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Star Decompositions of the Complete Split Graph

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Abstract

A graph is a discrete mathematical structure that consists of a set of vertices and a set of edges between pairs of vertices. A problem of interest in graph theory is that of graph decomposition, partitioning the set of edges into disjoint sets, producing subgraphs which are isomorphic to each other. Here we consider the problem of decomposing a class of graphs called complete split graphs into stars of a fixed size. We present conditions for the decomposition as well as an algorithm for the decomposition when it is possible.

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1 Introduction

A graph is a discrete structure in mathematics that consists of a set of vertices and a set of edges which connect pairs of vertices. The number of vertices in a graph is called its order, the number of edges is called its size, and the degree of a vertex is the number of edges incident to that vertex. This is not to be confused with the graphs of functions which are studied in calculus. We are concerned with the structure of graphs alone, and different drawings that represent the same adjacency relationships are treated as the same graph. In addition, in this paper, we only consider simple graphs. That is, there are no loops, and each pair of vertices has at most one edge between them. See below.

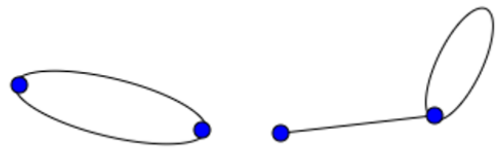

Figure 1: Example of multiple edges and a loop (not allowed)

A graph decomposition is a partitioning of the edges of a graph into disjoint sets in such a way that it produces subgraphs which are isomorphic to each other. In our case, we are interested in decomposing a graph into edge disjoint stars $K_{1,t}$, where t is some fixed number in the positive integers. Note that $K_{1,t}$, t-star, and S_t all denote the same graph. These names will be used interchangeably throughout this paper. Although the decomposition of more common graph classes such as complete graphs and complete bipartite graphs into stars is well-studied [1, 5], decomposition of the complete split graph into stars remains open and is the focus of this research.

A complete split graph can be thought of as a complete graph K_n with the edges of a smaller complete graph K_m removed. We denote this $K_n \setminus K_m$. Alternatively, we can treat the complete split graph as the join of a complete graph K_{n-m} and an independent set of m vertices. Below is an example of a complete split graph with complete subgraph of 4 vertices and independent set of 3 vertices.

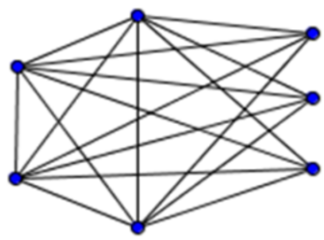

Figure 2: $K_7 \setminus K_3$

We are interested in finding a set of necessary and sufficient conditions for the decomposition of this class of graphs by considering various sizes of the clique, complete subgraph, in the split graph. For sufficient conditions, we provide algorithms for generating a decomposition. Note that in some cases, the decompositions we produce need not be unique. Our focus is simply showing that at least one exists.

2 Preliminary Cases

As with many problems in mathematics, trying to prove a general statement is difficult while certain special cases are easy to show. In this section, we consider a couple preliminary cases for the decomposition of the complete split graph into stars. To restate, our graph in question can be treated as the join of a clique of order $n - m$ and an independent set of order m.

For our first special case, let $m = n - 1$. Then $K_n \setminus K_m \cong K_{1,m}$, a star with m edges. Clearly this is decomposable into t-stars if and only if $t|m$. For the remainder of this paper, we ignore this trivial case and only consider $m < n - 1$.

Decomposition is also trivial for $t = 1$. This is because our star $K_{1,t}$ is simply an edge, and any complete split graph, or any graph for that matter, can easily be decomposed into its edges. Furthermore, due to a result by Kotzig [3], a connected graph is decomposable into paths of length 2 if and only if there are an even number of edges. Since a path of length 2 is isomorphic to $K_{1,2}$, this provides a necessary and sufficient condition for the decomposition of the complete split graph into 2-stars. Having taken care of these preliminary cases, we only consider decomposition into $K_{1,t}$'s where $t \geq 3$ from this point forward.

3 Necessary conditions

It is our goal here to find conditions for the decomposition of the complete split graph into stars in terms of the number of vertices in the clique and independent set that are both necessary and sufficient. We begin our work towards solving this problem in general by finding a necessary condition that holds for all cases.

Let K' denote the complete split graph with clique of order $n - m$ and independent set of order m. Then K' has $\binom{n}{2} - \binom{m}{2}$ edges. This is because K' is isomorphic to a complete graph on n vertices, which contains $\binom{n}{2}$ edges, with the $\binom{m}{2}$ edges of a complete graph of m vertices removed. By treating K' as the join of a clique and an independent set, we see that the number of edges is also equal to $\binom{n-m}{2} + m(n-m) = \frac{(n-m)(n-m-1)}{2} + m(n-m)$. Now, assume that K' has a t-star decomposition for some value t. Then it must be the case that t divides the total number of edges of K'. This gives us the following result, the proof of which follows immediately from the explanation just provided.

Proposition 1. *If a graph K' has a decomposition into $K_{1,t}$'s, then*

$$t| \left[\frac{(n-m)(n-m-1)}{2} + m(n-m)\right]$$

In all cases considered for the rest of this paper, we include the assumption that values of n, m, and t are chosen such that $t| \left[\frac{(n-m)(n-m-1)}{2} + m(n-m)\right]$ In all constructions, we also assume that values are chosen so that we never deal with the impossibility of a negative number of vertices or edges. For example, we cannot have $m > n$.

4 Casework and Conditions for Decomposition

In the preceding section, we presented a necessary condition for the star decomposition of the complete split graph. It comes as no surprise then that the next task is finding sufficient conditions for decomposition. To do this, we break the problem into four major cases. The first case is when $n - m = t$. The other cases are for $n - m < t, t < n - m < 2t$, and $n - m \geq 2t$. The reason for these separations will be explained later. For the first two cases, not only do we provide sufficient conditions, but rather we provide a set of necessary and sufficient conditions for star decomposition. For the final two cases, we present several sets of sufficient conditions and some details that hopefully will lead to an eventual characterization.

4.1 Case 1: $n - m = t$

Begin by letting $K' \cong K_n \setminus K_m$ where $n - m = t$. This is the easiest of the four major cases to consider since we are fixing the order of the clique to be equal to the size of the stars. Without further introduction, we present the following result along with its accompanying proof.

Proposition 2. *K' has an S_t-decomposition if and only if t is odd and $m \geq \frac{t+1}{2}$.*

Proof. Begin by supposing that K' has an S_t-decomposition. Then t must divide the number of edges of K'. Thus $t|[\frac{(n-m)(n-m-1)}{2} + m(n-m)]$. Since $n-m = t$, this means that the total number of stars in the decomposition is $\frac{t(t-1)}{2t} + \frac{mt}{t} = \frac{t-1}{2} + m$. The number of stars must be an integer so t is necessarily odd. Now, we can't have a star centered at every vertex in the independent set of size m, otherwise the remaining stars would have to use only the edges in the clique of size t. This is impossible because each vertex in the clique is adjacent to only $t - 1$ other vertices within the clique. Since some vertex in the independent set cannot be the center of a star, its incident edges can only be covered if it is an end of vertex of a star centered at each of the t vertices in the clique. Since there are at least t stars centered in the clique, we know that $t \leq \frac{t-1}{2} + m$. Therefore $m \geq \frac{t+1}{2}$. One direction of the proof is now complete.

Suppose now that t is odd and $m \geq \frac{t+1}{2}$. We perform the S_t-decomposition as follows. Let $m - \frac{t+1}{2}$ stars be centered in the independent set of size m with edges going to every vertex in the clique of order t. We know this number is at least 0 since $m \geq \frac{t+1}{2}$. The remaining stars will be centered in the clique. Note now that there are exactly $\frac{t+1}{2}$ vertices from the independent set whose edges have not been used yet. Let every vertex in the clique be the center of a star with edges going to these $\frac{t+1}{2}$ vertices. To complete the stars, each needs $\frac{t-1}{2}$ more edges, and the only edges left from K' are the $\frac{t(t-1)}{2}$ edges from the clique. It follows that if we can decompose K_t into t copies of $K_{1,\frac{t-1}{2}}$ then we are done. By a result from Yamamoto et al. [5], we are able to do this. (Note: Yamamoto's result will be looked at more closely later in this paper.) Therefore we have an S_t-decomposition of K', and the proof is complete. $\square$

4.2 Case 2: $n - m < t$

Moving on, we now let $K' \cong K_n \setminus K_m$ where $n - m < t$. As with the previous case, we begin by presenting our result, and provide justification in the form of its following proof.

Proposition 3. *K' has an S_t-decomposition if and only if $t | \frac{n+m-1}{2}$.*

Proof. Begin by supposing that K' has an S_t-decomposition. Then t must divide the number of edges of K'. Thus $t|[\frac{(n-m)(n-m-1)}{2} + m(n-m)]$. Since $n - m < t$, the vertices in the independent set all have degree at most $t - 1$ and can't be the center of any star. Thus all of the stars are centered in the clique of order $n - m$. Suppose for sake of contradiction that the stars centered at two of the vertices in the clique use a different number of edges from the clique. Without loss of generality, suppose that one uses a edges from the clique and another uses b. Since there are less than t vertices in this set, we have that a, $b < t$ and $a \neq b$. Then $m \equiv -a \pmod t$ and $m \equiv -b \pmod t$ since the number of edges used by the stars centered at these vertices must be divisible by t. Thus $a \equiv b \pmod t$, but both are less than t. So $a = b$, a contradiction. Thus all vertices use the same number of edges from the clique for the stars that they are centers of. This means that $(n-m)|\frac{(n-m)(n-m-1)}{2}$. Consequently, each of the $n - m$ vertices uses $\frac{n-m-1}{2}$ edges from the clique for its stars. Therefore $t|(\frac{n-m-1}{2} + m)$. Equivalently, $t|\frac{n+m-1}{2}$, and one direction of the proof is complete.

Suppose now that $t|\frac{n+m-1}{2}$. Then $n + m - 1 = (n - m - 1) + 2m$ is even. Thus $n - m - 1$ is even and $n - m$ is odd. Also, $m \equiv -\frac{n-m-1}{2} \pmod t$. Consider the clique of order $n-m$. Since $n-m < t$, no stars can be centered in the independent set of size m. Label the $n - m$ vertices $v_1, v_2, \ldots, v_{n-m}$. We will perform the decomposition as follows. As described in the previous part, each vertex must use the same number of edges from the clique when building the stars that it is the center of. For the stars centered at each vertex v_i, use the edges v_{i+k}, where $1 \leq i \leq n - m$ and $0 \leq k \leq \frac{n-m-1}{2}$. All of this is done modulo $n - m$. Since $m + \frac{n-m-1}{2} = tu$ for some positive integer u, we also have that $m = tu - \frac{n-m-1}{2}$. Let $w = u - 1$. Then we have $m = tw + (t - \frac{n-m-1}{2})$ where $w \geq 0$ is an integer. Since we have already used $\frac{n-m-1}{2}$ edges towards the stars, we can complete them and additional stars using w sets of t and the surplus $t - \frac{n-m-1}{2}$. This gives us an S_t-decomposition of K', and the proof is complete. □

4.3 Case 3: $n - m \geq 2t$

Let $K' \cong K_n \setminus K_m$ where $n - m \geq 2t$. Unlike, the preceding two cases, this one is not complete. That is, although we have sufficient conditions for our star decomposition, we do not have a set of conditions that is both necessary and sufficient. Before stating our results, we make note of the following two theorems which will be used in our proofs. Note that the variables in these theorems have been renamed to fit our needs.

Theorem 4. *[5] K_{n-m} is S_t-decomposable if and only if $n-m \geq 2t$ and $t|\frac{(n-m)(n-m-1)}{2}$.*

Theorem 5. *[4] Let $a \geq b \geq 1$ be integers. Then $K_{a,b}$ has an S_t-decomposition if and only if $a \geq t$ and $a \equiv 0 \pmod t$ when $t > b$ and $ab \equiv 0 \pmod t$ when $b \geq t$.*

In the above theorem, a and b represent the number of vertices in the clique and independent set without being assigned to either one of them. This is because whichever set has more vertices is irrelevant to the result. Now, we know that we can partition the edges of K' into K_{n-m} and $K_{n-m,m}$ since K' is the edge-disjoint union of these two graphs. Applying the two decomposition theorems above to these two components of K', we get the following result.

Proposition 6. *If $n - m \geq 2t$, $t|\frac{(n-m)(n-m-1)}{2}$, and $t|m(n-m)$, then K' is S_t-decomposable.*

Proof. Let $K' \cong K_n - K_m$ where $n - m \geq 2t$, $t|\frac{(n-m)(n-m-1)}{2}$, and $t|m(n-m)$. Begin by decomposing K' into a complete graph K_{n-m} and a complete bipartite graph $K_{n-m,m}$. By Theorem 4, K_{n-m} has a decomposition into edge disjoint copies of $K_{1,t}$. Similarly, by Theorem 5, $K_{n-m,m}$ has a decomposition into edge disjoint copies of $K_{1,t}$. Therefore every edge of K' belongs to exactly one star, and we have a $K_{1,t}$ decomposition as desired. □

The reader may observe that this looks similar to the converse of our necessary condition for decomposition; however, note that it is in fact different. For a counter example to the converse of Proposition 1 consider $K' \cong K_5 \setminus K_3$. This graph has 7 edges, but it does not have a decomposition into copies of $K_{1,7}$. This is obvious since the graph does not have enough vertices. The following special case follows immediately from the above proposition.

Corollary 7. *If t is odd, $t|n-m$, and $n - m \geq 2t$, then K' is S_t-decomposable.*

This tells us that if $n - m$ is sufficiently large with respect to an odd value of t, then all we need is for t to divide the number of vertices in the clique. When t is even, a slight modification is necessary which we account for below.

Corollary 8. *If $2t|n-m$, then K' is S_t-decomposable.*

Proof. Observe that if $2t|n-m$, then clearly $t|\frac{(n-m)(n-m-1)}{2}$ and $t|m(n-m)$. The rest of the proof follows immediately from Proposition 6. □

Note that our ability to derive the above conditions from Theorems 4 and 5 is why we initially made a distinction between the case for $n - m \geq 2t$ and the case for $t < n - m < 2t$. The following condition for decomposition presents an additional set of decomposable graphs and is proven with a slight adaptation of the proof for the analogous proposition used for the case where $n - m < t$. As such, the proof is not included.

Proposition 9. *If $t|\frac{n+m-1}{2}$, then K' is S_t-decomposable.*

The next result presents a construction technique unused in the previous results. Before stating it though, we mention the following theorem with the variables renamed to suit our needs. In some cases we cannot decompose a graph entirely into edge disjoint sets which are isomorphic to some subgraph H. In such a circumstance, a *maximum packing* is a partitioning into copies of H with the fewest leftover edges. Note that for a packing of edges on a graph, the *leave* of the graph is the set of leftover edges.

Theorem 10. *[2] Let $n, m, t \in \mathbb{Z}^+$ where $n - m \geq 2t$. Then there are $\lfloor \frac{\binom{n-m}{2}}{t} \rfloor$ t-stars in a maximum S_t-packing of K_{n-m}. Moreover, it is possible to have the leave of the graph be a star of size less than t.*

Proposition 11. *If $n-m$ is odd and $m \equiv -1 \pmod t$, then K' is S_t-decomposable.*

Proof. Consider K' where $n - m > 2t$, $n - m$ is odd, and $m \equiv -1 \pmod t$. To be more specific, let $n - m = kt + r'$ and $m = ht - 1$, where $h, k, r' \in \mathbb{Z}^+$, $k \geq 2, 1 \leq r' < t, h \geq 1$. If $t | \binom{n-m}{2}$, then $t | m(n - m)$ so that t can divide the total number of edges. By Proposition 6, we can do the decomposition. Suppose then that $t \nmid \binom{n-m}{2}$. It follows that $m(n - m) \equiv -\frac{(n-m)(n-m-1)}{2} \pmod t$. Since $m \equiv -1 \pmod t$, we can simplify and get that $n - m \equiv \frac{(n-m)(n-m-1)}{2} \pmod t$. We will use this fact shortly.

By Theorem 10, we can find a maximum packing of t-stars in the clique K_{n-m} with $\lfloor \frac{\binom{n-m}{2}}{t} \rfloor$ stars and a leave of $K_{1,r}$, with $0 \leq r < t$. Here $r = \binom{n-m}{2} - \lfloor \frac{\binom{n-m}{2}}{t} \rfloor t$.

Now, we consider the edges between the original clique and the independent set of order m. We do a packing of stars on these edges as follows. Let every vertex in the independent set be the center of k stars, with edges going to the same kt vertices in the clique. The edges left between the independent set and the clique form $K_{r',m}$. For the remaining r' vertices located in the clique, let each be the center of $(h - 1)$ stars with edges going to the same $(h - 1)t$ vertices in the independent set.

The remaining edges from K' that have yet to be used in stars form a $K_{1,r}$ inside the original clique and a $K_{r',t-1}$ between the clique and independent set. Since, $n - m \equiv \frac{(n-m)(n-m-1)}{2} \pmod t$ and we have defined $n - m$ so that $n - m \equiv r' \pmod t$ and $\binom{n-m}{2} \equiv r \pmod t$ with $r < t$ and $r' < t$, we have that $r = r'$. Suppose then that we let the r' vertices left in the $K_{r',t-1}$ be the same as the r vertices of degree 1 left in the clique. We then have $r = r'$ vertices of degree t. Since all of these vertices are pairwise nonadjacent, let each be the center of a $K_{1,t}$ using the remaining edges, and the decomposition is complete. Therefore K' with the assumed conditions is $K_{1,t}$-decomposable. $\square$

So far we have presented several sets of sufficient conditions for decomposition. We now attempt to use the information we have to find necessary and sufficient conditions for the decomposition where $n - m > 2t$.

4.3.1 Pursuing "Necessary and Sufficient" for $n - m \geq 2t$

Suppose K' is $K_{1,t}$-decomposable. Then $t | \binom{n-m}{2} + m(n - m)$ from our necessary condition. We want to break this into cases that we either know how to finish or ones that are impossible, causing a contradiction.

Case 1. Suppose $t | \binom{n-m}{2}$. Then $t | m(n - m)$, otherwise $t \nmid \binom{n-m}{2} + m(n - m)$, a contradiction. By Proposition 6, we know that the decomposition works for this situation, and we move on to the next case.

Case 2. Suppose $t \nmid \binom{n-m}{2}$. Since $t | \binom{n-m}{2} + m(n-m)$, this means that $m(n-m) \equiv -\frac{(n-m)(n-m-1)}{2} \pmod t$. We consider two possibilities for this.

Case 2.1. Suppose $n-m$ and t are relatively prime, that is, their greatest common divisor is 1. Then $m \equiv -\frac{n-m-1}{2} \pmod{t}$. This is equivalent to the statement that $t|\frac{n+m-1}{2}$. By Proposition 9, we know that this case is sufficient and we move on to the next case.

Case 2.2. Suppose then that $gcd(n-m,t) = d > 1$. Suppose further that $t \nmid \frac{n+m-1}{2}$, else we can do the decomposition according to Proposition 9. This is the case which we have not finished. We now list the facts that we know about this case that result from these conditions and not fitting into the other two cases:

- $t|\binom{n-m}{2} + m(n-m)$ (necessary for any decomposition)
- $m(n-m) \equiv -\frac{(n-m)(n-m-1)}{2} \pmod{t}$
- $n-m > 2t$ (all we're considering in this case)
- $t \nmid \frac{n+m-1}{2}$ (otherwise this is covered by Case 2.1)
- $t \nmid \binom{n-m}{2}$ (otherwise this is covered by Case 1)
- $gcd(n-m,t) = d > 1$
- $m \equiv -\frac{n-m-1}{2} \pmod{e'}$ where $t = de'$, $n-m = de$
- $gcd(\frac{n-m-1}{2}, t) = f > 1$
- $gcd(n-m, \frac{n-m-1}{2}) = h > 1$
- $n-m$ is odd
- t is composite

It is our hope in providing this list that it might help in the future with finding more conditions for decompositions or to achieve a contradiction, indicating that set of conditions previously stated is complete.

4.4 Case 4: $t < n-m < 2t$

For our fourth and final case, let $K' \cong K_n \setminus K_m$ where $t < n-m < 2t$. The following are sufficient conditions for the decomposition of K' into edge disjoint copies of $K_{1,t}$. As with Case 3, the results for this section have not been shown to be complete.

Proposition 12. *Let $t < n-m < 2t$. If $t|\frac{n+m-1}{2}$, then K' is S_t-decomposable.*

Proof. Let $K' \cong K_n \setminus K_m$. Suppose $t|\frac{n+m-1}{2}$. Now $\frac{n+m-1}{2} = \frac{n-m-1}{2} + m$. Since this must be an integer, $n-m-1$ is even and $n-m$ is odd. Label the vertices of the order $n-m$ clique $v_1, v_2, \ldots, v_{n-m}$. Suppose each v_i uses the same number of edges from the clique in stars that they are centers of. We we will distribute these edges as follow. Let stars be centered at v_i use edges v_iv_{i+k} where $1 \le i \le n-m$ and $1 \le k \le \frac{n-m-1}{2}$. Note that we calculate $i+k$ modulo $n-m$. Now, since $t|\frac{n+m-1}{2}$, there exists a positive integer u such that $\frac{n+m-1}{2} = m + \frac{n-m-1}{2} = tu$. Let $w = u-1$. Clearly $w \ge 0$ and we can write $m = tw + (t - \frac{n-m-1}{2})$. Note that since

$n - m < 2t$, we know that $t - \frac{n-m-1}{2}$ is a positive integer. We complete the star decomposition of K' as follows. Let every vertex in the clique be the center of w stars going to the w sets of t in the independent set. As mentioned earlier, each of the vertices in the clique is the center of a star using $\frac{n-m-1}{2}$ edges from the clique. These are $S_{\frac{n-m-1}{2}}$'s though. To complete them. Use the surplus $t - \frac{n-m-1}{2}$. Every edge of K' now appears in exactly one edge disjoint copy of $K_{1,t}$ and the proof is complete. □

We proved the following result earlier in this paper and restate it here to help with the proof of the next proposition. Note cautiously that this for K' in this statement, $n - m = t$.

Proposition 2. *Then K' is S_t-decomposable if and only if t is odd and $m \geq \frac{t+1}{2}$.*

Proposition 13. *Let $t < n - m < 2t$. If t is odd, $t|m$ and $n - m = t + 1$, then K' is S_t-decomposable.*

Proof. Let $K' \cong K_n \backslash K_m$. Suppose t is odd, $t|m$ and $n - m = t+1$. We decompose K' as fellows. Begin by noting that since $t|m$, there exists a positive integer k such that $m = tk$. Let a vertex v_1 from the independent set of order m be the center of a star. Since v_1 has degree $t + 1$, this star uses every edge incident to v_1 except for one. Let this edge be incident to u_1 in the clique. Let u_1 be the center of $k + 1$ stars. Since u_1 has degree $m + t = tk + t$, this covers every edge incident to u_1, including the final edge incident to v_1. We have now used every edge involving either u_1 or v_1. To complete the decomposition of K', it follows that we must now decompose the edges of $K' \setminus \{u_1, v_1\}$. Call this graph K''. Clearly $K'' \cong K_{n-1} \setminus K_{m-1}$. This is the join of a clique of order t and an independent set of order $kt - 1$. That is, $n - m = t$ and $m = kt - 1$. By Proposition 2, this is decomposable if and only if t is odd and $kt - 1 \geq \frac{t+1}{2}$. We have assumed that t is odd so the first condition is satisfied immediately. For the second condition to be satisfied, we need $(2k - 1)t \geq 3$. Since k is a positive integer, and we are considering $t \geq 3$, the proof is complete. □

As previously stated and similar to the case where $n-m \geq 2t$, we do not have a set of conditions which are both necessary and sufficient for this case. Completion of this task is left as potential work for the future.

5 Edge Orientation Approach

We now move away from the prior casework and look at the problem in a different way. In this section we approach the problem of finding star decompositions of the complete split graph by considering the use of orientations [2]. An *orientation* of a graph G is defined to be an assignment of a direction to each of its vertices. For each vertex $v \in V(G)$, the *outdegree* of v is defined to be the number of edges incident with v which are directed away from v. Before proceeding to our proposition and its proof, we first make note of the following lemma.

Lemma 14. *[1] Let G be any graph and $f : V(G) \to \mathbb{Z}$. Then G has an orientation in which each vertex $v \in V(G)$ has outdegree $f(v)$ if and only if for every $S \subseteq$*

$V(G)$,

$$\epsilon(S) \leq f(S)$$

with equality if $S = V(G)$.

The following proposition was proven in [2] but we include it here for the sake of completion and replace appropriate variables with the values for our problem.

Proposition 15. *[2] Let $G \cong K_n$ and $H \cong K_m$. Consider the graph $G \setminus H$. Let $d : V(G \setminus H) \to \mathbb{N}$ and $d_0 \in \mathbb{N}$. $G \setminus H$ has an orientation in which each $v \in H$ has outdegree $d(v)$, and each $v \in G \setminus H$ has outdegree d_0 if and only if*

$$(n-m)d_0 + d(H) = \binom{n-m}{2} + m(n-m)$$

Proof. Begin by assuming $G \setminus H$ has an orientation as defined above. Note that the left hand side of the above equation counts the total number of edges in $G \setminus H$ by adding up of the outdegrees of all of the vertices. The right hand side simply counts the total number of edges of the clique K_{n-m} and the complete bipartite portion $K_{n-m,m}$. Thus the two sides of the equation are simply two approaches to count the same thing.

Now, assume $(n-m)d_0 + d(H) = \binom{n-m}{2} + m(n-m)$. We apply Lemma 14 with $f(v) = \{d(v) \text{ for } v \in H, d_0 \text{ for } v \in G \setminus H\}$. Let $S \subseteq V(G)$. (We want to show $\epsilon(S) \leq f(S)$.) Let $I = S \cap (V(G) \setminus V(H))$ and $J = S \cap H$ with $|I| = i$ and $|J| = j$. Then $\epsilon(S) = \binom{i}{2} + ij$ and $f(S) = d_0 i + d(J)$. Then

$$\epsilon(S) \leq f(S) \Leftrightarrow \binom{i}{2} + ij \leq d_0 i + d(J)$$

$$\Leftrightarrow 0 \leq -\frac{1}{2}i^2 + (d_0 - j + \frac{1}{2})i + d(J)$$

We now consider the two cases $i = 0$ and $i = n-m$. If $i = 0$, then our inequality becomes $0 \leq d(J)$ which is clearly true. Now suppose $i = n-m$. Note that $(n-m)d_0 + d(J) = m(n-m) + \binom{n-m}{2} - d(H \setminus J)$. Thus

$$\epsilon(S) \leq f(S) \Leftrightarrow \binom{n-m}{2} + (n-m)j \leq (n-m)d_0 + d(J)$$

$$\Leftrightarrow \binom{n-m}{2} + \sum_{v \in J}(n-m) \leq m(n-m) + \binom{n-m}{2} - d(H \setminus J)$$

$$\Leftrightarrow \sum_{v \in J}(n-m) + d(H \setminus J) \leq m(n-m)$$

Now, we know the following to be true:

$$\sum_{v \in J}(n-m) + d(H \setminus J) \leq \sum_{v \in J}(n-m) + \sum_{v \in H \setminus J}(n-m) = m(n-m)$$

Thus $\epsilon(S) \leq f(S)$ for all $S \subseteq V(G)$. To finish the proof, suppose $S = V(G)$. Then

$$\epsilon(S) = \binom{n-m}{2} + m(n-m) \text{ by definition of } \epsilon$$

$$= (n - m)d_0 + d(H)$$
$$= f(S) \text{ by definition of } f$$

□

We are now ready to apply the concept of edge orientations on a graph to star decompositions. We say that a star decomposition of the complete split graph is *clique-balanced* if every vertex in the clique K_{n-m} is the center of the same number of t-stars.

The following corollary provides necessary and sufficient conditions for the clique-balanced star decomposition of the complete split graph and is a direct consequence of the preceding proposition.

Corollary 16. *$K_n \setminus K_m$ has a clique-balanced t-star decomposition if and only if $t|d(v)$ for all $v \in V(K_m)$ and $t|d_0$ where d_0 is the assigned outdegree of every vertex in the clique K_{n-m}.*

This fully characterizes those graphs which have a clique-balanced decomposition. Thus all other star decompositions have vertices in the clique which are the centers of different numbers of stars. If we remove the requirement that the outdegree is fixed for all vertices in the clique, then we can consider this general definition of decomposition once again. The following proposition directly relates the outdegrees of vertices for a given orientation to the decomposition of the graph into stars.

Proposition 17. *$K_n \setminus K_m$ has a t-star decomposition if and only if $t|d(v)$ for all $v \in V(K_n)$ where $d(v)$ denotes the outdegree of a vertex v.*

Proof. Begin by supposing that $K_n \setminus K_m$ has a t-star decomposition. Consider the edges of each of these stars. Assign an orientation to these edges such that every vertex is directed from the hub, center of the star, to the outer vertices. Thus the outdegree $d(v)$ of each vertex v is a multiple of t, possibly 0 in which case that vertex is the center of no stars. Therefore $t|d(v)$ for all $v \in V(K_n)$.

For the other direction, suppose we have an orientation on the edges of a graph such that $t|d(v)$ for all $v \in V(K_n)$. We decompose our graph into t-stars as follows. For each vertex v with positive outdegree, let that vertex v by the center of a star with edges precisely those directed edges from v. Since this number of edges must be a multiple of t, this gives a t-star decomposition as desired, and the proof is complete. □

We have now considered the decomposition of the complete split graph into fixed stars where every vertex of the clique was the center of the same number of stars. In such a case, we said the the graph $G \setminus H$ had a clique-balanced decomposition where $G \cong K_n$ and $H \cong K_m$. We can generalize this to say that a graph has a G'-balanced star decomposition if $V(G') \subseteq V(G)$, and all vertices in $V(G')$ are the center of the same number of stars. That is, we can find an orientation on the edges in which every vertex in $V(G')$ has the same outdegree.

6 Future Work

In this paper, we considered the fixed star decomposition of the complete split graph, which can be viewed as a complete graph with all edges of a clique removed or as the join of a clique and an independent set. We have shown necessary and sufficient conditions for the star decomposition where the number of vertices in the clique is less than or equal to the number of edges of the supposed star. A starting point for continued research would be to complete the characterization for the remaining cases.

This problem could also be considered more generally by removing the edges of some subgraph H belonging to another class of graphs than just cliques and trying to find a star decomposition.

Although entirely removing the restriction that we fix the size of stars to be some value t would not be interesting, we could consider the decomposition of a graph into stars of size t where t belongs to some finite subset of the positive integers.

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