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Consideration of dispersion and group velocity dispersion in the determination of velocities of electromagnetic propagation

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ABSTRACT

Electromagnetic (EM) propagation velocities play an important role in the determination of power and energy flow in materials and interfaces. It is well known that group and phase velocities need to be in opposition in order to achieve negative refractive index. Recently, we have shown that considerable differences may exist in phase, group and signal/energy velocities for normal and anomalous dispersion, especially near dielectric resonances. This paper examines the phase and group velocities in the presence of normal and anomalous dispersion, and group velocity dispersion (GVD), which requires introduction of the second order coefficient in the permittivity and permeability models.

Keywords: group and phase velocity, material dispersion, group velocity dispersion

1. INTRODUCTION

In studying materials (such as complex and metamaterials) and their interactions with EM waves, it is important to have a clear understanding of the nature of phase, group and energy velocities in the medium. In dispersive wave propagation, the phase velocity is different than the group velocity. In negative index materials (NIMs), the phase and group velocities are, furthermore, contradirectional. However, the concept of group velocity automatically implies a collection of frequencies, typically sidebands around a carrier frequency for EM and optics. Also when analyzing propagation of modulated EM waves, one needs to redefine the Poynting vector and the stored energy in terms of the modulation frequency or deviations around the carrier frequency in order to determine the energy velocity as a function of modulation frequency. In recent work, we have examined these three velocities and shown their differences during propagation in a dispersive material, with and without chirality, including only normal and anomalous dispersion [1,2].

Attempts to discriminate between the three velocities often center around either one frequency, or a heuristic extension to multiple frequencies by treating the frequency as a generalized variable. Examples of this generalized frequency approach (i.e., without discrimination between carrier frequency and modulation frequency) exist in the literature – refer to, for instance, the textbook by Haus [3] or the (otherwise) excellent analytical article by Bers that addresses some pertinent instructional aspects of the physical science of EM wave propagation [4]. Furthermore, any derivation of the three velocities must also incorporate the dispersion of material parameters such as permittivity, permittivity, and chirality in the neighborhood of the carrier. Admittedly, a few researchers have attempted to take what we consider a similar view, such as found in the work of Huang and Zhang [5]. A detailed analysis is provided by Stratton [6], who distinguishes between normal and anomalous dispersion, and their effect on the velocities, taking a specific dispersion relation as an example. Second derivatives have not been accounted for in any of the above works: incorporation of the second derivative would lead to group velocity dispersion (GVD) and effects thereof. This is discussed extensively in Agrawal [7]; however, there the velocities of interest are not compared. This paper examines the phase and group velocities in the presence of normal and anomalous dispersion, and group velocity dispersion (GVD), which requires introduction of the second order coefficient in the permittivity and permeability models.

2. PLANE WAVE SOLUTIONS IN SPECTRAL DOMAIN

The four vector EM fields $\overline{D}(\overline{r},t), \overline{B}(\overline{r},t), \overline{E}(\overline{r},t), \overline{H}(\overline{r},t)$ in Maxwell's equations are time varying functions. However, as is true for all constitutive relations, the ones relating $\widetilde{\overline{D}}(\overline{r},\omega), \widetilde{\overline{B}}(\overline{r},\omega), \widetilde{\overline{E}}(\overline{r},\omega), \widetilde{\overline{H}}(\overline{r},\omega)$ are valid in the frequency domain:

$$\tilde{\overline{D}}(\omega) = \tilde{\varepsilon}(\omega)\tilde{\overline{E}}(\omega), \tag{1}$$

$$\tilde{\overline{B}}(\omega) = \tilde{\mu}(\omega)\tilde{\overline{H}}(\omega), \tag{2}$$

where μ_0, ε_0 are the permeability and permittivity of free space, and $\tilde{\varepsilon}(\omega)$ and $\tilde{\mu}(\omega)$ are the frequency dependent electric permittivity and magnetic permeability parameters, often about a carrier frequency ω_0. In order to incorporate the constitutive relations into Maxwell's equations, we need to express each of the vector fields in terms of slowly time varying phasor fields as

$$\overline{C}(r,t) = \mathrm{Re}[\overline{C}_p(r,t)\exp j\omega_0 t], \tag{3}$$

where ω_0 is the carrier frequency, which has the Fourier transform

$$\tilde{\overline{C}}(\omega) = \frac{1}{2}[\tilde{\overline{C}}_p(\omega - \omega_0) + \tilde{\overline{C}}_p^{\ *}(-\omega - \omega_0)]. \tag{4}$$

Now, the material parameters appearing in the constitutive relations are specified in the frequency domain, and similarly must possess a time domain counterpart. Accordingly, we express them as

$$\begin{bmatrix} \varepsilon(t) \\ \mu(t) \end{bmatrix} = \left\{ \begin{bmatrix} \varepsilon_p(t) \\ \mu_p(t) \end{bmatrix} e^{j\omega_0 t} + c.c \right\}, \tag{5}$$

with their Fourier transforms

$$\begin{bmatrix} \tilde{\varepsilon}(\omega) \\ \tilde{\mu}(\omega) \end{bmatrix} = \begin{bmatrix} \tilde{\varepsilon}_p(\omega - \omega_0) + \tilde{\varepsilon}_p^{\ *}(-\omega - \omega_0) \\ \tilde{\mu}_p(\omega - \omega_0) + \tilde{\mu}_p^{\ *}(-\omega - \omega_0) \end{bmatrix}. \tag{6}$$

Equation (6) represents two shifted spectra of the permittivity and permeability, respectively around $\omega = \pm\omega_0$. Note that using (1), (3), (4) and (6), the phasor displacement field may be expressed in terms of the phasor electric field as

$$\tilde{\overline{D}} = \frac{1}{2}\left[\tilde{\overline{E}}_p(\omega - \omega_0)\ \tilde{\varepsilon}_p(\omega - \omega_0) + \tilde{\overline{E}}_p^{\ *}(-\omega - \omega_0)\ \tilde{\varepsilon}_p^{\ *}(-\omega - \omega_0) \right]. \tag{7}$$

A similar relationship can be established for the magnetic flux density and magnetic field. Dispersion of above material parameters in the frequency domain can be expressed as a (McLaurin) series expansion of the shifted spectrum around $\Omega \equiv \omega - \omega_0$. To *first order*, this leads to, after replacing $\Omega \to \omega - \omega_0$,

$$\begin{bmatrix} \tilde{\varepsilon}_p(\Omega) \\ \tilde{\mu}_p(\Omega) \end{bmatrix} = \begin{bmatrix} \tilde{\varepsilon}_p(\omega - \omega_0) \\ \tilde{\mu}_p(\omega - \omega_0) \end{bmatrix} \approx \begin{bmatrix} \tilde{\varepsilon}_{p0} + \Omega\tilde{\varepsilon}'_{p0} + \Omega^2\tilde{\varepsilon}''_{p0}/2 + \Omega^3\tilde{\varepsilon}'''_{p0}/6 + \dots \\ \tilde{\mu}_{p0} + \Omega\tilde{\mu}'_{p0} + \Omega^2\tilde{\mu}''_{p0}/2 + \Omega^3\tilde{\mu}'''_{p0}/6 + \dots \end{bmatrix}. \tag{8}$$

In order to determine possible propagation constants and the corresponding field solutions, we start from Maxwell's curl equations. For instance, from $\overline{\nabla} \times \overline{E} = -\partial \overline{B}/\partial t$, Fourier transforming and using the constitutive relations, we get for the phasor fields ($\omega > 0$) for the relation

$$\overline{k} \times \widetilde{\overline{E}}_p(\Omega) = \omega \widetilde{\mu}_p(\Omega) \widetilde{\overline{H}}_p(\Omega) . \tag{9}$$

Similarly, from the other Maxwell curl equation,

$$\overline{k} \times \widetilde{\overline{H}}_p(\Omega) = \omega \widetilde{\varepsilon}_p(\Omega) \widetilde{\overline{E}}_p(\Omega) . \tag{10}$$

The solutions for the above fields are obtained more readily by assuming a wave vector $\overline{k}(\Omega) = \widetilde{k}_z(\Omega) \hat{a}_z$ pointed in the Z-direction (which is arbitrary, and hence general, in an unbounded medium). This approach simplifies the resulting characteristic matrix to a 4×4 instead of a 6×6. The following set of homogeneous equations for the field components is then obtained from Eqs. (9,10):

$$\begin{aligned}
\widetilde{k}_z \, \widetilde{E}_{py} + \omega \, \widetilde{\mu}_p \, \widetilde{H}_{px} &= 0 \ , \\
\widetilde{k}_z \, \widetilde{E}_{px} - \omega \widetilde{\mu}_p \, \widetilde{H}_{py} &= 0 \ , \\
\omega \, \widetilde{\varepsilon}_p \, \widetilde{E}_{px} - \widetilde{k}_z \, \widetilde{H}_{py} &= 0 \ , \\
\omega \, \widetilde{\varepsilon}_p \, \widetilde{E}_{py} + \widetilde{k}_z \, \widetilde{H}_{px} &= 0 \ ,
\end{aligned} \tag{11a-d}$$

In Eqs. (11a-d), all tilded quantities are functions of Ω. Note that (11a) is identical to (11d), while (11b) and (11c) are identical, provided

$$\widetilde{k}_z = \widetilde{k}_{z1,2} = \pm\omega \sqrt{\widetilde{\mu}_p \, \widetilde{\varepsilon}_p} \ . \tag{12}$$

In Eq. (12), it may be noted that the RHS contains contribution to dispersion from two mechanisms: topological and material. The topological dispersion here is trivial, since we are considering propagation in an unbounded medium. Material dispersion is incorporated through the frequency dependence of the permittivity and permeability. Both types of dispersion can play a role in determining the group velocity, around the carrier frequency. In a medium with topological dispersion only, and no material dispersion, excursion of the generalized frequency ω around the carrier frequency ω_0 should be performed to determine the group velocity. This justifies why we have to retain the generalized frequency ω to begin with in the slowly varying phasor form of the Maxwell's equations Eqs. (9,10). It also turns out (to be seen later) that if instead of writing ω, one used ω_0, the expression for the group velocity does not converge to the standard nondispersive result in the absence of material dispersion.

3. PHASE AND GROUP VELOCITIES

The phase velocity corresponding to $\widetilde{k}_{z1}$ can be expressed as

$$\widetilde{v}_{p1}(\Omega) = \frac{1}{\widetilde{k}_{z1}/\omega} \approx \frac{1}{\sqrt{\widetilde{\varepsilon}_{p0}\widetilde{\mu}_{p0}} \left\{ 1 + \frac{1}{2}\left(\frac{\widetilde{\varepsilon}'_{p0}}{\widetilde{\varepsilon}_{p0}} + \frac{\widetilde{\mu}'_{p0}}{\widetilde{\mu}_{p0}}\right)\Omega + \frac{1}{4}\left(\frac{\widetilde{\varepsilon}''_{po}}{\widetilde{\varepsilon}_{p0}} + \frac{\widetilde{\mu}''_{p0}}{\widetilde{\mu}_{p0}} + 2\frac{\widetilde{\varepsilon}'_{po}}{\widetilde{\varepsilon}_{p0}}\frac{\widetilde{\mu}'_{p0}}{\widetilde{\mu}_{p0}}\right)\Omega^2 + ... \right\}} \hat{a}_z \ . \tag{13}$$

Next, we define

$$v_{g1} = \frac{1}{\dfrac{\partial \widetilde{k}_{z1}}{\partial \omega}} = \frac{1}{\dfrac{\partial \widetilde{k}_{z1}}{\partial \Omega}} \; , \tag{14}$$

where we express $\widetilde{k}_{z1}$ explicitly in terms of the sideband Ω:

$$\widetilde{k}_{z1} = (\omega_0 + \Omega)\sqrt{\widetilde{\mu}_p \, \widetilde{\varepsilon}_p} \tag{15}$$

and differentiate w.r.t. Ω to obtain:

$$\widetilde{v}_{g1}(\Omega) \approx \frac{1}{\sqrt{\widetilde{\varepsilon}_p \widetilde{\mu}_p} + (\omega_0 + \Omega)\dfrac{\partial}{\partial \Omega}\sqrt{\widetilde{\varepsilon}_p \widetilde{\mu}_p}} \, \hat{a}_z \, . \tag{16}$$

After considerable algebra and retaining up to Ω^2 yields:

$$\widetilde{v}_{g1}(\Omega) \approx$$

$$= \frac{1/\sqrt{\widetilde{\varepsilon}_{p0}\widetilde{\mu}_{p0}}}{\begin{bmatrix} \left\{1 + \dfrac{\omega_0}{2}\sqrt{\widetilde{\varepsilon}_{p0}\widetilde{\mu}_{p0}}\left(\dfrac{\widetilde{\varepsilon}'_{p0}}{\widetilde{\varepsilon}_{p0}} + \dfrac{\widetilde{\mu}'_{p0}}{\widetilde{\mu}_{p0}}\right)\right\} + \\ \left\{\left(\dfrac{\widetilde{\varepsilon}'_{p0}}{\widetilde{\varepsilon}_{p0}} + \dfrac{\widetilde{\mu}'_{p0}}{\widetilde{\mu}_{p0}}\right) + \dfrac{\omega_0}{2}\left(\dfrac{\widetilde{\varepsilon}''_{p0}}{\widetilde{\varepsilon}_{p0}} + \dfrac{\widetilde{\mu}''_{p0}}{\widetilde{\mu}_{p0}} + 2\dfrac{\widetilde{\varepsilon}'_{p0}}{\widetilde{\varepsilon}_{p0}}\dfrac{\widetilde{\mu}'_{p0}}{\widetilde{\mu}_{p0}}\right)\right\}\Omega + \\ \left\{\dfrac{3}{4}\left(\dfrac{\widetilde{\varepsilon}''_{p0}}{\widetilde{\varepsilon}_{p0}} + \dfrac{\widetilde{\mu}''_{p0}}{\widetilde{\mu}_{p0}} + 2\dfrac{\widetilde{\varepsilon}'_{p0}}{\widetilde{\varepsilon}_{p0}}\dfrac{\widetilde{\mu}'_{p0}}{\widetilde{\mu}_{p0}}\right) + \dfrac{3\omega_0}{2}\left(\dfrac{1}{6}\dfrac{\widetilde{\varepsilon}'''_{p0}}{\widetilde{\varepsilon}_{p0}} + \dfrac{1}{6}\dfrac{\widetilde{\mu}'''_{p0}}{\widetilde{\mu}_{p0}} + \dfrac{1}{2}\dfrac{\widetilde{\varepsilon}'_{p0}}{\widetilde{\varepsilon}_{p0}}\dfrac{\widetilde{\mu}''_{p0}}{\widetilde{\mu}_{p0}} + \dfrac{1}{2}\dfrac{\widetilde{\varepsilon}''_{p0}}{\widetilde{\varepsilon}_{p0}}\dfrac{\widetilde{\mu}'_{p0}}{\widetilde{\mu}_{p0}}\right)\right\}\Omega^2 + \dots \end{bmatrix}} \hat{a}_z \tag{17}$$

The following points pertain to discussion on comparison between the three velocities derived above:

1. Both velocities have explicit and implicit dependence on the carrier frequency ω_0. Implicit dependence is manifested through values of the permittivity and permeability, and their derivatives, at the carrier frequency.
2. In the non-dispersive limit (primed quantities are zero), we immediately observe that the velocities are equal. Note that both the velocities depend on the values of the permittivity $\widetilde{\varepsilon}_{p0}$ and permeability $\widetilde{\mu}_{p0}$ centered at the carrier frequency, and hence they all implicitly depend on the carrier frequency ω_0.
3. The group velocity has an explicit dependence on the carrier frequency ω_0 even when excursions (Ω) around the carrier frequency are not present. The reason for this is as follows. Note that in expressing $\widetilde{k}_z s$ in (12), we have incorporated the general frequency ω ($= \omega_0 + \Omega$) instead of simply the carrier ω_0 because taking the derivative of $\widetilde{k}_z$ w.r.t. ω *en route* to finding the group velocity using ω_0 instead of ω leads to a result that in the non-dispersive limit does not match the expected classical result. On the other hand, restoring the term ω in

$\widetilde{k}_z$ leads to the correct result for the nondispersive case. An interesting outcome of this approach is the persistence of ω_0 in the expression for group velocity under dispersion, as may be readily verified. This additional contribution is a result of the material dispersion of the permittivity and permeability in the neighborhood of the carrier ω_0, reflected through the operation of taking the partial derivative of the propagation constant *w.r.t.* the frequency ω.

4. Furthermore, the group velocity also has an explicit dependence on the carrier frequency ω_0 in the coefficient of the Ω term when excursions (Ω) around the carrier frequency are present. The reason is clear from the discussion in (2) above.
5. In the presence of dispersion, and in the presence of modulation, the two velocities are, in principle, different from each other.
6. In the presence of GVD, the group velocity gets an additional correction term in first order in Ω, while the phase velocity remains unaffected to first order. In second order, GVD affects both the phase and the group velocities, and moreover, the group velocity gets an additional contribution from third order dispersion.

It is intriguing to speculate on the conditions necessary to achieve contrapropagation of phase and group velocities ($v_p < 0, v_g > 0$) as desired in a NIM. Note that even in the absence of GVD, these conditions imply

$$1 + \frac{1}{2}\left(\frac{\widetilde{\varepsilon}'_{p0}}{\widetilde{\varepsilon}_{p0}} + \frac{\widetilde{\mu}'_{p0}}{\widetilde{\mu}_{p0}}\right)\Omega < 0, \tag{18}$$

$$\left\{1 + \frac{\omega_0}{2}\sqrt{\widetilde{\varepsilon}_{p0}\widetilde{\mu}_{p0}}\left(\frac{\widetilde{\varepsilon}'_{p0}}{\widetilde{\varepsilon}_{p0}} + \frac{\widetilde{\mu}'_{p0}}{\widetilde{\mu}_{p0}}\right)\right\} + \left\{\left(\frac{\widetilde{\varepsilon}'_{p0}}{\widetilde{\varepsilon}_{p0}} + \frac{\widetilde{\mu}'_{p0}}{\widetilde{\mu}_{p0}}\right) + \omega_0\left(\frac{\widetilde{\varepsilon}'_{p0}}{\widetilde{\varepsilon}_{p0}}\frac{\widetilde{\mu}'_{p0}}{\widetilde{\mu}_{p0}}\right)\right\}\Omega > 0. \tag{19}$$

It can be argued from the two conditions above that for NIM behavior as defined, $\dfrac{\widetilde{\varepsilon}'_{p0}}{\widetilde{\varepsilon}_{p0}}, \dfrac{\widetilde{\mu}'_{p0}}{\widetilde{\mu}_{p0}} < 0$. Now, writing

$\dfrac{\widetilde{\varepsilon}'_{p0}}{\widetilde{\varepsilon}_{p0}} = \alpha_\varepsilon, \dfrac{\widetilde{\mu}'_{p0}}{\widetilde{\mu}_{p0}} = \alpha_\mu$ and taking $\alpha_\varepsilon = \alpha_\mu = -\alpha, \alpha \in \mathfrak{R}^+$, it follows from (18), (19) that

$$\alpha\Omega > 1, \ \omega_0\alpha(\alpha\Omega - 1) > 2\alpha\Omega - 1. \tag{20}$$

The latter condition yields two ranges for α, one of which is discarded based on the first condition. This leads to the final condition

$$\alpha > \frac{\omega_0 + 2\Omega + \sqrt{\omega_0^2 + 4\Omega^2}}{2\omega_0\Omega} \approx \frac{1}{\omega_0} + \frac{1}{\Omega}, \ \Omega << \omega_0. \tag{21}$$

While NIM behavior can be achieved through first order dispersion, the inclusion of second and higher order effects is certainly expected to add further flexibilities to attainment of NIM behavior. This will be investigated in the future.

4. CONCLUDING REMARKS

We have examined the phase and group velocities in the presence of normal and anomalous dispersion, and group velocity dispersion (GVD), requiring introduction of the second (and higher) order coefficients in the permittivity and permeability models. Differences between phase and group velocities due to the higher order terms have been indicated.

A case study has been made for obtaining NIM behavior using first order dispersion, which can be extended to include GVD. The derivation of the energy velocity requires computation of the Poynting vector and the stored energies, which involve Fourier transform operations on functions of a component of the slowly time varying electric field phasor. While in the case of only first order dispersion, the resulting energy velocity simplified to only parametric dependence (due to cancellation of like spectral terms in the Poynting vector and the stored energy), the result in the more general case is more involved, and will also be explored in the future.

REFERENCES

[1] Banerjee, P.P. and Chatterjee, M.R., "Negative index in the presence of chirality and material dispersion", J. Opt. Soc. Amer. B **26,** 194-202 (2009).
[2] Chatterjee, M.R. and Banerjee, P.P., "Velocities of EM propagation in dispersive media: a simplified spectral overview", submitted for publication in Amer. J. Phys. (2010).
[3] Haus, H. [Waves and Fields in Optoelectronics], Prentice-Hall, New York, (1983).
[4] Bers, A., "Note on group velocity and energy propagation", Amer. J. Phys. **68,** 482-485 (2000).
[5] Huang, C-C and Zhang, Y-Z, "Poynting vector, energy density and energy velocity in anomalous dispersion medium", Phys. Rev. A **65,** 015802 (2002).
[6] Stratton, J.A. [Electromagnetic Theory], IEEE Press, Piscataway, (2007).
[7] Agrawal, G.P. [Nonlinear Fiber Optics], Academic, San Diego, (2007).