for quantum physics. It would be nice to know whether such difficulties are inherit in the systems.

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