1995

A Note on Reordering Ordered Topological Spaces and the Existence of Continuous, Strictly Increasing Functions

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A NOTE ON REORDERING ORDERED TOPOLOGICAL SPACES AND THE EXISTENCE OF CONTINUOUS, STRICTLY INCREASING FUNCTIONS

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To appear in Topology Proceedings.

I. Introduction

The origin of this paper is in a question that was asked of the author by Michael Wellman, a computer scientist who works in artificial intelligence at Wright Patterson Air Force Base in Dayton, Ohio. He wanted to know if, starting with $\mathbb{R}^n$ and its usual topology and product partial order, he could linearly reorder every finite subset and still obtain a continuous function from $\mathbb{R}^n$ into $\mathbb{R}$ that was strictly increasing with respect to the new order imposed on $\mathbb{R}^n$. It is the purpose of this paper to explore the structural characteristics of ordered topological spaces which have this kind of behavior. In order to state more clearly the questions that we will consider, we must define some terms.

By an *order* we mean a relation that is transitive and asymmetric ($p < q \Rightarrow q \not< p$). Unless otherwise stated, we will denote the order of an ordered set X as $<$. Two distinct elements, p and q, of X are *comparable* if $p < q$ or $q < p$, and are *incomparable* otherwise. An order is called a *linear order* if every two distinct elements of the set are comparable. A *chain* is a subset of an ordered set that is linearly ordered. An *antichain* of an ordered set X is a subset of X in which every two distinct elements are incomparable. If p and q are elements of X then $[p, q] = \{r \in X : p \leq r \leq q\}$. A subset A of X is *convex* if $[p, q] \subseteq A$ for all $p, q \in X$.

An *ordered topological* space is an ordered set with a topology. There need be no connection between the order and the topology. So $\mathbb{R}^2$ with its usual ordering (i.e. $\langle a, b\rangle \leq \langle c, d\rangle \Leftrightarrow a \leq c$ and $b \leq d$) is an ordered topological space, as is $\mathbb{R}^2$ with the usual topology and the lexicographic ordering, and $\mathbb{R}^2$ with the discrete topology and the usual ordering. We will use the word space when discussing an ordered set with a topology, and the word set when only an order is involved.

Let $\langle X, <_X\rangle$ and $\langle Y, <_Y\rangle$ be ordered spaces and let $f : X \to Y$. We will say that f is *strictly increasing* if $f(p) <_Y f(q)$ for all $p, q \in X$ with $p <_X q$. If X and Y are linearly ordered and f is also onto, then we say that X and Y are *order isomorphic*. The function f is called *increasing* if $p \leq_X q$ implies that $f(p) \leq_Y f(q)$.

The author wishes to thank the referee and Professor Herden for their suggestions improving the quality of this paper.

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

Let $\langle X, < \rangle$ be an ordered set. An order $\prec$ of X is called an *extension* of $<$ if $< \subseteq \prec$. It is an old theorem of Szpilrajn [S] that every order has a linear extension.

An ordered space $\langle X, <, \mathcal{T} \rangle$ is a *pliable space* if for every extension $\prec$ of $<$ there is a continuous, strictly $\prec$-increasing function $f : X \to \mathbb{R}$. In light of Szpilrajn's Theorem, we need only consider linear extensions of the order of X. If X is just an ordered set with no topology and every linear extension of X is order isomorphic to a subset of $\mathbb{R}$, then we say that X is a *pliable set*. When is an ordered set pliable? When is an ordered topological space pliable? These are the questions that we will answer in Section II.

Wellman's original question involved the reordering not of the whole space, but of a subset. If $Y \subseteq X$ then we will use $<_Y$ to denote the restriction of $<$ to Y. If $\prec_Y$ is a linear extension of $<_Y$ on Y, then $\prec$ will denote the order on X that is obtained by taking the transitive closure of $< \cup \prec_Y$. The transitive closure of a relation R will be denoted $TC[R]$. A subset Y of an ordered space $\langle X, < \rangle$ is called a *pliable subset* of X if for every extension $\prec_Y$ of $<_Y$ there is a continuous strictly $\prec$-increasing function $f : X \to \mathbb{R}$. Again, we need consider only linear extensions of $<_Y$. When is a subset Y of an ordered space X a pliable subset of X? This question will be discussed in Sections IV and V.

The remainder of this introduction is a review of terms and theorems that we will use in exploring the questions stated above. In Section III, we will extend our results for ordered spaces to preordered spaces.

Notions of separability in ordered sets will play an important role in our discussions. The most basic that we will use is the concept of order separability that can be found in Birkhoff's book [B]. An ordered set X is *order separable* if there is countable subset C of X such that for every $p, q \in X$ with $p < q$ there is $r \in C$ such that $p \leq r \leq q$. This is all we need to know about a linearly ordered space to be sure that it looks like a subset of the real line.

Theorem 1 (Birkhoff, 1948). *A linearly ordered set is order isomorphic to a subset of $\mathbb{R}$ if and only if it is order separable.*

We want to know what sorts of ordered spaces not only have continuous strictly increasing functions into $\mathbb{R}$, but have the much stronger property that every time we linearly reorder them in a way that is consistent with the existing order, we can still get such a function.

For which ordered topological spaces, X, is there a continuous, strictly increasing function $f : X \to \mathbb{R}$? We need some definitions and notation before giving the answers to this question. Let X be an ordered set. For every $p \in X$ we will set $d(p) = \{q \in X : q \leq p\}$ and $i(p) = \{q \in X : p \leq q\}$. We will use this older version of the more modern notation $\downarrow(p)$ and $\uparrow(p)$ to better match the notation $D(A)$ and $I(A)$ given below. Also, $d'(p) = d(p) \setminus \{p\} = \{q \in X : q < p\}$ and $i'(p) = i(p) \setminus \{p\} = \{q \in X : p < q\}$. In order to differentiate between these types of sets for different orders, we will sometimes write the order in current use as a subscript, such as $d_\prec(p)$ or $d_<(p)$. So, if Y is a subset of an ordered set X and $\prec_Y$ is an extension of $<_Y$, then $d_\prec(q) = \{p \in X : p \prec q\}$. If no subscript appears, the operation always refers to the original order of the ordered set. Similarly if A is a subset of X then $d(A) = \cup_{p \in A} d(p)$ and $i(A) = \cup_{p \in A} i(p)$. A is called *decreasing* (*increasing*) if $d(A) = A$ ($i(A) = A$). Note that if Y is a subset of X, $A \subseteq X$, and $\prec_Y$ is an extension of $<_Y$, then $d_{\prec_Y}(A) = A \cup \{p \in Y : \exists q \in A \cap Y (p \preceq_Y q)\}$.

If $A \subseteq B \subseteq X$ then A is *cofinal* in B if $B \subseteq d(A)$, and A is *coinitial* in B if $B \subseteq i(A)$.

If $d(p)$ and $i(p)$ are closed for all elements p of an ordered space X then the order of X is said to be *continuous*. The notation $D(A)$ will represent the smallest closed decreasing subset of X that contains A, and $I(A)$ will represent the smallest closed increasing subset of X that contains A. X is called *weakly continuous* if $D(p) \cap I(q) = \emptyset$ for every $p, q \in X$ with $p < q$.

A pair $\langle A, B \rangle$ of subsets of X is called a *corner* of X if A is decreasing, B is increasing, and $A \cap B = \emptyset$. The pair $\langle A, B \rangle$ is said to be closed (open) if both A and B are closed (open). We will say that a pair $\langle C, D \rangle$ (not necessarily a corner) can be expanded to a pair $\langle E, F \rangle$ if $C \subseteq E$ and $D \subseteq F$. In that case we say that $\langle E, F \rangle$ expands $\langle C, D \rangle$.

The following definition is due to Nachbin [N]. An ordered topological space X is called *normally ordered* if every closed corner of X can be expanded to an open corner of X. Note that a continuous linearly ordered space is normally ordered.

Theorem 2 (Nachbin, 1965). *An ordered space X is normally ordered if and only if for every closed corner $\langle A, B \rangle$ of X, there is a continuous increasing function $f : X \to [0, 1]$ such that $A \subseteq f^{-1}(0)$ and $B \subseteq f^{-1}(1)$.*

This theorem is at the same time too strong and too weak. It is too strong because we don't need to separate every pair of sets that can make up a closed corner. It is too weak because the function is only increasing and not necessarily strictly increasing. Let us deal with the weakness first. In [H1] Herden defines an ordered space X to be *Nachbin separable* if there is a countable collection $\mathcal{C}$ of closed corners of X such that for every $p, q \in X$ with $p < q$ there is $\langle A, B \rangle \in \mathcal{C}$ such that $p \in A$ and $q \in B$. He then obtains the following theorem.

Theorem 3 (Herden, 1989). *If a normally ordered space X is Nachbin separable then there is a continuous strictly increasing function $f : X \to \mathbb{R}$.*

One way to get the countable collection of closed corners needed in the preceding theorem is to use a separability condition in an ordered space that is continuous. But order separability is insufficient for our purpose. We need to separate with more than a single point. An ordered space X is *weakly order separable* if there is a countable subset C of X such that for every $p, q \in X$ with $p < q$ there are $r, s \in C$ such that $p \leq r < s \leq q$. We will discuss the relations between various separability conditions in Section IV. For the moment, we will only observe the seemingly contradictory fact that weak order separability implies order separability.

Theorem 4. *If a weakly order separable, normally ordered space X is weakly continuous, then there is a continuous, strictly increasing function $f : X \to \mathbb{R}$.*

In dealing with the strength of Theorem 1, Herden uses a different approach, constructing a structure similar to that used in the proof of Urysohn's Lemma. For details of the following sketch, see [H1] and [H2]. A set $\mathcal{S}$ of open decreasing subsets of an ordered space X is said to be a *separable system* on X if it satisfies the following properties.

(1) $\exists A, B \in \mathcal{S} (\overline{A} \subseteq B)$.
(2) $\forall A, B \in \mathcal{S} \left(\overline{A} \subseteq B \Rightarrow \exists C \in \mathcal{S} (\overline{A} \subseteq C \wedge \overline{C} \subseteq B)\right)$.

Let $\mathbf{S} = \{\mathcal{S}_\alpha : \alpha \in A\}$ be a collection of separable systems on an ordered space X and let $Y \subseteq X$. We will say that $\mathbf{S}$ *splits the elements of* Y if for every $p, q \in Y$ with $p < q$ there is $\alpha \in A$ such that $p \in E$ and $q \notin E$ for every $E \in \mathcal{S}_\alpha$. If $Y = X$ then $\mathbf{S}$ is called a splitting collection of separable systems.

Theorem 5 (Herden, 1989). *Let X be an ordered space. There is a continuous, strictly increasing function $f : X \to \mathbb{R}$ if and only if there is a countable splitting collection of separable systems on X.*

II. Pliable Spaces

In attacking the problem of determining which ordered spaces are pliable, we shall see that the problem breaks nicely into two parts, one topological and the other order-theoretic. The topological part is easily taken care of by a property which greatly strengthens the concept of a continuous order. The order-theoretic part is more interesting, with connections to Suslin's Problem. We will look at the topological part first.

When reorderings give rise to continuous, strictly increasing functions, these functions can be used to show that certain kinds of subsets must be closed. The set of points in X which are less than or equal to a point p in some linear extension $\prec$ must be closed, since it will be the inverse image of the closed set $(-\infty, f(p)]$ under a continuous function f. The next two lemmas will assist us in constructing some desired extensions.

Lemma 6. *Let X be an ordered set and $Y \subseteq X$. If $\prec_Y$ extends $<_Y$ then $\prec$ is an order and for every $p, q \in X$, $p \prec q$ if and only if $p < q$ or there are $r, s \in Y$ such that $p \leq r \prec_Y s \leq q$.*

Proof. We prove the second part first. Clearly, if $p < q$ or if there are $r, s \in Y$ such that $p \leq r \prec_Y s \leq q$, then $p \prec q$. Assume that $p \prec q$ and that $p \not< q$. There is $n \in \omega$ and $\sigma \in {}^{n+2}X$ such that $\sigma(0) = p$, $\sigma(n+1) = q$, and, for every $m \in n+1$, either $\sigma(m) < \sigma(m+1)$ or $\sigma(m) \prec_Y \sigma(m+1)$. We will induct on the length of σ to obtain our result.

If $n = 0$ then $p, q \in Y$ and $p \prec_Y q$. Let $n \in \omega$ and assume that if $\sigma \in {}^{n+2}X$ such that for every $m \in n+1$ either $\sigma(m) < \sigma(m+1)$ or $\sigma(m) \prec_Y \sigma(m+1)$, and if $\sigma(0) \not< \sigma(n+1)$, then there are $r, s \in Y$ such that $\sigma(0) \leq r \prec_Y s \leq \sigma(n)$. Let $\sigma \in {}^{n+3}X$ such that for every $m \in n+2$ either $\sigma(m) < \sigma(m+1)$ or $\sigma(m) \prec_Y \sigma(m+1)$. Also assume that $\sigma(0) \not< \sigma(n+2)$. If $\sigma(0) < \sigma(n+1)$ then $\sigma(n+1) \not< \sigma(n+2)$. Therefore $\sigma(n+1) \prec_Y \sigma(n+2)$ and we can set $r = \sigma(n+1)$ and $s = \sigma(n+2)$. If $\sigma(0) \not< \sigma(n+1)$ then there are $r, t \in Y$ such that $\sigma(0) \leq r \prec_Y t \leq \sigma(n+1)$. If $\sigma(n+1) < \sigma(n+2)$ then we can set $s = t$. Otherwise, $t \prec_Y \sigma(n+1)$, so we can set $s = \sigma(n+1)$.

To show that $\prec$ is an order, we need only show that for no $p \in X$ is $p \prec p$. But if $p \prec p$ then there are $r, s \in Y$ such that $p \leq r \prec_Y s \leq p$. It follows that $s \leq r$, contradicting the fact that $\prec_Y$ is an extension of $<_Y$.

Lemma 7. *Let X be an ordered set. If A is a decreasing subset of X then there is an extension $\prec$ of $<$ such that if $p \in A$ and $q \in X \setminus A$ then $p \prec q$. If A is an increasing subset of X then there is an extension $\prec$ of $<$ such that if $p \in X \setminus A$ and $q \in A$ then $p \prec q$.*

Proof. Let A be a decreasing subset of X and let R be the set of all $\langle p, q \rangle$ such that $p \in A$ and $q \in X \setminus A$. Set $\prec = TC[< \cup R]$. We need only show that there is no $p \in X$ such that

$p \prec p$. Let $n \in \omega$ and $\sigma \in {}^{n+2}X$ such that for every $m \in n+1$ either $\sigma(m) < \sigma(m+1)$ or $\sigma(m) \in A$ and $\sigma(m+1) \notin A$. If $\sigma(0) \not< \sigma(n+1)$, there must be $m \in n+1$ such that $\sigma(m) \in A$ and $\sigma(m+1) \notin A$. Let j be the least such m, and k the greatest. Then $\sigma(0) \leq \sigma(j)$, so $\sigma(0) \in A$ because $\sigma(j) \in A$ and A is decreasing. Also, $\sigma(k+1) \leq \sigma(n)$, so $\sigma(n) \notin A$ because $\sigma(k+1) \notin A$ and $X \setminus A$ is increasing. Thus $\sigma(0) \neq \sigma(n+1)$.

The case when A is increasing is done in a similar fashion. □

The following property is what is needed to take care of the topological part of our problem. An ordered space X is called *extremely continuous* if every decreasing (increasing) subset of X that has a maximal (minimal) element is closed.

Lemma 8. *Let X be a extremely continuous ordered space. Then X is normally ordered and every extension of $<$ on X is continuous.*

Proof. Let $\prec$ be an extension of $<$. Let $p \in X$. Then $d_\prec(p)$ is $<$-decreasing and p is $<$-maximal in $d_\prec(p)$. Also, $i_\prec(p)$ is $<$-increasing and p is $<$-minimal in $i_\prec(p)$. Thus both sets are closed, and $\prec$ is continuous.

Now let $\langle A, B\rangle$ be a closed corner in X. If $X = A \cup B$, then A and B are also open, so assume that there is $p \in X \setminus (A \cup B)$.

Set $E = X \setminus (A \cup d'(p))$ and $F = X \setminus (B \cup i'(p))$. Then p is a minimal element of E and a maximal element of F. Therefore $d(F)$ and $i(E)$ are closed. Then $U = X \setminus i(E)$ is an open decreasing set and $V = X \setminus d(F)$ is an open increasing set. Now $A \cap i(E) = \emptyset$ because A is decreasing, and $B \cap d(F) = \emptyset$ because B is increasing. Therefore $A \subseteq U$ and $B \subseteq V$. Finally, $U \cap V = X \setminus [i(E) \cup d(F)]$. But $E \cup F \subseteq i(E) \cup d(F)$ and

$$
\begin{aligned}
E \cup F &= [X \setminus (A \cup d'(p))] \cup [X \setminus (B \cup i'(p))] \\
&= X \setminus [(A \cap B) \cup (A \cap i'(p)) \cup (d'(p) \cap B) \cup (d'(p) \cup i'(p))] \\
&= X.
\end{aligned}
$$

Thus $U \cap V = \emptyset$, and X is normally ordered.

Theorem 9. *An ordered space $\langle X, <, \mathcal{T}\rangle$ is pliable if and only if it is extremely continuous and $\langle X, <\rangle$ is a pliable set.*

Proof. If $\langle X, <, \mathcal{T}\rangle$ is a pliable space then $\langle X, <\rangle$ is a pliable set. Let A be a decreasing subset of X with a maximal element q. We can find a extension $\prec$ of $<$ such that $q \prec r$ for all $r \in X \setminus A$ and $p \preceq q$ for all $p \in A$. Since X is a pliable space, there is a continuous, strictly $\prec$-increasing function $f : X \to \mathbb{R}$. But then $A = f^{-1}[(-\infty, f(q)]]$, so A is closed. The dual case, when A is increasing and has a minimal element, is proved in a similar fashion. Thus X is extremely continuous.

Now assume that $\langle X, <\rangle$ is a pliable set and that $<$ is extremely continuous. Let $\prec$ be a linear extension of $<$. By Theorem 1 and Lemma 8, $\langle X, \prec, \mathcal{T}\rangle$ is an order separable, continuous ordered space, and there is a continuous, strictly $\prec$- increasing function $f : X \to \mathbb{R}$ by Theorem 4. □

So the problem now becomes: when is an ordered set pliable? We have seen that in the case of a linearly ordered set, what is needed is order separability. That this is not sufficient for sets that are not linearly ordered is shown by the following example. In this and other examples $\mathbb{Q}$ is the set of rational numbers, and $\mathbb{P}$ is the set of irrationals.

Example. Let $X = \mathbb{Q} \cup (\mathbb{P} \times 2)$ where $2 = \{0, 1\}$, and let $<_0$ be the usual order on $\mathbb{R}$. Define $<$ on X by setting $x < y$ if and only if one of the following properties holds.

(1) $x, y \in \mathbb{Q}$ and $x <_0 y$.
(2) $x \in \mathbb{Q}$, $y = \langle z, j\rangle \in \mathbb{P} \times 2$, and $x <_0 z$.
(3) $x = \langle z, j\rangle \in \mathbb{P} \times 2$, $y \in \mathbb{Q}$, and $z <_0 y$.
(4) $x = \langle w, i\rangle \in \mathbb{P} \times 2$, $y = \langle z, j\rangle \in \mathbb{P} \times 2$, and $w <_0 z$.

Then X is a weakly order separable set, but is not pliable.

Proof. The set $\mathbb{Q}$ is weakly order dense in X. Let $\prec$ be any linear extension of $<$. The sets $\{\langle z, 0\rangle, \langle z, 1\rangle\}$ for $z \in \mathbb{P}$ form an uncountable collection of pairwise disjoint intervals, so there can be no strictly $\prec$-increasing function from $\langle X, \prec\rangle$ into $\mathbb{R}$.

It is to avoid these large collections of pairwise disjoint intervals that we introduce the notion of a collection of nonoverlapping sets, similar to the notion of cellularity. A relation R will be called *order-consistent* if $TC[R]$ is an order. A collection $\mathcal{A}$ of subsets of an ordered set X is called a *collection of nonoverlapping subsets* of X if and only if it satisfies the following conditions.

(1) $\mathcal{A}$ is a collection of pairwise disjoint sets, each having at least two elements.
(2) The relation $\left\{\langle A, B\rangle \in \mathcal{A}^2 : A \neq B \text{ and } \exists p \in A \exists q \in B (p < q)\right\}$ is order- consistent.

If we wish, we may assume that the elements of the collection $\mathcal{A}$ each have exactly two elements. The previous example shows how these collections of nonoverlapping sets can be used to create collections of pairwise disjoint intervals. If every linear extension of the existing order is to be order isomorphic to a subset of $\mathbb{R}$, we must keep these collections countable. We will do that by means of a cardinal function. For any ordered set X, $\nu(X)$ is the supremum of the cardinalities of all nonoverlapping collections of subsets of X. When the order of X is to be emphasized, it can included as in $\nu(X, <)$. Note that in a linearly ordered set, saying that $\nu(X) \leq \omega$ is equivalent to saying that every collection of pairwise disjoint intervals is at most countable.

Lemma 10. *Let $\langle X, <\rangle$ be an ordered set and let $\prec$ be a linear extension of $<$. If $\nu(\langle X, <\rangle) \leq \omega$ then $\nu(\langle X, \prec\rangle) \leq \omega$.*

Proof. If $\mathcal{A}$ is a collection of nonoverlapping subsets of X under $\prec$, then it is also such a collection under $<$.

Lemma 11. *If an ordered set X is pliable, then $\nu(X) \leq \omega$.*

Proof. Denote the order of X by $<$. Assume that $\nu(X) > \omega$. Let $\mathcal{A}$ be an uncountable collection of nonoverlapping subsets of X. We may assume that $|A| = 2$ for every $A \in \mathcal{A}$. The transitive closure $<_1$ of the relation $\left\{\langle A, B\rangle \in \mathcal{A}^2 : A \neq B \wedge \exists p \in A \exists q \in B(p < q)\right\}$ is an order. Now let $<_2$ be a linear ordering of $\mathcal{A}$ that extends $<_1$. Define $<_3$ on $Y = \cup\mathcal{A}$ by $p <_3 q$ if and only if $p < q$ or there are $A, B \in \mathcal{A}$ such that $p \in A$, $q \in B$, and $A <_2 B$. It is easy to show that $<_3$ is an order that extends $<_Y$. Then $<_4 = TC[< \cup <_3]$ is an order on X by Lemma 6. But if $\prec$ is a linear extension of $<_4$, then $\{[p, q] : \exists A \in \mathcal{A}(p, q \in A) \text{ and } p \prec q\}$ is an uncountable collection of pairwise disjoint intervals in $\langle X, \prec\rangle$, so X is not pliable.

If Suslin's Hypothesis (SH) is assumed and $\nu(X) \leq \omega$, then, since linear extensions on X will inherit this property, X will be a pliable set. Thus we have the following theorem.

Theorem 12. *SH* $\Leftrightarrow$ *"an ordered set* X *is pliable if and only if* $\nu(X) \leq \omega$*."*

Equivalence is obtained by noting that a Suslin line S would be an ordered set with $\nu(S) \leq \omega$, but which is not pliable.

Corollary 13. *SH* $\Leftrightarrow$ *"an ordered space* X *is pliable if and only if* X *is extremely continuous and* $\nu(X) \leq \omega$*.*

Of course, in linearly ordered sets, the way to eliminate the need for a special axiom is to inject order separability into the situation. The same is true in nonlinear ordered sets, as the next theorem shows.

Theorem 14. *If an ordered set* X *is order separable and* $\nu(X) \leq \omega$*, then* X *is pliable.*

Proof. Let X be an order separable ordered set with $\nu(X) \leq \omega$. Let $\prec$ be a linear extension of $<$ on X. Then $\nu(X, \prec) \leq \omega$ and $\langle X, \prec \rangle$ can have at most countably many disjoint intervals containing at least two elements. For the sake of brevity, let us call a subset of X special if it is convex in $\prec$, has at least two elements, and is an antichain in $<$. Let $\mathcal{A}$ be a maximal collection of pairwise disjoint special subsets of X. Each element of $\mathcal{A}$ is countable, because it is an antichain in $\langle X, < \rangle$. $\mathcal{A}$ itself must also be countable because $\nu(X, \prec) \leq \omega$. Thus $A = \cup \mathcal{A}$ is countable. Let B be an order-dense subset of $\langle X, < \rangle$ and set $C = A \cup B$. Let $p, q \in X$ such that $p \prec q$. If $p < q$ then there is $r \in B$ such that $p \leq r \leq q$, or $p \preceq r \preceq q$. Assume that $p \not< q$. If $[p, q]$ is not an antichain in $<$, then there are $r, s \in X$ such that $p \preceq r < s \preceq q$. Since B is order dense in $\langle X, < \rangle$, there is $t \in B$ such that $r \leq t \leq s$. Then $p \preceq t \preceq q$. If $[p, q]$ is an antichain in $<$, then there is $r \in [p, q]$ such that $r \in A$, otherwise $\mathcal{A}$ is not maximal. Therefore C is order dense in $\langle X, \prec \rangle$.

Lemma 15. *If every linear extension of an ordered set* X *is order separable, then* X *is order separable.*

Proof. Assume that X is not order separable. Let $\beta \in \omega_1$ and assume that for every $\alpha \in \beta$, Y_α is a countable subset of X. Since X is not order separable, there are $p_\beta, q_\beta \in X$ such that $p_\beta < q_\beta$ and $[p_\beta, q_\beta] \cap [\cup_{\alpha \in \beta} Y_\alpha] = \emptyset$. Set $Y_\beta = \{p_\beta, q_\beta\}$ and set $Y = \cup_{\alpha \in \omega_1} Y_\alpha$.

Now define a linear extension of $<_Y$. For every $\beta \in \omega_1$ let $Z_\beta = \cup_{\alpha < \beta} Y_\alpha$ and set $\prec_0 = \emptyset$. Let $\gamma \in \omega_1$ with $\gamma > 0$ and assume that $\{\prec_\beta : \beta \in \gamma\}$ satisfies the following properties

(1) $\forall \beta \in \gamma$, $\prec_\beta$ is a linear extension of $<_{Z_\beta}$.
(2) $\forall \alpha, \beta \in \gamma (\alpha \in \beta \Rightarrow \prec_\alpha \subseteq \prec_\beta)$

If γ is a limit ordinal, then set $\prec_\gamma = \cup_{\alpha \in \gamma} \prec_\alpha$. Assume that $\gamma = \delta + 1$ for some $\delta \in \omega_1$. Define $\prec_\gamma$ on Z_γ by $r \prec_\gamma s$ if and only if one of the following conditions is satisfied.

(1) $r, s \in Z_\delta$ and $r \prec_\delta s$.
(2) $r, s \in Y_\delta$ and $r < s$.
(3) $r \in Y_\delta$, $s \in Z_\delta$, and $\exists t \in Z_\gamma (p_\delta < t \wedge t \preceq_\delta s)$.
(4) $r \in Z_\delta$, $s \in Y_\delta$, and $\forall t \in Z_\delta (p_\delta < t \Rightarrow r \prec_\delta t)$.

Then $\prec_\gamma$ is a linear extension of $<_{Z_\gamma}$ and it extends $\prec_\alpha$ for all $\alpha \in \gamma$.

Now set $\prec = \cup_{\alpha \in \omega_1} \prec_\alpha$. This is a linear extension of $<_Y$. If C is a countable subset of Y then there is $\beta \in \omega_1$ such that $C \subseteq Z_\beta$. But there is no $r \in Z_\beta$ such that $p_\beta \preceq_{\beta+1} r \preceq_{\beta+1} q_\beta$, so C cannot be order dense in Y. Now $\prec$ can be extended to a linear ordering of X that is also an extension of $<$. The now linearly ordered set X then has a subset, Y, which is not order separable.

Theorem 16. *An ordered set X is pliable if and only if X is order separable and $\nu(X) \leq \omega$.*

Proof. If X is pliable, then $\nu(X) \leq \omega$ and X is order separable by Lemma 15. The other direction is given by Theorem 14.

Corollary 17. *An ordered space X is a pliable space if and only if it is extremely continuous and order separable, and $\nu(X) \leq \omega$.*

What do pliable sets look like? First note that it is a direct consequence of having $\nu(X) \leq \omega$ that every antichain in a pliable set is countable. We can extend this slightly as the following proposition.

Proposition 18. *If X is a pliable set and $A \subseteq X$ then there are countable subsets, B and C of A such that $A \subseteq d(B)$ and $A \subseteq i(C)$.*

Proof. Let $A \subseteq X$ and assume that if $B \subseteq A$ and $A \subseteq d(B)$ then B is uncountable. Let $\beta \in \omega_1$ and assume that $\{p_\alpha : \alpha \in \beta\} \subseteq A$. Since $\{p_\alpha : \alpha \in \beta\}$ is countable, there is $p_\beta \in A \setminus d\left(\{p_\alpha : \alpha \in \beta\}\right)$. Now the set of all pairs $\{p_\alpha, p_{\alpha+1}\}$, where α is a limit ordinal in ω_1, is an uncountable collection of nonoverlapping subsets of X, contradicting the hypothesis that X is pliable. The dual part is proved the same way.

Another easy observation that will be useful later is that in pliable sets, as in linear sets, order separability and weak order separability are equivalent.

One possible approach considered by the author to show that pliable sets must be order separable was to write the set as a union of chains, each of which must be order separable by pliability, and use $\nu(X)$ to extend the separability to the entire set. To do this, one needs a countable number of chains, so the following questions arose, which are still unanswered.

Question 1. If X is pliable, is it the union of a countable number of chains?

Question 2. If $\nu(X) \leq \omega$, is X the union of a countable number of chains?

If the answer to this question is yes, then the requirement in Theorems 14 and 16 that X be order separable can be replaced by a requirement that every chain in X be order separable.

Question 3. If X is order separable and every antichain in X is countable, is X the union of a countable number of chains?

For related results see [A], [DM], [K 1-8], [MWL], [P], and [Pr].

If we include a topology with the order, we get an even more rigid structure. The next lemma shows that a limit point p of X must be a limit point of $d(p)$ or of $i(p)$.

Lemma 19. *If X is a pliable space and $p \in X$ then p is not in the closure of $X \setminus (d(p) \cup i(p))$.*

Proof. Let X be a pliable space and $p \in X$. Set $B = X \setminus (d(p) \cup i(p))$. Let A be a maximal antichain of B and let $q \in A$. We will show that p is not a limit point of either

$B \cap d(A)$ or $B \cap i(A)$. First define an extension $\prec$ of $<$ such that $p \prec q$ and $q \prec r$ for all $r \in A \setminus \{q\}$. Since X is pliable, there is a continuous, strictly $\prec$-increasing function $f : X \to \mathbb{R}$. Then $f^{-1}[(-\infty, f(q))]$ is a neighborhood of p that misses $i(A)$, so p is not a limit point of $B \cap i(A)$. A similar proof shows that p is not a limit point of $B \cap d(A)$, so p is not a limit point of B. $\square$

We next see that the set of limit points in a pliable space must form a chain.

Proposition 20. *If X is a pliable space, p is a limit point of X, and $q \in X$, then p and q are comparable.*

Proof. Let X be pliable space and let $p, q \in X$ such that p and q are incomparable. We will show that p is not a limit point of either $d(p)$ or $i(p)$. Then by Lemma 19, p cannot be a limit point at all. By Lemma 7 we can define an extension $\prec_1$ of $<$ so that $q \prec_1 p$ and $r \prec_1 q$ for all $r \in d'(p)$, and we can define an extension $\prec_2$ of $<$ so that $p \prec_2 q$ and $q \prec_2 r$ for all $r \in i'(p)$. The functions arising from the pliability of X will then show that p is not a limit point of $d(p)$ or $i(p)$. $\square$

So the limit points of X will form a backbone for the space, with an at most countable number of bulges consisting of isolated points.

III. Pliable Preordered Spaces

In the applications of the social sciences, a preorder rather than an order is used. A *preorder* is a relation that is reflexive and transitive. If $\lesssim$ is a preorder on a set X and $p, q \in X$, then $p < q$ means that $p \lesssim q$ and $q \not\lesssim p$. If $p \lesssim q$ and $q \lesssim p$ then we say that $p \sim q$. The definitions given in the introduction can be restated in terms of a preordered, rather than an ordered, set or space by replacing $\leq$ by $\lesssim$. We do want to require that a strictly increasing function on a preordered set also be increasing. The definition of a extremely continuous ordered space can similarly be extended to one for a extremely continuous preordered space.

It is easy to see that if X is a preordered set, $f : X \to \mathbb{R}$ is increasing, and $p, q \in X$ such that $p \sim q$, then $f(p) = f(q)$. This collapse of a set onto a single point allows us to use ordered sets to obtain corresponding theorems for preordered sets. This principle, used in this section to obtain a characterization of pliable preordered sets and spaces, can also be used to extend the results of sections IV and V to preordered sets. We will first redefine a collection of nonoverlapping sets for preordered sets, as it is slightly different than the corresponding definition in ordered sets. Let X be a preordered space and let $\mathcal{A}$ be a collection of subsets of X. $\mathcal{A}$ is a collection of nonoverlapping subsets of X if and only if the following conditions are satisfied.

(1) $\mathcal{A}$ is a collection of pairwise disjoint subsets of X, each of which has at least two elements.
(2) The relation $\left\{\langle A, B\rangle \in \mathcal{A}^2 : A \neq B \wedge \exists p \in A \exists q \in B (p < q)\right\}$ is order consistent.
(3) $\forall A, B \in \mathcal{A} (p \in A, q \in B, p \neq q \Rightarrow p \nsim q)$.

The last condition prohibits sets containing elements which, as far as the preorder and strictly increasing functions are concerned, are really the same.

To construct an ordered set from a preordered set, we need only follow the example set by a strictly increasing function and collapse the equivalence classes under the relation $\sim$ to a single point. Let $\langle X, \lesssim, \tau\rangle$ be a preordered topological space and, for every $p \in X$, let $\hat{p} = \{q \in X : p \sim q\}$. The *quotient* $\widehat{X}$ *of* X *generated by* $\sim$ is the ordered space $\langle \widehat{X}, <, \sigma\rangle$ where

(1) $\widehat{X} = \{\hat{p} : p \in X\}$,
(2) $< = \left\{\langle \hat{p}, \hat{q}\rangle \in \widehat{X}^2 : p < q\right\}$, and
(3) σ is the quotient topology on $\widehat{X}$.

Lemma 21. *Let X be a preordered space and let $\widehat{X}$ be the quotient of X.*

(1) *X is extremely continuous if and only if $\widehat{X}$ is extremely continuous.*
(2) *X is order separable if and only if $\widehat{X}$ is order separable.*
(3) *$\nu(X) \leq \omega$ if and only if $\nu(\widehat{X}) \leq \omega$.*
(4) *X is pliable if and only if $\widehat{X}$ is pliable.*

Proof.

Part 1. Assume that X is extremely continuous. Let $\hat{A}$ be an increasing subset of $\widehat{X}$ having a minimal element $\hat{p}$. Let $A = \left\{q \in X : \hat{q} \in \hat{A}\right\}$. If $q \in A$ and $r \in X$ such that $q \lesssim r$, then $\hat{q} \leq \hat{r}$, so $\hat{r} \in \hat{A}$. Then $r \in A$. Thus A is increasing. If $q \in A$ then $\hat{q} \in \hat{A}$, so $\hat{q} \not< \hat{p}$. Thus $q \not< p$ and, since $p \in A$, p is a minimal element of A. So A is closed because X is strongly continuous. Since $\widehat{X}$ has the quotient topology, $\hat{A}$ must be closed in $\widehat{X}$. The proof of the case when $\hat{A}$ is a decreasing subset of $\widehat{X}$ having a maximal element is similar.

Assume that $\widehat{X}$ is extremely continuous. Let A be an increasing subset of X having a minimal element p. Let $\hat{A} = \{\hat{q} : q \in A\}$. If $\hat{q} \in \hat{A}$ and $\hat{r} \in \widehat{X}$ such that $\hat{q} \leq \hat{r}$, then $q \lesssim r$, so $r \in A$. Then $\hat{r} \in \hat{A}$ and $\hat{A}$ is increasing. If $\hat{q} \in \hat{A}$ then $q \in A$, so $q \not< p$. Thus $\hat{q} \not< \hat{p}$ and $\hat{p}$ is a minimal element of $\hat{A}$. So $\hat{A}$ is closed because $\widehat{X}$ is extremely continuous. Since $\widehat{X}$ has the quotient topology, A must be a closed subset of X. The case when A is a decreasing subset of X with a maximal element is done the same way.

Part 2. Assume that X is order separable and let C be a countable subset of X that is order dense in X. Let $\widehat{C} = \{\hat{p} : p \in C\}$. Clearly $\widehat{C}$ is countable. Let $\hat{p}, \hat{q} \in \widehat{X}$ with $\hat{p} < \hat{q}$. Then $p < q$ in X so there is $r \in C$ such that $p \lesssim r \lesssim q$. But then $\hat{r} \in \widehat{C}$ and $\hat{p} \leq \hat{r} \leq \hat{q}$. Therefore $\widehat{C}$ is order dense in X.

Assume that $\widehat{X}$ is order separable and let $\widehat{C}$ be a countable order dense subset of $\widehat{X}$. Let g be a choice function on $\widehat{C}$ and set $C = \left\{g(\hat{p}) : \hat{p} \in \widehat{C}\right\}$. Then C is a countable subset of X. Let $p, q \in X$ with $p < q$. Then $\hat{p} < \hat{q}$, so there is $\hat{r} \in \widehat{C}$ such that $\hat{p} \leq \hat{r} \leq \hat{q}$. Then $p \lesssim g(r) \lesssim q$. Therefore C is order dense in X.

Part 3. Assume that $\nu(X) \leq \omega$. Let $\hat{\mathcal{A}}$ be a collection of nonoverlapping sets in $\widehat{X}$. Let g be a choice function on $\cup\hat{\mathcal{A}}$. For every $\hat{A} \in \hat{\mathcal{A}}$ set $A = \left\{g(\hat{p}) : \hat{p} \in \hat{A}\right\}$. Set $\mathcal{A} = \{A : \hat{A} \in \hat{\mathcal{A}}\}$. We will show that $\mathcal{A}$ is a collection of nonoverlapping subsets of X.

Let $p, q \in \cup\mathcal{A}$ with $p \neq q$. There are $\hat{r}, \hat{s} \in \cup\hat{\mathcal{A}}$ such that $p = g(\hat{r})$ and $q = g(\hat{s})$. Since $p \neq q$, we have that $\hat{r} \neq \hat{s}$. Then $p \nsim q$.

Let $A, B \in \mathcal{A}$ such that $A \cap B \neq \emptyset$. Let $p \in A \cap B$ and let $\hat{q} \in \hat{A}$ and $\hat{r} \in \widehat{B}$ such that $g(\hat{q}) = p = g(\hat{r})$. This means that $\hat{q} = \hat{r}$, so $\hat{A} \cap \widehat{B} \neq \emptyset$. But $\hat{\mathcal{A}}$ is a collection of pairwise disjoint sets, so $\hat{A} = \widehat{B}$. Therefore $A = B$.

Let $A \in \mathcal{A}$. Since $|\hat{A}| \geq 2$ it must be true that $|A| \geq 2$.

Finally, let $R = \{\langle A, B\rangle \in \mathcal{A}^2 : \exists p \in A \exists q \in B (p < q)\}$. Set $\widehat{R}$ equal to the set of all $\langle \hat{A}, \widehat{B}\rangle \in \hat{\mathcal{A}}^2$ such that there exist $\hat{p} \in \hat{A}$ and $\hat{q} \in \widehat{B}$ with $\hat{p} < \hat{q}$. If $\langle A, B\rangle \in R$ then there are $p \in A$ and $q \in B$ such that $p < q$. Let $\hat{r} \in \hat{A}$ and $\hat{s} \in \widehat{B}$ such that $p = g(\hat{r})$ and $q = g(\hat{s})$. Then $r \sim p < q \sim s$, so $\hat{r} < \hat{s}$. Thus $\langle \hat{A}, \widehat{B}\rangle \in \widehat{R}$. On the other hand, if $\langle \hat{A}, \widehat{B}\rangle \in \widehat{R}$ then there are $\hat{p} \in \hat{A}$ and $\hat{q} \in \widehat{B}$ such that $\hat{p} < \hat{q}$. Then $g(\hat{p}) \in A$, $g(\hat{q}) \in B$, and $g(\hat{p}) \sim p < q \sim g(\hat{q})$. Thus $\langle A, B\rangle \in R$. Therefore $\langle A, B\rangle \in TC[R]$ if and only if $\langle \hat{A}, \widehat{B}\rangle \in TC[\widehat{R}]$. It follows that, since $TC[\widehat{R}]$ is an order, so is $TC[R]$.

Since $\mathcal{A}$ is a collection of nonoverlapping subsets of X and $\nu(X) \leq \omega$, $\mathcal{A}$ must be countable. Clearly $|\hat{\mathcal{A}}| = |\mathcal{A}|$, so $\hat{\mathcal{A}}$ is countable. Thus $\nu(\widehat{X}) \leq \omega$.

Assume that $\nu(\widehat{X}) \leq \omega$. Let $\mathcal{A}$ be a collection of nonoverlapping subsets of X. For every $A \in \mathcal{A}$ let $\hat{A} = \{\hat{p} : p \in A\}$ and set $\hat{\mathcal{A}} = \{\hat{A} : A \in \mathcal{A}\}$. We will show that $\hat{\mathcal{A}}$ is a collection of nonoverlapping subsets of X.

Let $\hat{A}, \widehat{B} \in \hat{\mathcal{A}}$ such that $\hat{A} \cap \widehat{B} \neq \emptyset$. Let $\hat{p} \in \hat{A} \cap \widehat{B}$. There are $q \in A$ and $r \in B$ such that $\hat{p} = \hat{q}$ and $\hat{p} = \hat{r}$. But then $q \sim r$. By Property 3 of the definition of a collection of nonoverlapping subsets of a preordered space, $q = r$. Then, by Property 1 of that definition, $A = B$. Therefore $\hat{A} = \widehat{B}$.

Let $\hat{A} \in \hat{\mathcal{A}}$. By Property 2 of the definition above, there are $p, q \in A$ such that $p \neq q$. Then, by Property 3, $p \nsim q$. Thus $\hat{p}, \hat{q} \in A$ and $\hat{p} \neq \hat{q}$.

Let $\widehat{R} = \left\{\langle \hat{A}, \widehat{B}\rangle \in \hat{\mathcal{A}}^2 : \exists \hat{p} \in \hat{A} \exists \hat{q} \in \widehat{B} (\hat{p} < \hat{q})\right\}$ and set R equal to the set of all $\langle A, B\rangle$ in $\mathcal{A}^2$ such that there exist $p \in A$ and $q \in B$ with $p < q$. If $\langle \hat{A}, \widehat{B}\rangle \in \widehat{R}$ then there are $\hat{p} \in \hat{A}$ and $\hat{q} \in \widehat{B}$ such that $\hat{p} < \hat{q}$. Then there are $r \in A$ and $s \in B$ such that $r \sim p < q \sim s$. Therefore $\langle A, B\rangle \in R$. On the other hand, if $\langle A, B\rangle \in R$ then there are $p \in A$ and $q \in B$ such that $p < q$. Then $\hat{p} \in \hat{A}$, $\hat{q} \in \widehat{B}$, and $\hat{p} < \hat{q}$. Therefore $\langle \hat{A}, \widehat{B}\rangle \in \widehat{R}$. It follows that, since $TC[R]$ is an order, so is $TC[\widehat{R}]$.

Now $\hat{\mathcal{A}}$ is countable because $\nu(\widehat{X}) \leq \omega$. It is not hard to see that $|\mathcal{A}| = |\hat{\mathcal{A}}|$, so $\mathcal{A}$ is countable. Therefore $\nu(X) \leq \omega$.

Part 4. Assume that X is pliable. Let $\prec$ be an extension of $<$ on $\widehat{X}$. Define $\precsim$ on X by $\precsim = \lesssim \cup \{\langle p, q\rangle \in X^2 : \hat{p} \prec \hat{q}\}$. We will show that $\precsim$ is a linear extension of $\lesssim$ on X.

To see that $\precsim$ is a preorder, let $p, q \in X$ such that $p \precsim q$ and $q \precsim r$. If $p \lesssim q$ and $q \lesssim r$ then $p \lesssim r$, so $p \precsim r$. If $p \lesssim q$ and $\hat{q} \prec \hat{r}$ then $\hat{p} \leq \hat{q}$, so $\hat{p} \prec \hat{r}$ and $p \precsim r$. If $\hat{p} \prec \hat{q}$ and $q \lesssim r$ then $\hat{q} \leq \hat{r}$, so $\hat{p} \prec \hat{r}$ and $p \precsim r$. If $\hat{p} \prec \hat{q}$ and $\hat{q} \prec \hat{r}$ then $\hat{p} \prec \hat{r}$, so $p \precsim r$. Thus $\precsim$ is transitive. Clearly $\precsim$ is reflexive, so it is a preorder. Clearly $\precsim$ is a an extension of $\lesssim$.

Since X is pliable, there is a continuous, strictly $\precsim$- increasing function $f : X \to \mathbb{R}$. Define $\hat{f} : \widehat{X} \to \mathbb{R}$ by $\hat{f}(\hat{p}) = f(p)$. Let $\hat{p}, \hat{q} \in \widehat{X}$. If $\hat{p} = \hat{q}$ then $p \sim q$. Since f is increasing, $f(p) = f(q)$. Thus $\hat{f}(\hat{p}) = \hat{f}(\hat{q})$, and $\hat{f}$ is really a function. If $\hat{p} \prec \hat{q}$ then $p \prec q$, so $\hat{f}(\hat{p}) = f(p) < f(q) = \hat{f}(\hat{q})$. Now let U be an open subset of $\mathbb{R}$. Then $f^{-1}[U] = \left\{p \in X : \hat{p} \in \hat{f}^{-1}[U]\right\}$. Since f is continuous, $f^{-1}[U]$ is open, so $\hat{f}^{-1}[U]$ is also open. Therefore $\widehat{X}$ is pliable.

Assume that $\widehat{X}$ is pliable. Let $\precsim$ be an extension of $\lesssim$ on X. Define $\prec$ on $\widehat{X}$ by $\hat{p} \prec \hat{q}$ if and only if $p \prec q$. We have already seen that $\prec$ must be an order. We will show that it extends $<$.

Let $\hat{q}, \hat{q} \in \widehat{X}$ with $\hat{p} < \hat{q}$. Then $p < q$ and, since $\precsim$ extends $\lesssim$, $p \prec q$. Thus $\hat{p} \prec \hat{q}$, and $\prec$ extends $<$.

Since $\widehat{X}$ is pliable, there is a continuous, strictly $\prec$-increasing function $\hat{f} : \widehat{X} :\rightarrow \mathbb{R}$. Define $f : X \rightarrow \mathbb{R}$ by $f(p) = \hat{f}(\hat{p})$. Let $p, q \in X$. If $p = q$ then $\hat{p} = \hat{q}$, so $f(p) = \hat{f}(\hat{p}) = \hat{f}(\hat{q}) = f(q)$. If $p \prec q$ then $\hat{p} \prec \hat{q}$, so $f(p) = \hat{f}(\hat{p}) < \hat{f}(\hat{q}) = f(q)$. Let U be an open subset of $\mathbb{R}$. Then $f^{-1}[U] = \left\{p \in X : \hat{p} \in \hat{f}^{-1}[U]\right\}$. But $\hat{f}^{-1}[U]$ is open in $\widehat{X}$, so $f^{-1}[U]$ is open in X. Thus f is continuous and X is pliable.

Corollary 22. *A preordered space X is pliable if and only if it is strongly continuous and order separable, and $\nu(X) \leq \omega$.*

IV. Pliable Subsets of Ordered Spaces

We now turn our attention to the task of trying to characterize pliable subsets of ordered spaces. In this section, we will make the ordered space as general as possible, and consequently will have results for a very limited class of subsets. In the absence of additional structure such as a normal ordering, we must make use of Herden's separable systems. One can actually use these systems to obtain a characterization of pliability or pliable subsets in terms of the existence of a separable system that distinguishes points under the extended order. Such a characterization does not make it easier to tell when a space or a subset is pliable, and does not reveal much of the underlying structure of the space. We will therefore pursue a goal of finding properties that will give rise to these systems, rather than looking for the systems themselves. It is hoped that the properties for which we search will be easier to discern than the systems to which they give rise. We cannot avoid using separable systems altogether, because there must obviously be a continuous, strictly increasing increasing function from the ordered set X into $\mathbb{R}$. Unless we want to pile extra conditions onto X, which we do not wish to do in this section, we must assume the existence of a separable system on X. We will reshape the sets in this system to transform it into a separable system with respect to the extended order.

We will work with a particular type of separable system that Herden calls a linear separable system. A separable system $\mathcal{S}$ is a *linear separable system* if for every $E, F \in \mathcal{S}$ with $E \neq F$, either $\overline{E} \subseteq F$ or $\overline{F} \subseteq E$. This is actually a slight variation of Herden's definition, which only requires that $E \subseteq F$ or $F \subseteq E$. If we define $E < F$ to mean that $E \neq F$ and $\overline{E} \subset F$, then $\mathcal{S}$ is a linearly ordered set. If $\mathcal{S}$ is countable, then it is order isomorphic to a subset of $\mathbb{Q}$. In fact, we can write $\mathcal{S} = \{E_r : r \in \mathbb{Q}\}$ with the property that if $r, s \in \mathbb{Q}$ with $r < s$ then $\overline{E}_r \subseteq E_s$.

Let $\mathcal{S} = \{E_r : r \in \mathbb{Q}\}$ be a linear separable system on on ordered space X and let $Y \subseteq X$. $\mathcal{S}$ is said to *distinguish* the elements of Y if for every $p, q \in X$ there are $r, s \in \mathbb{Q}$ such that $r < s$, $p \in E_r$, and $q \notin E_s$. If $Y = X$ then $\mathcal{S}$ is called a distinguishing linear separable system. A collection $\mathbf{S} = \{\mathcal{S}_\alpha : \alpha \in A\}$ of linear separable systems, where $\mathcal{S}_\alpha = \{E_{\alpha,r} : r \in \mathbb{Q}\}$, is said to distinguish the elements of Y if for every $p, q \in Y$ with

$p < q$ there are $\alpha \in A$ and $r, s \in \mathbb{Q}$ such that $r < s$, $p \in E_{\alpha,r}$, and $q \notin E_{\alpha,s}$. If $Y = X$ then $\mathbf{S}$ is called a distinguishing collection of linear separable systems.

Lemma 23. *If there is a distinguishing countable collection $\{\mathcal{E}_n : n \in \omega\}$ of countable linear separable systems on an ordered space X then there is a splitting collection $\{\mathcal{F}_n : n \in \omega\}$ of separable systems on X.*

Proof. For every $n \in \omega$, let $\mathcal{E}_n = \{E_{n,r} : r \in \mathbb{Q}\}$. Let P be the set of ordered pairs $\langle r, s\rangle \in \mathbb{Q}^2$ such that $r < s$ and, for every $n \in \omega$ and $\langle r, s\rangle \in P$ set $\mathcal{F}_{n,r,s} = \{E_{n,t} : r < t < s\}$.

We will first show that $\mathcal{F}_{n,r,s}$ is a separable system. Let $t, u \in (r, s) \cap \mathbb{Q}$ such that $t < u$. Then $E_{n,t}, E_{n,u} \in \mathcal{F}_{n,r,s}$ and $\overline{E}_{n,t} \subseteq E_{n,u}$. Now let $E_{n,t}, E_{n,u} \in \mathcal{F}_{n,r,s}$ with $\overline{E}_{n,t} \subseteq E_{n,u}$. If $t < u$ then $\overline{E}_{n,t} \subseteq E_{n,u}$. If $u < t$, then, choosing $u \leq v \leq t$, we have $\overline{E}_{n,t} \subseteq E_{n,u} \subseteq E_{n,v}$ and $\overline{E}_{n,v} \subseteq \overline{E}_{n,t} \subseteq E_{n,u}$. Thus $\mathcal{F}_{n,r,s}$ satisfies both conditions for being a separable system.

Now let $p, q \in X$ with $p < q$. There is $n \in \omega$ and there is $\langle r, s\rangle \in P$ such that $p \in E_{n,r}$ and $q \notin E_{n,s}$. If $E_{n,t} \in \mathcal{F}_{n,r,s}$ then $p \in E_{n,r} \subseteq E_{n,t}$ and $q \notin E_{n,s} \supseteq E_{n,t}$. Thus $\{\mathcal{F}_{n,r,s} : n \in \omega \text{ and } \langle r, s\rangle \in P\}$ is the desired splitting collection of separable systems

Theorem 24. *Let X be an ordered space. A finite subset Y of X is a pliable subset of X if and only if the following properties are satisfied.*

1. *There is a distinguishing countable linear separable system on X.*
2. *For every subsets, A and B, of Y such that $d(A) \cap B = \emptyset$ there is a countable linear separable system $\mathcal{K}_{A,B}$ on X such that for every $K \in \mathcal{K}_{A,B}$, $A \subseteq K$ and $B \cap K = \emptyset$.*

Proof. Since the theorem is trivially true when Y has less than two elements, we may assume that Y has at least two. Let $\prec_Y$ be a linear extension of $<_Y$ and let $Y = \{y_i : i \in n+2\}$ where $y_i \prec_Y y_{i+1}$ for $i \in n+1$. We will first show that Property 2 can be used to generate a countable linear separable system $\mathcal{F}$ on $\langle X, \prec\rangle$ that distinguishes that points of Y.

For every $i \in n+1$ let $\mathcal{K}_i$ be a countable linear separable system on X such that for every $K \in \mathcal{K}_i$, $\{y_j : j \leq i\} \subseteq K$ and $\{y_j : i < j\} \cap K = \emptyset$. We may write each $\mathcal{K}_i$ as $\{K_{i,r} : r \in \mathbb{Q}\}$ where $r < s$ implies that $\overline{K}_{i,r} \subseteq K_{i,s}$.

Let z be an irrational. For every $r \in \mathbb{Q} \cap (z, z+n+1)$ there is a unique $\alpha(r) \in \mathbb{Q} \cap (z, z+1)$ and a unique $\beta(r) \in n+1$ such that $r = \alpha(r) + \beta(r)$. Set $F_r = \bigcap_{j=\beta(r)}^{n} K_{j,r}$. Clearly each F_r is open and is $<$-decreasing in X. Now set $\mathcal{F} = \{F_r : r \in \mathbb{Q} \cap (z, z+n+1)\}$. We next show that each F_r is $\prec$- decreasing. To do this, we will show that for every $i \in n+2$ and $r \in \mathbb{Q} \cap (z, z+n+1)$, $y_i \in F_r$ if and only if $i \leq \beta(r)$.

If $i \leq \beta(r)$ then $y_i \in K$ for every $K \in \mathcal{K}_j$ and $j \geq \beta(r)$. Thus $y_i \in \bigcap_{j=\beta(r)}^{n} K_{j,r} = F_r$. If $y_i \in F_r = F_{\alpha(r)+\beta(r)}$ then $y_i \in K_{\beta(r),r}$. Thus $i \leq \beta(r)$. It follows that F_r is $\prec$-decreasing.

Let $r, s \in \mathbb{Q} \cap (z, z+n+1)$ with $r < s$. We will show that $\overline{F}_r \subseteq F_s$. As above, we can write $r = \alpha(r) + \beta(r)$ and $s = \alpha(s) + \beta(s)$. If $\beta(s) < \beta(r)$ then $s = \alpha(s) + \beta(s) < z + 1 + \beta(s) < \alpha(r) + \beta(r) = r$. So $\beta(r) \leq \beta(s)$. Now $\overline{F}_r = \overline{\bigcap_{j=\beta(r)}^{n} K_{j,r}} \subseteq \bigcap_{j=\beta(r)}^{n} \overline{K}_{j,r} \subseteq \bigcap_{j=\beta(s)}^{n} \overline{K}_{j,r} \subseteq \bigcap_{j=\beta(s)}^{n} K_{j,s} = F_s$. Therefore $\mathcal{F}$ is a linear separable system on $\langle X, \prec\rangle$. Furthermore, if $i, j \in n+2$ with $i < j$ then, choosing $r, s \in \mathbb{Q} \cap (z, z+1)$ such that $r < s$, we have $y_i \in F_{r+i}$, $y_j \in F_{s+j}$, and $r + i < s + j$. Thus $\mathcal{F}$ distinguishes the elements of Y under $\prec$.

Let $\mathcal{G}$ be the set of all F_r with $r \in \mathbb{Q} \cap (z, z+1)$ and let $\mathcal{H}$ be the set of all F_r with $r \in \mathbb{Q} \cap (z+n, z+n+1)$. We can reindex $\mathcal{G}$ and $\mathcal{H}$ as $\{G_r : r \in \mathbb{Q}\}$ and $\{H_r : r \in \mathbb{Q}\}$ respectively, preserving the property that if $r, s \in \mathbb{Q}$ with $r < s$ then $\overline{G}_r \subseteq G_s$ and $\overline{H}_r \subseteq H_s$. We can similarly reindex $\mathcal{F}$ as $\{F'_r : r \in \mathbb{Q}\}$, also requiring that if $r \in \mathbb{Q} \cap (z+1, z+n)$ then $F'_r = F_r$.

Let $\mathcal{E} = \{E_r : r \in \mathbb{Q}\}$ be a distinguishing countable linear separable system on X. For every $r \in \mathbb{Q}$ set $A_r = E_r \cap G_r$ and $B_r = E_r \cup H_r$. Set $\mathcal{A} = \{A_r : r \in \mathbb{Q}\}$ and $\mathcal{B} = \{B_r : r \in \mathbb{Q}\}$. Let $M = \left\{\langle s, t\rangle \in \mathbb{Q}^2 : 0 < t - s < 1\right\}$. For every $r \in \mathbb{Q}$ and every $\langle s, t\rangle \in M$ let $C_{r,s,t} = (E_r \cap F'_{r+t}) \cup F'_{r+s}$. Set $\mathcal{C}_{s,t} = \{C_{r,s,t} : r \in \omega\}$. We will show that $\{\mathcal{A}, \mathcal{B}, \mathcal{F}\} \cup \{\mathcal{C}_{r,s,t} : r \in \mathbb{Q} \text{ and } \langle s, t\rangle \in M\}$ is a distinguishing collection of countable linear separable systems on $\langle X, \prec\rangle$. The result then follows from Lemma 23.

Obviously the elements of the systems contained in the collection given above are open and $<$-decreasing. We will check that they are also $\prec$-decreasing. Let $r \in \mathbb{Q}$. In each of the following cases let $p, q \in X$ such that $p \prec q$ and $p \not< q$. Then there are $i, j \in n+2$ such that $p \leq y_i \prec_Y y_j \leq q$. Of course, $0 < j$ and $i < n+1$.

Assume that $q \in A_r$. This means that $y_j \in A_r$. But $A_r \subseteq F_s$ for some $s \in \mathbb{Q} \cap (z, z+1)$. Since $\beta(s) = 0$, we have $F_s \cap Y \subseteq \{y_0\}$, a contradiction. Therefore $q \notin A_r$ and A_r is $\prec$-decreasing.

Assume that $q \in B_r$. Now $H_r = F_s$ for some $s \in \mathbb{Q} \cap (z+n, z+n+1)$. Since $\beta(s) = n$, we have $i \leq \beta(s)$ and $y_i \in F_s$. Therefore $y_i \in B_r$ which means that $p \in B_r$. Therefore B_r is $\prec$-decreasing.

Let $\langle s, t\rangle \in M$ and assume that $q \in C_{r,s,t}$. Since F'_{r+s} is already $\prec$-decreasing, we may assume that $q \in E_r \cap F'_{r+t}$. If $r+s > z+n$ then $F'_{r+s} = F_u$ for some $u \in \mathbb{Q} \cap (z+n, z+n+1)$. Now $i \leq n \leq \beta(u)$, so $y_i \in F_u$. Thus $y_i \in C_{r,s,t}$ and $p \in C_{r,s,t}$. If $r+t < z+1$ then $F'_{r+t} = F_u$ for some $u \in \mathbb{Q} \cap (z, z+1)$. But then $\beta(u) = 0 < j$, so $y_j \notin F_u$, a contradiction. Let $k \in \mathbb{Z}$ such that $z+k+1 < r+t < z+k+2$. Then $z+k < r+s$ because $t - s < 1$. In order that $y_j \in F'_{r+t}$, it must be true that $j < k+1$. But then $i < k$, so $y_i \in F'_{r+s}$. Therefore $y_i \in C_{r,s,t}$, and hence $p \in C_{r,s,t}$. Thus $C_{r,s,t}$ is $\prec$-decreasing.

We already know that $\mathcal{F}$ is a countable linear separable system on $\langle X, \prec\rangle$. Now we will show that $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}_{s,t}$ are all countable linear separable systems on $\langle X, \prec\rangle$. Let $r, s \in \mathbb{Q}$ such that $r < s$. Then $\overline{A}_r \subseteq \overline{E}_r \cap \overline{G}_r \subseteq E_s \cap G_s = A_s$ and $\overline{B}_r = \overline{E}_r \cup \overline{H}_r \subseteq E_s \cup H_s = B_s$. Thus $\mathcal{A}$ and $\mathcal{B}$ are countable linear separable systems on $\langle X, \prec\rangle$. Fix $\langle t, u\rangle \in M$. Then $\overline{C}_{r,t,u} \subseteq (\overline{E}_r \cap \overline{F}'_{r+u}) \cup \overline{F}_{r+t} \subseteq (E_s \cap F'_{s+u}) \cup F'_{s+t} = C_{s,t,u}$. So $\mathcal{C}_{t,u}$ is a countable linear separable system on $\langle X, \prec\rangle$.

It remains to show that $\{\mathcal{F}, \mathcal{A}, \mathcal{B}\} \cup \{\mathcal{C}_{r,s} : \langle r, s\rangle \in M\}$ is distinguishing. Let $p, q \in X$ with $p \prec q$. If $p \not< q$ then there are $i, j \in n+2$ such that $i < j$, $p \leq y_i$, and $y_j < q$. By choosing $r, s \in \mathbb{Q}$ such that $z+i < r < s < z+i+1$ we get that $p \in F_r$ and $q \notin F_s$. So we may assume that $p < q$. We may also assume that there are not $t, u \in \mathbb{Q}$ such that $t < u$, $p \in F'_t$, and $q \notin F'_u$. Because $\mathcal{E}$ is a distinguishing linear separable system on $\langle X, <\rangle$, we can find $r, s \in \mathbb{Q}$ such that $r < s$, $p \in E_r$, and $q \notin E_s$. There are three cases to consider.

First, assume that for every $t \in \mathbb{Q}$, $q \in F'_t$. Then $p \in F'_t$ for every $t \in \mathbb{Q}$. It follows that $p \in E_r \cap G_r = A_r$ and $q \notin E_s \cap G_s = A_s$. So $\mathcal{A}$ distinguishes p and q.

Next, assume that for every $t \in \mathbb{Q}$, $p \notin F'_t$. Then for every $t \in \mathbb{Q}$ we have $q \notin F'_t$. It follows that $p \in E_r \cup H_r = B_r$ and $q \notin E_s \cup H_s = B_s$. So $\mathcal{B}$ distinguishes p and q.

Finally, assume that there are $t, u \in \mathbb{Q}$ such that $q \notin F'_t$ and $p \in F'_u$. By our assumption above, $t \leq u$. We also know that if $v < t$ then $p \notin F'_v$, and if $u < v$ then $q \in F'_v$. Let $S = \sup_{\mathbb{R}}\{v \in \mathbb{Q} : q \notin F'_v\}$ and $I = \inf_{\mathbb{R}}\{v \in \mathbb{Q} : p \in F'_v\}$. Then $t \leq S \leq I \leq u$. But if $S < I$ then there are $v, w \in \mathbb{Q}$ such that $S < v < w < I$. This means that $q \in F'_v$ and $p \notin F'_w$, a contradiction. Therefore $S = I$. Choose $\langle v, w\rangle \in M$ with $r+v < I < r+w$. Now pick $x \in \mathbb{Q}$ such that $r < x < s$ and $x+v < I$. Since $I < r+w$, we have $p \in F'_{r+w}$. But we already know that $p \in E_r$, so $p \in (E_r \cap F'_{r+w}) \cup F'_{r+v} = C_{r,v,w}$. Since $x+v < S$, we have $q \notin F'_{x+v}$. But $x < s$ and $q \notin E_s$, so $q \notin E_x$. Therefore $q \notin (E_x \cap F'_{x+w}) \cup F'_{x+v} = C_{x,v,w}$. Thus $\mathcal{C}_{v,w}$ separates p and q. $\square$

By giving X the added structure of a normally ordered space, we can greatly simplify the requirements for Y to be a pliable subset of X.

Theorem 25. *Let X be a normally ordered space. A finite subset Y of X is a pliable subset of X if and only if the following conditions are satisfied.*

(1) *There is a distinguishing countable linear separable system on X.*
(2) *For every subsets A and B of Y, if $d(A) \cap B = \emptyset$, then $D(A) \cap I(B) = \emptyset$.*

Proof. The normality of X can be used to define the separable systems $\mathcal{K}_{A,B}$ used in Theorem 24. $\square$

The methods used in this section will not generalize to infinite subsets. It is still an open question whether similar characterizations can be given for infinite subsets. It is possible that no theorem can be obtained which improves on just saying that Y is pliable if and only if for every extension $\prec_Y$ of $<$ there is a splitting separable system on X with respect to $\prec$.

V. Pliable Subsets of Normally ordered Spaces

In this section, we improve on the work in the previous section in the sense that we will obtain a characterization of pliable subsets without any restrictions on their cardinality. This is done at the cost of placing some strong restrictions on the type of space being considered, and how the subset sits in the space. Some of our restrictions will be necessary, anyway, while others occur naturally in familiar spaces. Rather than using Herden's approach to the existence of continuous, strictly increasing functions, we will return to Nachbin's approach. Our spaces will be normally ordered, the orders will be weakly continuous, and we will need a type of separability. We need to build these structures in such a way that they will be preserved when the space is reordered. Let us begin by considering the problem of making sure that $\langle X, \prec\rangle$ is normally ordered for every extension $\prec_Y$ of $<_Y$. The first lemma shows that such a space must be normally ordered.

Lemma 26. *Let X be an ordered space and $Y \subseteq X$. If $\langle X, \prec\rangle$ is normally ordered for every linear extension $\prec_Y$ of $<_Y$ then X is normally ordered.*

Proof. Let $\langle A, B\rangle$ be a closed corner of X. By Lemma 7 there is a linear extension $\prec_Y$ of $<_Y$ such that if $p \in A \cap Y$, $q \in Y \setminus (A \cup B)$, and $r \in B \cap Y$, then $p \prec_Y q \prec_Y r$. Now A is decreasing and B is increasing under $\prec$. Since $\langle X, \prec\rangle$ is normally ordered, $\langle A, B\rangle$ can be expanded to an open corner $\langle U, V\rangle$ in $\langle X, \prec\rangle$, which is also an open corner in $\langle X, <\rangle$. $\square$

Lemma 26 also shows that saying "$\langle X, \prec\rangle$ is normally ordered for every extension $\prec_Y$ of $<_Y$" is equivalent to saying "$\langle X, \prec\rangle$ is normally ordered for every linear extension $\prec_Y$ of $<_Y$".

If $\langle X, \prec\rangle$ is to be normally ordered for every linear extension $\prec_Y$ of $<_Y$, then any pair $\langle A, B\rangle$ which can be a closed corner in any reordering, must be expandable to an open corner in the existing order. The following lemma outlines some types of pairs of subsets of Y which must have this property.

Lemma 27. *Let X be an ordered space and let Y be a pliable subset of X. Let $A, B \subseteq Y$ such that $d(A) \cap B = \emptyset$.*

1. *If A does not have a finite cofinal subset and B does not have a finite coinitial subset, then $\langle A, B\rangle$ can be expanded to an open corner $\langle U, V\rangle$ in X. Moreover, U and V can be chosen so that $U \cap Y = d(A) \cap Y$ and $V \cap Y = i(B) \cap Y$.*
2. *If $Y \setminus [d(A) \cup i(B)] \neq \emptyset$ then $\langle A, B\rangle$ can be expanded to an open corner $\langle U, V\rangle$ in X. Moreover, if $r, s \in Y \setminus [d(A) \cup i(B)]$ such that $r \leq s$, then we can choose U and V so that $U \cap Y \subseteq [d(A) \cup d(r)] \cap Y$ and $V \cap Y \subseteq [i(B) \cup i(s)] \cap Y$.*
3. *If A has at least two maximal elements or B has at least two minimal elements, then $\langle A, B\rangle$ can be expanded to an open corner $\langle U, V\rangle$. Moreover, if A has at least two maximal elements, then U and V can be chosen so that $U \cap Y = d(A) \cap Y$, and if B has at least two minimal elements, then U and V can be chosen so that $V \cap Y = i(B) \cap Y$.*
4. *If A has a maximal element and B has a minimal element, then $\langle A, B\rangle$ can be expanded to an open corner in X.*

Proof.

Part 1. Assume that A does not have a finite cofinal subset and B does not have a finite coinitial subset. Then by Proposition 18 there is a cofinal subset $\{p_n : n \in \omega\}$ of A such that if $m < n$ then $p_n \nleq p_m$ and there is a coinitial subset $\{q_n : n \in \omega\}$ of B such that if $m < n$ then $q_m \nleq q_n$. Let $\prec_Y$ be an extension of $<_Y$ that satisfies the following two conditions. First, if $m, n \in \omega$ and $m < n$ then $p_m \prec_Y p_n \prec_Y q_n \prec_Y q_m$. Second, if $r \in Y \setminus [d(A) \cup i(B)]$ and $n \in \omega$, then $p_n \prec_Y r \prec_Y q_n$. That such an extension exists follows from Lemmas 6 and 7, as does the existence of the other extensions that we will define in this proof. Let $f : X \to \mathbb{R}$ be continuous and strictly $\prec$-increasing. Let $x = \lim_{n\to\infty} f(p_n)$ and $y = \lim_{n\to\infty} f(q_n)$. Then $\left\langle f^{-1}[(-\infty, x)], f^{-1}[(y, \infty)]\right\rangle$ is an open corner in X that expands $\langle A, B\rangle$. Also, if $r \in Y \setminus d(A)$ then $x \leq f(r)$, so $r \notin f^{-1}[(-\infty, x)]$. If $r \in Y \setminus i(B)$ then $f(r) \leq y$, so $r \notin f^{-1}[(y, \infty)]$. Therefore $f^{-1}[(-\infty, x)] \cap Y = d(A) \cap Y$ and $f^{-1}[(y, \infty)] \cap Y = i(B) \cap Y$.

Part 2. Let $r \in Y \setminus [d(A) \cup i(B)]$. Let $\prec_Y$ be an extension of $<_Y$ such that if $p \in A$ and $q \in B$ then $p \prec_Y r \prec_Y q$. Let $f : X \to \mathbb{R}$ be continuous and strictly $\prec$-increasing. Then $\left\langle f^{-1}[(-\infty, r)], f^{-1}[(r, \infty)]\right\rangle$ is an open corner in X that expands $\langle A, B\rangle$.

Let $r, s \in Y \setminus [d(A) \cup i(B)]$ such that $r \leq s$. Define a linear extension $\prec_Y$ of $<_Y$ that satisfies the following properties.

1. If $p \in d(A)$ then $p \prec_Y r$.
2. If $q \in i(B)$ then $s \prec_Y q$.
3. If $p \in Y \setminus [d(A) \cup d(r)]$ then $r \prec_Y p$.

(4) If $q \in Y \setminus [i(B) \cup i(s)]$ then $q \prec_Y s$.

Let $f : X \to \mathbb{R}$ be continuous and strictly $\prec_Y$- increasing. Set $U = f^{-1}[(-\infty, f(r))]$ and $V = f^{-1}[(f(s), \infty)]$. Then $\langle U, V \rangle$ is an open corner in $\langle X, < \rangle$ that expands $\langle A, B \rangle$. Let $p \in U \cap Y$. Then $f(p) < f(r)$. If $p \in Y \setminus [d(A) \cup d(r)]$ then $r \prec_Y p$, so $f(r) < f(p)$. Thus $p \in [d(A) \cup d(r)] \cap Y$. Similarly, if $q \in V \cap Y$ then $q \in [i(B) \cup i(s)] \cap Y$.

Part 3. Assume that A has two maximal elements r and s. By Part 2, $\langle A \setminus \{r\}, B \rangle$ can be expanded to an open corner $\langle U_1, V_1 \rangle$ in X, and $\langle A \setminus \{s\}, B \rangle$ can be expanded to an open corner $\langle U_2, V_2 \rangle$ in X. Then $\langle U_1 \cup U_2, V_1 \cap V_2 \rangle$ is an open corner in X that expands $\langle A, B \rangle$. The dual case when B has two minimal elements is done the same way.

To establish the second claim of Part 3, define an extension $\prec_Y^r$ of $<_Y$ such that if $p \in [d(A) \cap Y] \setminus \{r\}$ then $p \prec_Y^r r$ and if $q \in Y \setminus d(A)$ then $r \prec_Y^r q$. Let $f : X \to \mathbb{R}$ be continuous and strictly $\prec^r$-increasing, and set $U_r = f^{-1}[(-\infty, f(r))]$. Now define an extension $\prec_Y^s$ of $<_Y$ such that if $p \in [d(A) \cap Y] \setminus \{s\}$ then $p \prec_Y^s s$ and if $q \in Y \setminus d(A)$ then $s \prec_Y^s q$. Let $g : X \to \mathbb{R}$ be continuous and strictly $\prec^s$-increasing, and set $U_s = g^{-1}[(-\infty, g(s))]$. The set $U = U_r \cup U_s$ is an open decreasing subset of X and $d(A) \cap Y \subseteq U$. If $p \in U \cap Y$ then either $f(p) < f(r)$ or $g(p) < g(s)$, so $p \in d(A) \cap Y$. If B has two minimal elements, then we can in the same manner obtain an open increasing subset V of X such that $V \cap Y = i(B) \cap Y$.

Part 4. Let r be a maximal element of A and let s be a minimal element of B. Let $\prec_Y$ be an extension of $<_Y$ such that if $p \in A$ and $q \in B$ then $p \preceq_Y r \prec_Y s \preceq_Y q$. Let $f : X \to \mathbb{R}$ be continuous and strictly $\prec$-increasing. Let $x, y \in (f(r), f(s))$ such that $x < y$. Then $\langle f^{-1}[(-\infty, x)], f^{-1}[(y, \infty)] \rangle$ is an open corner in X that expands $\langle A, B \rangle$.
□

We will say that a subset Y of an ordered space X is *nicely embedded* in X if Y satisfies Properties 1-3 of Lemma 27.

Lemma 27 says that most reasonable pairs of subsets of Y can be separated in X. In fact, if $\langle A, B \rangle$ cannot be expanded to an open corner in X, then $Y \setminus [d(A) \cup i(B)] = \emptyset$ and either A has a maximum element while B has no minimal element, or A has no maximal element while B has a minimum element. The following example is, therefore, typical of what occurs if $\langle A, B \rangle$ cannot be expanded to an open corner. It also shows that Lemma 27 cannot be improved.

Example. Let X be the set consisting of the points $\langle 0, 0 \rangle$ and $\langle 1, 0 \rangle$, together with the sequences $\{\langle 0, 2^{-n} \rangle : n \in \omega\}$ and $\{\langle 2^{-m}, 2^{-n} \rangle : m, n \in \omega\}$. We will set $\langle a, b \rangle < \langle c, d \rangle$ in X if and only if $a = c = 0$ and $b < d$. Let $\mathcal{G}$ be the topology on $X \setminus \{\langle 0, 0 \rangle, \langle 1, 0 \rangle\}$ inherited from $\mathbb{R}^2$. For every $k \in \omega$ let $E_k = \{\langle 0, 0 \rangle\} \cup \{\langle 2^{-m}, 2^{-n} \rangle : m, n \in \omega \text{ and } m, n > k\}$ and $F_k = \{\langle 1, 0 \rangle\} \cup \{\langle 0, 2^{-n} \rangle : n \in \omega \text{ and } n > k\}$. Give X the topology generated by $\{E_k : k \in \omega\} \cup \{F_k : k \in \omega\} \cup \mathcal{G}$. Set $Y = \{\langle 0, 0 \rangle\} \cup \{\langle 0, 2^{-n} \rangle : n \in \omega\}$, $A = \{\langle 0, 0 \rangle\}$, and $B = \{\langle 0, 2^{-n} \rangle : n \in \omega\}$. Then X is a normally ordered space, Y is a pliable subset of X, $d(A) \cap B = \emptyset$, and $\langle A, B \rangle$ cannot be expanded to an open corner of X.

Proof. Note that A has a maximum element, B has no minimal elements, and $A \cup B = Y$. That $d(A) \cap B = \emptyset$ is obvious from the definitions. We also have the stronger property that $D(A) \cap B = \emptyset$ and $A \cap I(B) = \emptyset$, because $D(A) = \{\langle 0, 0 \rangle, \langle 1, 0 \rangle\}$ and $I(B) = B \cup \{\langle 1, 0 \rangle\}$.

That Y is a pliable subset of X follows from the facts that Y is already a chain in X and the function $f : X \to \mathbb{R}$ given by $f(\langle x, y\rangle) = y$ is continuous and strictly increasing.

If U is an open subset of X with $A \subseteq U$ then there is $k \in \omega$ such that $E_k \subseteq U$. Let V be an open subset of X such that $B \subseteq V$. Then there is $j \in \omega$ such that if $j < n$ then $\langle 2^{-n}, 2^{-k-1}\rangle \in V$. But $\langle 2^{-n}, 2^{-k-1}\rangle \in E_k$, so $U \cap V \neq \emptyset$. Thus $\langle A, B\rangle$ cannot be expanded to an open corner in X.

To show that X is normally ordered, let $\langle H, K\rangle$ be a closed corner in X. Then $\langle 1, 0\rangle$ cannot be in both H and K.

If $\langle 1, 0\rangle \notin H$ then there is $i \in \omega$ such that $F_i \cap H = \emptyset$. But H is decreasing, so $Y \cap H = \emptyset$. It follows that there is $j \in \omega$ such that if $m > j$ then $\langle 2^{-m}, 2^{-n}\rangle \notin H$ for all $n \in \omega$. Let $W = \{\langle 1, 0\rangle\} \cup Y \cup \{\langle 2^{-m}, 2^{-n}\rangle : m, n \in \omega \text{ and } m > j\}$. Then W is open and increasing in X and $H \subseteq X \setminus W$. Since $X \setminus W$ is discrete, $\langle H, K \cup W\rangle$ is an open corner in X that expands $\langle H, K\rangle$.

If $\langle 1, 0\rangle \notin K$ then there is $i \in \omega$ such that $F_i \cap K = \emptyset$. Since K is increasing, and H is decreasing, there are $j, k \in \omega$ such that $j \leq k$ and if $m < j$ and $k < n$ then $\langle 0, 2^{-m}\rangle \notin H$ and $\langle 0, 2^{-n}\rangle \notin K$. Now we can find $l \in \omega$ such that if $m > l$ and $n < j$ then $\langle 2^{-m}, 2^{-n}\rangle \notin H$ and if $m > l$ and $k < n$ then $\langle 2^{-m}, 2^{-n}\rangle \notin K$. Set $U = \{\langle 0, 2^{-n}\rangle : n > k\} \cup \{\langle 2^{-m}, 2^{-n}\rangle : m > l \text{ and } n > k\}$ and set V equal to the union of the sets $\{\langle 0, 2^{-n}\rangle : n < j\}$ and $\{\langle 2^{-m}, 2^{-n}\rangle : m > l \text{ and } n < j\}$. Then $\langle F_i \cup U \cup H, V \cup K\rangle$ is an open corner in X that expands $\langle H, K\rangle$. □

Simply being able to expand a closed corner to an open corner is not sufficient. In going to the larger open corner, we may add elements of Y. These new elements could cause the open sets , which were decreasing and increasing under the original order of X, to be no longer decreasing and increasing under the reordering. One of the important points of Lemma 27 is that the open corners in cases 1-3 add no new elements of Y. But note that the pairs in Lemma 27 were not closed corners in X. When we expand one of these to an open corner, say $\langle U, V\rangle$, we might even have $p \in U \cap Y$ and $q \in V \cap Y$ such that $q \prec_Y p$ for some extension $\prec_Y$ of $<_Y$. To avoid this situation, we will assume that we can find an open corner expanding the closed corner in which it is impossible to reorder the elements of Y in such a way. A subset Y of a normally ordered space X is called a *narrow* subset of X if for every closed corner $\langle A, B\rangle$ in X there is an open corner $\langle U, V\rangle$ in X that expands $\langle A, B\rangle$ such that if $p \in U \cap (Y \setminus A)$ and $q \in V \cap (Y \setminus B)$ then $p < q$. But not all pliable subsets of normally ordered spaces are narrow, as the following example shows.

Example. Let $X = \mathbb{R}$ and give X the usual topology. Define an order $<$ on X as follows. Let $N = \{n \in \mathbb{Z} : n < -1\}$, $P = \{n \in \mathbb{Z} : n > 1\}$, $S = \{-1 - \frac{1}{n} : n \in N\}$, and $T = \{1 - \frac{1}{n} : n \in P\}$. Set $m < -1 - \frac{1}{m}$ for all $m \in N$ and set $1 - \frac{1}{n} < n$ for all $n \in P$. Set $m < n$ for all $m \in N$ and all $n \in P$ such that $m \neq -n$. This is trivially an order since if $p, q \in X$ with $p < q$ then there is no $r \in X$ with $q < r$. Let $Y = N \cup P$. Then X is a normally ordered space, Y is a pliable subset of X which is not narrow, and for every linear extension $\prec_Y$ of $<_Y$, $\langle X, \prec\rangle$ is normally ordered.

Proof. The idea behind the example is to create a space having two antichains, P and N, so that $I(N) \setminus i(N) \neq \emptyset$ and $D(P) \setminus d(P) \neq \emptyset$, but which are sufficiently entwined by the order that not many elements of N can be made larger than elements of P by reordering Y.

Set $Z = N \cup P \cup S \cup T \cup \{-1, 1\}$. Note that if $p, q \in X$ and $p < q$ then $p, q \in N \cup P \cup S \cup T \subset Z$. We will first show that if $\prec_Y$ is a linear extension of $<_Y$, then $\langle X, \prec \rangle$ is normally ordered.

Suppose that $\langle A, B \rangle$ is a closed corner in $\langle X, \prec \rangle$ such that $Z \subseteq A \cup B$. Since X is a normal topological space, there are disjoint open subsets, U and V, of X such that $A \subseteq U$ and $B \subseteq V$. Let $p, q \in X$ such that $q \in U$ and $p \prec q$. Then $p, q \in Z$ and $q \in A$ because $B \cap U = \emptyset$. But A is decreasing with respect to $\prec$, so $p \in A$. Thus U is decreasing with respect to $\prec$. Similarly, V in increasing in $\prec$. Therefore $\langle U, V \rangle$ is an open corner in $\langle X, \prec \rangle$ that expands $\langle A, B \rangle$. So for the remainder of the proof that $\langle X, \prec \rangle$ is normally ordered, we need only show that if $\langle A, B \rangle$ is a closed corner in $\langle X, \prec \rangle$ then it can be expanded to a closed corner of $\langle X, \prec \rangle$ having Z in the union of its components. We will work under the assumption that $Z \not\subseteq A \cup B$.

First notice the following fact about the extended order. If there are $m \in N$ and $n \in P$ such that $n \prec_Y m$, then $m = -n$. In that case, if $j \in N \setminus \{m\}$ and $k \in P \setminus \{n\}$ then $j < n \prec_Y m < k$. Assume that $\{-1, 1\} \cap B = \emptyset$. Then $Z \setminus B$ is a closed $\prec$-decreasing subset of X. Thus $\langle A \cup (Z \setminus B), B \rangle$ is a closed corner in $\langle X, \prec \rangle$ expanding $\langle A, B \rangle$ such that $Z \subseteq A \cup (Z \setminus B) \cup B$.

Assume that $-1 \notin B$ and $1 \in B$. Since $1 \in \overline{T}$, there is $K \in \mathbb{Z}$ such that if $n \in P$ and $n > K$ then $1 + \frac{1}{n} \notin A$. Since A is decreasing, $A \cap \{n \in P : K < n\} = \emptyset$. We can choose K large enough so that if there is $m \in P$ such that $m \prec -m$ then $m < K$. Set $B' = B \cup \{1 + \frac{1}{n} : n \in P \text{ and } K < n\} \cup i_{\prec}(\{n \in P : K < n\})$. Then B' is a closed $\prec$-increasing subset of X and $A \cap B' = \emptyset$. Also, $Z \setminus B'$ is a closed $\prec$-decreasing subset of X. Thus $\langle A \cup (Z \setminus B'), B' \rangle$ is a closed corner in $\langle X, \prec \rangle$ that expands $\langle A, B \rangle$.

Assume that $1 \notin B$ and $-1 \in B$. Since $-1 \in \overline{S}$, there is $K \in N$ such that if $n \in N$ and $n < K$ then $-1 - \frac{1}{n} \notin A$. Set $B' = B \cup \{-1 - \frac{1}{n} : n \in N \text{ and } n < K\}$. Then B' is a closed, $\prec$-increasing subset of X and $A \cap B' = \emptyset$. Also, $Z \setminus B'$ is a closed $\prec$-decreasing subset of X. Thus $\langle A \cup (Z \setminus B'), B' \rangle$ is a closed corner in $\langle X, \prec \rangle$ that expands $\langle A, B \rangle$.

Finally, if $\{-1, 1\} \subseteq B$ then $\{-1, 1\} \cap A = \emptyset$, which is just the dual problem of our first case.

We can now show that Y is a pliable subset of X by showing that for every linear extension $\prec_Y$ of $<_Y$, $\langle X, \prec \rangle$ is continuous and weakly order separable. That X is weakly order separable under $\prec$ follows from the fact that the order on X is itself countable, as is Y. So it remains to show that $\langle X, \prec \rangle$ is continuous.

If $p \notin N \cup P \cup S \cup T$ then $d_{\prec}(p) = i_{\prec}(p) = \{p\}$, which is closed. If $p \in N$ then $i_{\prec}(p) \subseteq N \cup P \cup \{-1 - \frac{1}{p}\}$ and there is $n \in P$ such that $d_{\prec}(p) \subseteq \{n, 1 - \frac{1}{n}\}$. If $p \in P$ then $d_{\prec}(p) \subseteq N \cup P \cup \{1 - \frac{1}{p}\}$ and there is $n \in N$ such that $i_{\prec}(p) \subseteq P \cup \{n, -1 - \frac{1}{n}\}$. If $p \in S$ then $i_{\prec}(p) = \{p\}$ and $d_{\prec}(p) = \{p\} \cup d_{\prec}\left(\frac{-1}{1+p}\right)$. If $p \in T$ then $d_{\prec}(p) = \{p\}$ and $i_{\prec}(p) = \{p\} \cup i_{\prec}\left(\frac{1}{1-p}\right)$. In any case, both $i_{\prec}(p)$ and $d_{\prec}(p)$ are closed. Lemma 26 now implies that $\langle X, < \rangle$ is normally ordered.

But Y is not a narrow subset of X. Let $\langle U, V \rangle$ be an open corner in X that expands $\langle \{-1\}, \{1\} \rangle$. Then there is $m \in P$ such that if $n \in P$ and $m < n$ then $-n \in U$ and $n \in V$. But $-n \not< n$. □

Narrow subsets can be useful in certain situations, as is shown in the next lemma. We will use the following notion. A corner $\langle A, B\rangle$ of an ordered space X is said to have a *countable corner basis* if there is a countable collection $\mathcal{C}$ of open corners in X expanding $\langle A, B\rangle$ such that if $\langle U, V\rangle$ is an open corner in X expanding $\langle A, B\rangle$ then there is $\langle E, F\rangle \in \mathcal{C}$ such that $\langle U, V\rangle$ expands $\langle E, F\rangle$. We can always write $\mathcal{C}$ as $\{\langle E_n, F_n\rangle : n \in \omega\}$ where $\langle E_n, F_n\rangle$ expands $\langle E_{n+1}, F_{n+1}\rangle$ for all $n \in \omega$.

Lemma 28. *Let Y be a subset of an ordered set X and let $\langle A, B\rangle$ be a closed corner in X having a countable corner basis, $\{\langle E_n, F_n\rangle : n \in \omega\}$. If neither $(Y \setminus A) \cap E_0$ nor $(Y \setminus B) \cap F_0$ contains an infinite antichain and if $\langle X, \prec\rangle$ is normally ordered for every extension $\prec_Y$ of $<_Y$ then there is $n \in \omega$ such that $p < q$ for all $p \in (Y \setminus A) \cap E_n$ and $q \in (Y \setminus B) \cap F_n$.*

Proof. Assume that for every $n \in \omega$ we can find $p_n \in (Y \setminus A) \cap E_n$ and $q_n \in (Y \setminus B) \cap F_n$ such that $p_n \not< q_n$. Set $H = \{p_n : n \in \omega\}$ and $K = \{q_n : n \in \omega\}$. Since all antichains in $(Y \setminus A) \cap E_0$ are finite, H contains an infinite chain $\{p_{\sigma(n)} : n \in \omega\}$. We may assume that either $p_{\sigma(n+1)} < p_{\sigma(n)}$ for all $n \in \omega$, or $p_{\sigma(n)} < p_{\sigma(n+1)}$ for all $n \in \omega$. Now $\{q_{\sigma(n)} : n \in \omega\}$ contains no infinite antichains, so it must have an infinite chain $\{q_{\tau(n)} : n \in \omega\}$. Again we may assume that either $q_{\tau(n+1)} < q_{\tau(n)}$ for all $n \in \omega$, or that $q_{\tau(n)} < q_{\tau(n+1)}$ for all $n \in \omega$.

Assume that for all $n \in \omega$, $p_{\tau(n+1)} < p_{\tau(n)}$. Let $m, n \in \omega$ such that $m < n$. If $p_{\tau(m)} < q_{\tau(n)}$ then $p_{\tau(n)} < p_{\tau(m)} < q_{\tau(n)}$, a contradiction. Thus $p_{\tau(m)} \not< q_{\tau(n)}$.

We will next define an extension $\prec_Y$ of $<_Y$ such that for every $n \in \omega$, $q_{\tau(n)} \prec_Y p_{\tau(n)}$. Set $\prec_0 = <_Y$. Let $m \in \omega$ and assume that $\prec_m$ is an extension of $<_Y$ and that for every $n \in \omega$ with $m \le n$, $p_{\tau(n)} \not\prec_m q_{\tau(n)}$. By Lemmas 6 and 7, $\prec_{m+1} = TC\left[\prec_m \cup \left\{\langle q_{\tau(m)}, p_{\tau(m)}\rangle\right\}\right]$ is an order on Y. Let $n \in \omega$ such that $m + 1 \le n$. Since $p_{\tau(n)} \not\prec_m q_{\tau(n)}$ and $p_{\tau(m)} \not< q_{\tau(n)}$, we have $p_{\tau(n)} \not\prec_{m+1} q_{\tau(n)}$. By induction, $\prec_Y = \bigcup_{n\in\omega} \prec_n$ is the desired extension of $<_Y$.

In either of the above cases, we can further require that if $p \in Y \cap A$ and $q \in Y \setminus A$, or if $p \in Y \setminus B$ and $q \in Y \cap B$, then $p \prec_Y q$. Then $\langle A, B\rangle$ is a closed corner in $\langle X, \prec\rangle$. If $\langle U, V\rangle$ is an open corner in $\langle X, \prec\rangle$ that expands $\langle A, B\rangle$ then there is $m \in \omega$ such that $\langle U, V\rangle$ expands $\langle E_m, F_m\rangle$. Choose $n \in \omega$ such that $\tau(n) > m$. Then $\langle U, V\rangle$ expands $\langle E_{\tau(n)}, F_{\tau(n)}\rangle$ but $q_{\tau(n)} \prec_Y p_{\tau(n)}$, a contradiction. Thus $\langle A, B\rangle$ cannot be expanded to an open corner in $\langle X, \prec\rangle$, contradicting the hypothesis that $\langle X, \prec_Y\rangle$ is normally ordered. Thus Y must be narrow. $\square$

Any closed corner of $[0, 1]^n$ with its usual order and topology will satisfy the conditions of Lemma 28. So if Y is a subset of $[0, 1]^n$ and all linear extensions on Y result in normally ordered spaces, then Y is narrow. It is an open question whether subsets of $\mathbb{R}$ have this property.

We see in the next theorem that narrowness and nice embeddedness are sufficient to make the extended space normally ordered.

Theorem 29. *If Y is a narrow, nicely embedded subset of a normally ordered space X and $\prec_Y$ is a linear extension of $<_Y$, then $\langle X, \prec\rangle$ is normally ordered.*

Proof. Let $\langle A, B\rangle$ be a closed corner in $\langle X, \prec\rangle$. Then $\langle A, B\rangle$ is a closed corner in $\langle X, <\rangle$, so there is an open corner $\langle U_1, V_1\rangle$ in $\langle X, <\rangle$ expanding $\langle A, B\rangle$ such that if $p \in (Y \setminus A) \cap U_1$ and $q \in (Y \setminus B) \cap V_1$ then $p < q$. Since $\langle X, <\rangle$ is normally ordered, there is an open corner $\langle U_2, V_2\rangle$ in $\langle X, <\rangle$ expanding $\langle A, B\rangle$ such that $D(U_2) \subseteq U_1$ and $I(V_2) \subseteq V_1$.

Let $A' = d_{\prec_Y}(U_2) \setminus U_2$ and $B' = i_{\prec_Y}(V_2) \setminus V_2$. Now $A' \cap V_1 = B' \cap U_1 = \emptyset$, so $D(A') \cap I(V_2) = I(B') \cap D(U_2) = \emptyset$. Let U_3 be an open $<$-decreasing subset of X such that $A' \subseteq U_3$ and $U_3 \cap V_2 = \emptyset$. Let V_3 be an open $<$-increasing subset of X such that $B' \subseteq V_3$ and $U_2 \cap V_3 = \emptyset$.

If there is not a finite subset of A' that is cofinal in A' and there is not a finite subset of B' that is coinitial in B' then, by Lemma 27, $\langle A', B' \rangle$ can be expanded to an open corner in $\langle X, < \rangle$.

Assume that A' has a finite cofinal subset E. If A' has two maximal elements then, by Lemma 27, $\langle A', B' \rangle$ can be expanded to an open corner in $\langle X, < \rangle$. If A' has a maximum element p, then there is $r \in U_2 \cap Y$ such that $p \prec_Y r$. Then $r \in Y \setminus [d(A') \cup i(B')]$, so, by Lemma 27, $\langle A', B' \rangle$ can be expanded to an open corner in $\langle X, < \rangle$. It follows in a similar fashion that if B' has a finite coinitial subset then $\langle A', B' \rangle$ can be expanded to an open corner in $\langle X, < \rangle$.

In any case, we can find an open corner $\langle U_4, V_4 \rangle$ in $\langle X, < \rangle$ that expands $\langle A', B' \rangle$. Furthermore, Lemma 27 says that we can choose U_4 and V_4 so that $U_4 \cap Y \subseteq [U_2 \cup d(A')] \cap Y$ and $V_4 \cap Y \subseteq [V_2 \cup i(B')] \cap Y$.

Set $U = U_2 \cup (U_3 \cap U_4)$ and $V = V_2 \cup (V_3 \cap V_4)$. Then $\langle U, V \rangle$ is an open corner in $\langle X, < \rangle$. Let $p, q \in Y$ with $p \prec_Y q$ and $q \in U$. If $q \in U_2$ and $p \notin U_2$, then $p \in A' \subseteq U_3 \cap U_4$. If $q \in U_4$ then either $q \in U_2$ or $q \in d(A')$. In either case, $p \in U$. It follows from Lemma 6 that U is decreasing under $\prec$. A similar argument shows that V is increasing under $\prec$, so $\langle X, \prec \rangle$ is normally ordered. $\square$

The next step in our quest for a partial characterization of pliable subsets is some type of separability condition. Before searching for the appropriate condition, we will review some existing types of separability and compare their strengths and weaknesses.

In the introduction, we defined order separability and weak order separability. The reason for the incongruity of naming the stronger property "weak" is that it was being compared to different, and much stronger property which was also referred to as order separability. To avoid confusion, we will name it strong order separability.

An ordered space X is *strongly order separable* if there is a countable subset C of X such that for every $p, q \in X$ with $p < q$ there is $r \in C$ such that $p < r < q$. This more closely resembles the topological idea of density. $\mathbb{Q}$ is strongly order dense, while $\mathbb{Z}$ is only weakly order dense. But $\mathbb{R}^2$ does not satisfy any of the three properties given so far. This is because all three properties require that elements of the dense subset appear in the intervals between related points of X. So the fact that $\{[\langle x, -1\rangle, \langle x, 1\rangle] : x \in \mathbb{R}\}$ is an uncountable collection of pairwise disjoint intervals in $\mathbb{R}^2$ makes it impossible for $\mathbb{R}^2$ to satisfy any of the properties. We will therefore try to find a weaker condition that holds in $\mathbb{R}^n$, but which is strong enough to give us continuous, strictly increasing functions, as weak order separability did.

In an ordered set, strong order separability implies weak order separability, which implies order separability. None of the arrows reverse, except in linearly ordered spaces, where weak order separability and order separability are all equivalent.

One bit of behavior that we will require of our separability condition is that it is preserved when we reorder a pliable subset. Here order separability and weak order separability do very well, while strong order separability is a failure.

Proposition 30. *Let Y be a pliable subset of an ordered space X and let $\prec_Y$ be an extension of $<_Y$. If $\langle X, < \rangle$ is (weakly) order separable, so is $\langle X, \prec \rangle$.*

Proof. Since $\langle Y, <_Y \rangle$ is a pliable space, $\langle Y, \prec_Y \rangle$ is weakly order separable. Let A be a countable subset of Y that is weakly order dense in $\langle Y, \prec_Y \rangle$.

Assume that $\langle X, < \rangle$ is order separable and let B be a countable order dense subset of X. Set $C = A \cup B$. Let $p, q \in X$ with $p \prec q$. If $p < q$ then there is $r \in B$ such that $p \leq r \leq q$. If $p \not< q$ then there are $s, t \in Y$ such that $p \leq s \prec_Y t \leq q$. Then there is $r \in A$ such that $s \preceq_Y r \preceq_Y t$. In either case, $r \in C$ and $p \preceq r \preceq q$, so C is order dense in $\langle X, \prec \rangle$.

Assume that $\langle X, < \rangle$ is weakly order separable and let B be a countable weakly order dense subset of X. Set $C = A \cup B$. Let $p, q \in X$ with $p \prec q$. If $p < q$ then there are $r, s \in A$ such that $p \leq r < s \leq q$. If $p \not< q$ then there are $t, u \in Y$ such that $p \leq t \prec_Y u \leq q$. There are then $r, s \in B$ such that $t \preceq_Y r \prec_Y s \preceq_Y u$. In either case, $r, s \in C$ and $p \preceq r \prec s \preceq q$. Therefore C is weakly order dense in $\langle X, \prec \rangle$. □

That strong order separability is not preserved by the reordering of pliable subsets is easily shown by the two element antichain $\{0, 1\}$ with the discrete topology. It is a pliable space and is trivially strongly order separable, but by setting $0 < 1$ we get an ordered space that is not strongly order separable.

We have seen that weak order separability works nicely to generate continuous, strictly increasing functions. The following example shows that order separability does not.

Example. Let $X = \omega_1 + 1$. Isolate all elements of the set ω_1 and let the point ω_1 have as its neighborhood base the collection of all sets of the form $\{\omega_1\} \cup A$ where $\omega_1 \setminus A$ is countable. Define an order $<$ on X by setting $\alpha < \omega_1$ for every $\alpha \in \omega_1$. Then X is a continuous, order separable, normally ordered space, but there is no continuous, strictly increasing, real-valued function on X.

Proof. If $\alpha \in \omega_1$ then $d(\alpha) = \{\alpha\}$ and $i(\alpha) = \{\alpha\} \cup \{\omega_1\}$, both of which are closed. Also, $d(\omega_1) = X$ and $i(\omega_1) = \{\omega_1\}$ are closed, so X is continuous. The set $\{\omega_1\}$ is order dense in X, so X is order separable. Let $\langle A, B \rangle$ be a closed corner in X. If $B = \emptyset$ then $\langle X, \emptyset \rangle$ is an open corner of X that expands $\langle A, B \rangle$. If $B \neq \emptyset$ then $\omega_1 \notin A$. Thus A must be countable. So $\langle A, X \setminus A \rangle$ is an open corner in X that expands $\langle A, B \rangle$, and X is normally ordered.

Let $f : X \to \mathbb{R}$ be a strictly increasing function. We will show that it cannot be continuous. Now $f[\omega_1] \subseteq (-\infty, f(\omega_1))$. We may assume that $f(\omega_1) = 0$. There must be $n \in \omega$ such that $f^{-1}[(-\infty, -2^{-n})]$ is uncountable. But then $X \setminus f^{-1}[(-2^{-n}, \infty)]$ is uncountable, so $f^{-1}[(-2^{-n}, \infty)]$ is not open in X. Hence f is not continuous. □

The consequence of all this is that none of our current separability conditions will do. The following new property will work for us. It differs from the previous properties in that they were totally order theoretic in nature, while our new condition combines the order and the topology. Because of this, it neither implies nor is implied by any of the previous conditions.

An ordered space X is said to be *quasiseparable* if there is a countable subset C of X such that for every nonminimal (nonmaximal) $p \in X$ and every open increasing (decreasing) neighborhood U of p there is $r \in C \cap U$ such that $r \leq p$ ($p \leq r$). The set C is said to be it quasidense in X. Then $\mathbb{Q}^n$ is quasidense in $\mathbb{R}^n$ with its usual order and topology. The condition that there be some $q \in X$ with $q < p$ (or $p < q$) is there only to allow the consideration of some spaces having large sets of maximal or minimal elements. For example, $\left\{\langle x, y\rangle \in \mathbb{R}^2 : y \leq -x\right\}$ is quasiseparable with this condition, but not without it, because we cannot find a countable number of elements of the space to go above the elements of the antichain $\left\{\langle x, y\rangle \in \mathbb{R}^2 : y = -x\right\}$ of maximal elements.

Lemma 31. *If the set A of minimal elements and the set B of maximal elements of a subset Y of a quasiseparable ordered space X are countable and if $\prec_Y$ is an extension of $<_Y$, then $\langle X, \prec\rangle$ is quasiseparable.*

Proof. Let C be a countable quasidense subset of X. Set $D = A \cup B \cup C$. Let $p \in X$ and let U be a $\prec$-decreasing neighborhood of p. Assume that there is $q \in X$ such that $p \prec q$. If $p < q$ then there is $r \in C$ such that $p \preceq r$ and $r \in U$. If $p \not< q$ then there are $r, s \in Y$ such that $p \leq r \prec s \leq q$. If $p < r$ then we have the previous case, so assume that $p = r$. Then $p \in Y$. If p is maximal in $\langle Y, <_Y\rangle$ then $p \in B$ and we can use it as the desired element of D. If p is not maximal in $\langle Y, <_Y\rangle$, then we are back to the first case. The proof of the dual case is similar.

Theorem 32. *If X is a normally ordered, weakly continuous, quasiseparable ordered space then there is a continuous, strictly increasing, real-valued function on X.*

Proof. Let C be a countable quasidense subset of X and let E be the set of all $\langle p, q\rangle \in C^2$ such that $\langle D(p), I(q)\rangle$ is a closed corner in X. We will show that if $r, s \in X$ with $r < s$ then there is $\langle p, q\rangle \in E$ such that $r \in D(p)$ and $s \in I(q)$. The result then follows by Theorem 4.

Let $r, s \in X$ with $r < s$. Then $\langle D(r), I(s)\rangle$ is a closed corner in X. Since X is normally ordered, there is an open corner $\langle U, V\rangle$ in X that expands $\langle D(p), I(q)\rangle$ such that $D(U) \cap I(V) = \emptyset$. Now pick $p, q \in C$ such that $r \leq p$, $q \leq s$, $p \in U$, and $q \in V$. Then $\langle p, q\rangle \in E$, $r \in D(p)$, and $s \in I(q)$. $\square$

As is indicated by Theorem 32, the last piece in our puzzle is weak continuity. Now if Y is a pliable subset of an ordered space X and $\prec_Y$ is an extension of $<_Y$ then $\langle X, \prec_Y\rangle$ will be weakly continuous. For if $p \prec q$ and $f : X \to \mathbb{R}$ is continuous and strictly $\prec$-increasing, then $\left\langle f^{-1}\left[(-\infty, f(p))\right], f^{-1}\left[[f(q), \infty)\right]\right\rangle$ is a closed corner in X. Since we cannot know ahead of time whether Y is pliable, we will need to make sure that if A could be $D_\prec(p)$ and B could be $I_\prec(q)$ for some $\prec$, then $A \cap B = \emptyset$. To do this, we will use the following definitions. Let A and B be subsets of an ordered set X. We will write $A \uparrow B$ if there is an extension $\prec_{A\cup B}$ of $<_{A\cup B}$ in which A is cofinal. We will write $A \downarrow B$ if there is an extension $\prec_{A\cup B}$ of $<_{A\cup B}$ in which A is coinitial.

We need three more lemmas to assist us in proving the main result of this section. The first one is reminiscent of the property of extreme continuity.

Lemma 33. *Let Y be a pliable subset of an ordered space X and let $A, B \subseteq Y$. If A has a maximal element, then $D(A) \cap Y = d_{<_Y}(A)$. If B has a minimal element, then $I(B) \cap Y = i_{<_Y}(B)$. If A has a maximal element, B has a minimal element, and $d(A) \cap B = \emptyset$, then $D(A) \cap I(B) = \emptyset$.*

Proof. Let q be a maximal element of A. By Lemmas 6 and 7, there is an extension $\prec_Y$ of $<_Y$ such that $p \prec_Y q$ when $p \in d_{<_Y}(A) \setminus \{q\}$ and $q \prec_Y p$ when $p \in Y \setminus d_{<_Y}(A)$. Since Y is pliable in X, there is a continuous, strictly $\prec$-increasing function $f : X \to \mathbb{R}$. Then $D(A) \cap Y \subseteq f^{-1}[(-\infty, f(q)]] \cap Y = d_{<_Y}(A)$. The proof of the second conclusion is similar.

Now assume that A has a maximal element q, B has a minimal element r, and $d(A) \cap B = \emptyset$. By Lemmas 6 and 7, there is an extension $\prec_Y$ of $<_Y$ such that if $p \in A \setminus \{q\}$ and $s \in B \setminus \{r\}$, then $p \prec q \prec r \prec s$. Let $f : X \to \mathbb{R}$ be continuous and strictly $\prec$-increasing. Then $D(A) \subseteq f^{-1}[(-\infty, f(q)]]$, $I(B) \subseteq f^{-1}[[f(r), \infty)]$, and $f^{-1}[(-\infty, f(q)]] \cap f^{-1}[[f(r), \infty)] = \emptyset$. □

We want to ensure that our spaces exhibit this behavior, so we will say that a subset Y of an ordered space X is *extremely continuous in* X if for every $A, B \subseteq Y$, $D(A) \cap Y = d_{<_Y}(A)$ whenever a subset A of Y has a maximal element, $I(A) \cap Y = i_{<_Y}(A)$ whenever A has a minimal element, and $D(A) \cap I(B) = \emptyset$ whenever A has a maximal element, B has a minimal element, and $d(A) \cap B = \emptyset$.

Lemma 34. *Let Y be a pliable subset of an ordered space X. If $\prec_Y$ is an extension of $<_Y$ and $A \subseteq X$ then $D_\prec(A) = D(d_{\prec_Y}(D(A)))$ and $I_\prec(A) = I(i_{\prec_Y}(I(A)))$.*

Proof. Since $\prec$ extends $<$, we have $D(d_{\prec_Y}(D(A))) \subseteq D_\prec(A)$. In order to reverse the subset, we will show that $D(d_{\prec_Y}(D(A)))$ is $\prec$- decreasing. Let $p \in X$ and $q \in D(d_{\prec_Y}(D(A)))$ such that $p \prec q$. If $p < q$ then $p \in D(d_{\prec_Y}(D(A)))$, so assume that $p \not< q$. That means that there are $r, s \in Y$ such that $p \leq r \prec s \leq q$ and $r \not< s$. Now $s \in D(d_{\prec_Y}(D(A)))$, so either $s \in D[d_{\prec_Y}(D(A)) \cap Y]$ or $s \in D(A)$.

If $r \notin d_{\prec_Y}(D(A)) \cap Y$ then r is maximal in $[d_{\prec_Y}(D(A)) \cap Y] \cup \{r\}$, so, by Lemma 33, $D([d_{\prec_Y}(D(A)) \cap Y] \cup \{r\}) \cap Y = [d_{\prec_Y}(D(A)) \cap Y] \cup d_{\prec_Y}(r)$. Thus, if s is an element of the set $D(d_{\prec_Y}(D(A)) \cap Y)$ then $s \in d_{\prec_Y}(D(A))$. But if $s \in D(A)$ then $r \in d_{\prec_Y}(D(A))$. So in either case, $r \in d_{\prec_Y}(D(A))$. Therefore $p \in D(d_{\prec_Y}(D(A)))$.

The proof that $I_\prec(A) = I(i_{\prec_Y}(I(A)))$ is similar.

Lemma 35. *Let X be an ordered set and let $A, B, E, F \subseteq X$. If $A \uparrow B$, $E \downarrow F$, and $d(A \cup B) \cap (E \cup F) = \emptyset$, then there is an extension $\prec$ of $<$ such that A is cofinal in $A \cup B$, E is coinitial in $E \cup F$, and, if $p \in A \cup B$ and $q \in E \cup F$, then $p \prec q$.*

Proof. Recall that $A \uparrow B$ means that there is an extension $\prec_{A\cup B}$ of $<_{A \cup B}$ in which A is cofinal, and $E \downarrow F$ means that there is an extension $\prec_{E \cup F}$ of $<_{E\cup F}$ in which E is coinitial.

Let $\prec^1_{A\cup B}$ be an extension of $<_{A\cup B}$ in which A is cofinal. By Lemma 6, $\prec^1 = TC[< \cup \prec^1_{A\cup B}]$ is an order on X. Clearly A is cofinal in $A \cup B$ under $\prec^1$. Since $d(A\cup B) \cap (E \cup F) = \emptyset$, we have $\prec^1_{E\cup F} = <_{E\cup F}$. Also, $d_{\prec^1}(A \cup B) \cap (E \cup F) = \emptyset$. Thus $E \downarrow F$ in $\prec^1$ and there is an extension $\prec^2_{E\cup F}$ of $\prec^1_{E\cup F}$ in which E is coinitial. By Lemma 6, $\prec^2 = TC[\prec^1 \cup \prec^2_{E\cup F}]$ is an order on X. Clearly E is coinitial in $E \cup F$ under $\prec^2$ and, since $d_{\prec^1}(A \cup B) \cap (E \cup F) = \emptyset$, $d_{\prec^2}(A \cup B) \cap (E \cup F) = \emptyset$. Thus, by Lemma 7, there is an extension $\prec$ of $\prec^2$ such that if $p \in A \cup B$ and $q \in E \cup F$, then $p \prec q$.

Lemma 36. *Let Y be a pliable subset of an ordered space X and let $p, q \in X$ such that $p < q$.*

(1) *If $r \in D(p) \cap Y$ and $s \in I(q) \cap Y$ then $r < s$.*

(2) *Let* $A, B \subseteq Y$. *If* $d_{<_Y}(A) \cap [I(q) \cup B] = \emptyset$, $i_{<_Y}(B) \cap [D(p) \cup A] = \emptyset$, $[D(p) \cap Y] \uparrow A$, *and* $[I(q) \cap Y] \downarrow B$, *then* $D(D(p) \cup A) \cap I(I(q) \cup B) = \emptyset$.

Proof.

Part 1. Assume that there are $r \in D(p) \cap Y$ and $s \in I(q) \cap Y$ such that $r \not< s$. Let $\prec_Y$ be an extension of $<_Y$ such that $s \prec_Y r$. Then $r \in D_\prec(p) \cap I_\prec(q)$, a contradiction.

Part 2. By Lemma 35 there is an extension $\prec_Y$ of $<_Y$ such that $A \cup d_{\prec_Y}(D(p))$ and $B \cup i_{\prec_Y}(I(q))$. Then

$$\begin{aligned} D(D(p) \cup A) \cap I(I(q) \cup B) &= D(d_{\prec_Y}(D(p))) \cap I(i_{\prec_Y}(I(q))) \\ &= D_{\prec_Y}(p) \cap I_{\prec_Y}(q) \\ &= \emptyset. \end{aligned}$$

□

Let us say that X *is inherently weakly continuous with respect to* Y if $D(D(p) \cup A) \cap I(I(q) \cup B) = \emptyset$ for every $p, q \in X$ with $p < q$ and every $A, B \subseteq Y$ such that $d_{<Y}(A) \cap [I(q) \cup B] = \emptyset$, $i_{<Y}(B) \cap [D(p) \cup A] = \emptyset$, $[D(p) \cap Y] \uparrow A$, and $[I(q) \cap Y] \downarrow B$. If X is inherently weakly continuous with respect to Y then, by taking $A = B = \emptyset$, X must be weakly continuous.

Theorem 37. *A narrow subset* Y *of a normally ordered, quasiseparable ordered space* X *is pliable if and only if the following conditions are satisfied.*

(1) $\langle Y, <_Y \rangle$ *is a pliable space.*
(2) Y *is nicely embedded in* X.
(3) Y *is extremely continuous in* X.
(4) X *is inherently weakly continuous with respect to* Y.

Proof. We have already seen that Properties 1-4 are necessary. It remains to show that they are sufficient. Assume that the properties are satisfied. Let $\prec_Y$ be a linear extension of $<_Y$. Now $\langle X, \prec \rangle$ is normally ordered by Theorem 29. We will show that it is quasiseparable, then that it is weakly continuous. Our result then follows by Theorem 32.

Since $\langle Y, <_Y \rangle$ is a pliable space, the set of minimal elements of Y and the set of maximal elements of Y are both countable. By Lemma 31, $\langle X, \prec \rangle$ is quasiseparable.

Let $p, q \in X$ with $p \prec q$. Assume that $p < q$. Let $A = d_{\prec_Y}(D(p)) \cap Y = d_{\prec_Y}(D(p) \cap Y)$ and $B = i_{\prec_Y}(I(q)) \cap Y = i_{\prec_Y}(I(q) \cap Y)$. Assume that $A \cap B \neq \emptyset$ and let $t \in A \cap B$. There are $u \in D(p) \cap Y$ and $v \in I(q) \cap Y$ such that $v \preceq_Y t \preceq_Y u$, which contradicts the fact that, by Lemma 36, $u < v$. Therefore $A \cap B = \emptyset$.

Now $I(q) \cap Y \subseteq B$, so $d_{<_Y}(A) \cap [I(q) \cup B] = A \cap B = \emptyset$. Also, $D(p) \cap Y \subseteq A$ so $i_{<_Y}(B) \cap [D(p) \cup A] = B \cap A = \emptyset$. Since X is inherently weakly continuous with respect to Y, we know that

$$\begin{aligned} D_\prec(p) \cap I_\prec(q) &= D(d_{\prec_Y}(D(p))) \cap I(i_{\prec_Y}(I(q))) \\ &= D(D(p) \cup A) \cap I(I(q) \cup B) \\ &= \emptyset. \end{aligned}$$

If $p \not\prec q$ then there are $r, s \in Y$ such that $p \leq r \prec_Y s \leq q$. By Lemma 33, $D(r) \cap Y = d_{<_Y}(r)$ and $I(s) \cap Y = i_{<_Y}(s)$. So $d_{\prec_Y}(D(r)) = d_{\prec_Y}(r)$ and $i_{\prec_Y}(I(s)) = i_{\prec_Y}(s)$. Again using Lemma 33 we get

$$\begin{aligned} D_{\prec}(p) \cap I_{\prec}(q) &\subseteq D_{\prec}(r) \cap I_{\prec}(s) \\ &= D\left(d_{\prec_Y}\left(D(r)\right)\right) \cap I\left(i_{\prec_Y}\left(I(s)\right)\right) \\ &= D\left(d_{\prec_Y}(r)\right) \cap I\left(i_{\prec_Y}(s)\right) \\ &= \emptyset. \end{aligned}$$

So $\langle X, \prec \rangle$ is weakly continuous. $\square$

To illustrate one consequence of this theorem, let $n \in \omega$ with $1 < n$ and let $X = \mathbb{R}^n$ with its usual topology and order. Let Y be the set of points in X having integers for all its coordinates. Then X is a normally ordered quasiseparable ordered space. Since Y is discrete, it is a narrow subset of X. If $A, B \subseteq Y$ and $d(A) \cap i(B) = \emptyset$ then $D(A) = d(A)$ and $I(B) = i(B)$, so Y is nicely embedded and extremely continuous in X. And X is inherently weakly continuous with respect to Y. Thus Y is a pliable subset of X.

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