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Results on Some Generalizations of Interval Graphs

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Abstract
An interval graph is the intersection graph of a family of intervals on the real line. Interval graphs are a well-studied class of graphs. Path graphs are a generalization of interval graphs and are defined to be the intersection graphs of a family of paths in a tree. In this thesis, we study path graphs which are representable in a subdivided $K_{1,3}$. Our main results are a characterization theorem and a polynomial time algorithm for recognition of this class of graphs. The second section of this thesis provides a bound for a graph parameter, the boxicity of a graph, for intersection graphs of subtrees of subdivided $K_{1,n}$. Finally, we characterize k-trees that are path graphs.

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Abstract

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1 Introduction

It is well known that chordal graphs (CH) are the intersection graphs of subtrees of a tree. Path graphs (P) are the intersection graph of subpaths of a tree. Interval graphs (I) are the intersection graphs of subpaths of a path. These three classes of graphs satisfy the inclusion relation

$$I \subset P \subset CH.$$

These classes of graphs have been well studied and are very well understood. Many graph problems that are algorithmically difficult in general are fast to solve on these graphs. These graphs also have many applications to problems in operations research [1]. This thesis considers two extensions of the idea of interval graphs.

The first generalization builds on work done by Prisner in [2]. An interval graph is the intersection graph of a family of intervals on the real line. If we instead allow intervals to exist along exactly two of the positive vertical axis and the positive and negative horizontal axes of the Cartesian plane, we get a class of intersection graphs we call the $3 - star$ path graphs (3SP). These graphs satisfy the following inclusion relation

$$I \subset 3SP \subset P \subset CH.$$

This thesis proposes a characterization theorem and a polynomial time recognition algorithm for this class of graphs.

The second generalization of interval graphs we study involves the boxicity of graphs. The boxicity of a graph is a measure of how close a given graph is to being an interval graph. Boxicity is defined as the minimum number of dimensions n such that a graph has an intersection representation of axis-parallel $n-$dimensional boxes. The importance of studying boxicity is that graphs with low boxicity often admit faster solutions to generally hard to solve graph problems.

2 Background and Definitions

Throughout this thesis we let $G = (V, E)$ be a finite connected and undirected graph. The set V is a set of vertices (or nodes) and E is a set of pairs of vertices called edges. Two vertices are said to be adjacent provided their is an edge between the two of them. For two adjacent vertices $u, v \in V(G)$, we let the edge between them be called uv. In this thesis, the notation $G\backslash C$ indicates the induced subgraph of G formed by removing the vertices in set C from the graph. A graph is a complete graph if every pair of distinct vertices is adjacent. The neighborhood of a vertex v, denoted $N(v)$ is the set of all vertices $u \in V(G)$ such that u is adjacent to v. The closed neighborhood $N[v]$ is the set $N(v) \cup \{v\}$. An independent set S of a graph is a set of vertices no two of which are adjacent. A clique in a graph is a complete subgraph of a graph. A maximal clique is a clique that is not the proper subset of another clique.

A cycle in a graph is a set of vertices $v_1, \dots v_n$, $n \geq 3$ such that $v_i v_{i+1} \in E(G)$ and $v_1 v_n \in E(G)$. A tree is a graph that does not contain any cycles. A chordal graph is one in which every cycle of length at least four has a chord; a chord is an edge connecting two non-consecutive vertices of a cycle. An intersection graph for a family of sets is a graph in which each vertex corresponds to a set in the family and there is an edge between two vertices provided their associated sets have a non-empty intersection. When discussing the set in the family associated with a vertex of the graph, we often call the set the representation of the vertex or we say the vertex is mapped to that set. It is well known that chordal graphs are the intersection graphs of a family of subtrees of a tree [3]. We will work with a few subclasses of chordal graphs in this thesis, the interval graphs and path graphs. An interval graph is the intersection graph of intervals on the real line. Interval graphs are also those graphs that have intersection representations as subpaths of a path [4]. A graph is a path graph provided it is the intersection graph of a family of paths in a tree.

Gavril has shown in [3] that every chordal graph G has an intersection representation as a family of subtrees of a tree, T, where $V(T)$ is the set of maximal cliques of G; such a tree is called a clique tree of G. An asteroidal triple in a graph is a set of three pairwise non-adjacent vertices such that there exists a path between every pair that avoids a neighbor of the third. Lekkerkerker and Boland have characterized interval graphs as the chordal graphs that do not contain an asteroidal triple [5].

We say a set S of vertices of a graph G separates $v_1, v_2 \in V(G)$ if v_1, v_2 lie in different components of $G \setminus S$. A set of vertices is called a separator provided it separates some vertices of the graph. If a set S of vertices is a separator in a graph but no subset of S is a separator, we say S is a minimal separator. Given an interval graph with at least 3 maximal cliques, there is one clique that separates the other two.

In this thesis we look into a subclass of path graphs with a specific structure. Given a graph G and two adjacent vertices $u, v \in V(G)$, we say H is a subdivision of G if H is obtained by removing the edge uv and adding a new vertex, w with w adjacent to only u and v in H. We call a graph an $n - star$ if it can be obtained by repeated subdivision of the complete bipartite graph, $K_{1,n}$. The node of degree n in an $n - star$ is called the middle clique or middle node of the $n - star$. A branch of an $n - star$ is a path obtained by removing the middle node from the $n - star$. We call a graph an $n - star$ graph provided it is the intersection graph of a family of subtrees of an $n - star$. We will look at those graphs that are the intersection graph of subpaths of an $n - star$.

Lastly, in the following discussion we use partial orders to explicitly characterize some graphs. A partial order is a binary relation $\leq$ defined on a set P that is reflexive, transitive, and antisymmetric. A strict partial order is a binary relation $<$ on a set P that is irreflexive, transitive, and antisymmetric. A linear ordering is a partial order where every pair of elements is comparable. A chain is a linearly ordered subset of a partial order. A chain covering of a partial order is a partitioning of the partial order into chains. An element $a \in P$ of a partial order is said to be a maximal element if for all other elements $b \in P$, we have $a \not\leq b$. When we refer to a top element of a chain L in a partial order we mean the element $p \in L$ such that, for all $q \in L$, with $q \neq p$, we have $q < p$. Similarly, a bottom element is an element $p \in L$ such that, for all $q \in L$, with $q \neq p$, we have $p < q$. Note that the top element of a chain may or may not be maximal in the partial order itself.

3 Polynomial Time Algorithm for Recognition of 3-star Path Graphs

In this section we use a partial order first defined by Prisner in recognizing $n-star$ graphs [2]. We call this partial order the middle clique partial order. Let G be an $n - star$ graph with middle clique C. Let $Q = \{q_1, \ldots q_p\}$ be the set of connected components of $G \setminus C$. Define two sets $T(q_i)$ and $B(q_i)$ in the following manner. $T(q_i)$ is the set of all vertices in the middle clique C that are neighbors of *every* vertex in the component q_i. $B(q_i)$ is the set of all vertices in C that are neighbors of *any* vertex in q_i. We say two components q_i and q_j are equivalent if $T(q_i) = B(q_i) = T(q_j) = B(q_j)$. We let R be the set of all equivalence classes under this definition. Prisner has shown the relation defined by $r_i, r_j \in R$ have $r_i < r_j$ provided $B(r_i) \subseteq T(r_j)$ is a poset. An important observation about this partial order is that, given an $n - star$ graph G with middle clique C, a chain in this partial order corresponds to a branch in an $n - star$ representation with middle clique C. We use this fact in the proof of the next theorem.

The following theorem will justify our approach to the recognition algorithm for this class of graphs. Our theorem and algorithm are written for the case of subpaths of a $3 - star$. At the end of this section we describe how we would generalize this result to recognition of graphs which are the intersection graph of subpaths of a $k - star$ for a fixed value k.

Theorem 3.1. *Let G be an $n-star$ graph. G is representable as the intersection graph of subpaths of a 3-star if and only if the middle clique partial order for some middle clique C can be covered with 3 chains with top elements X, Y, Z such that $N[X] \cap N[Y] \cap N[Z] = \varnothing$.*

Proof. $\Rightarrow$ Suppose G is representable as the intersection graph of subpaths of a $3-star$. Let T be such a representation with middle clique C. Recall that a branch in T is equivalent to a chain in the middle clique partial order, $(P, <)$. Since T has 3 branches, $(P, <)$ can be covered with 3 chains. Let the top elements of these three chains be X, Y, Z. Since no vertex v of G is mapped to a subtree of T, for each $v \in V(G)$, v is not in one of $N[X], N[Y]$, or $N[Z]$. Thus $N[X] \cap N[Y] \cap N[Z] = \varnothing$.

$\Leftarrow$ Suppose G is an $n - star$ graph with middle clique C. Let $(P, <)$ be the middle clique partial order for middle clique C. Suppose $(P, <)$ has a covering with three chains with top elements X, Y, Z such that $N[X] \cap N[Y] \cap N[Z] = \varnothing$. Recall that a chain in this partial order can be a branch in a representing with middle clique C. Form a tree T with middle clique C and the three chains as branches. Since this is a clique tree representation for G, each vertex is mapped to a connected subtree. Lastly, since $N[X] \cap N[Y] \cap N[Z] = \varnothing$, each vertex of G must be mapped to a subpath or a single node in this tree T. Therefore T is a subpath representation and G is a $3 - star$ path graph. $\square$

The next theorem is useful in our recognition algorithm. One main step of our algorithm is finding a chain covering of a partial order when the top elements of the chains are dictated. We now show how to do this.

Theorem 3.2. *Let $(P, <)$ be a partial order with $X = \{p_1, \ldots p_n\} \subset P$. For $a, b \in P$, we define $a \prec b$ if and only if $a < b$ and $a \notin X$. Then,*

1. *$(P, \prec)$ is a partial order.*
2. *$(P, <)$ has an n-chain covering with the top element of each chain a member of X if and only if $(P, \prec)$ has an n-chain covering.*

Proof. Let $(P, <)$ be a poset and let $(P, \prec)$ be the relation defined above. We first prove that $(P, \prec)$ is a poset.

Proof of (1).

Antisymmetry: Suppose $a \prec b$ for some $a, b \in P$. By definition, $a < b$. Thus, $b \not< a$ and $b \not\prec a$.

Transitivity: Let $a, b, c \in P$ with $a \prec b$ and $b \prec c$. By definition, $a < b$ and $b < c$ and since $<$ is transitive, we know $a < c$. Since $a \prec b$, it follows that $a \notin X$. Finally, by definition $a < c$ and $a \notin X$ implies $a \prec c$.

Irreflexivity: Since for all $a \in P$, $a \not< a$, it follows readily that $a \not\prec a$.

Proof of (2).

$\Rightarrow$ Suppose that $(P, <)$ has a covering with n chains, $L_1, \ldots L_n$ such that the top of each L_i is p_i. We claim that each of these chains under the relation $<$ is a chain under the relation $\prec$. Let $L_i = a_m < a_{m-1} < \cdots < a_1 < p_i$. By the definition of $\prec$, we have $a_i \prec a_{i-1}$ for each i and $a_1 \prec p_i$. Thus, this list is a chain under $\prec$ and therefore $(P, \prec)$ has a covering with n chains.

$\Leftarrow$ Suppose that $(P, \prec)$ has a covering with n chains, $L_1, \ldots L_n$. Each p_i is maximal with respect to $\prec$ so if p_i is in chain L_i it must be the top element of that chain. Since each p_i is maximal, there is exactly one p_i in each chain, else we would have $p_i \prec p_j$ for some i and j. Further, for $a, b \in P$, $a \prec b$ implies $a < b$ so each L_i is a chain with respect to $<$. Thus, $(P, <)$ has a covering with n chains where some p_i is the top of some chain. $\square$

The first algorithm is the implementation of the preceding theorem.

Algorithm 1 Existence of 3-chain covering with specified top elements

Input: Partial order $(P, <)$ and three elements, $X, Y, Z \in P$.
Output: 'Yes' if there exists a 3-chain covering of $(P, <)$ with top elements X, Y, Z. 'No' else.
Construct partial order $(P, \prec)$ from $(P, <)$.
if $(P, \prec)$ has a 3-chain cover **then**
    Print 'Yes'
else
    Print 'No'
end if

Felsner, Raghavan, and Spinrad have shown how to recognize if a partial order has a $3-chain$ covering in linear time. Thus, Algorithm 1 may run in polynomial time. The next algorithm is the main algorithm in this section. Algorithm 2 shows how to recognize intersection graphs of subpaths of a 3-star in polynomial time.

Algorithm 2 Recognition of intersection graphs of subpaths of a 3-star

Input: Graph G
Output: 'Yes' if G has an intersection representation of subpaths of a 3-star, 'No' else.

if G is not chordal **then**
    Print 'No'
else
    Find all maximal cliques of G
    for All maximal cliques C of G **do**
        Q $\leftarrow$ set of components in $G \setminus C$
        Generate middle clique partial order $(P, <)$ on Q
        if $\forall q \in Q : G[C \cup q]$ is an interval graph **then**
            for All $\{X, Y, Z\} \subseteq Q$ **do**
                if $N[X] \cap N[Y] \cap N[Z] = \varnothing$ **and** $(P, <)$ can be covered with three chains with X, Y, Z top elements **then**
                    Print 'Yes'
                end if
            end for
        end if
    end for
    Print 'No'
end if

Theorem 3.3. *Algorithm 2 outputs 'Yes' if and only if G is an intersection graph of subpaths of a 3-star. Further, Algorithm 2 runs in polynomial time.*

Proof. Let G be a graph. Our Algorithm will output 'Yes' if a middle clique C and three components X, Y, Z of $G \setminus C$ are found that satisfy Theorem 3.1. Since we test all possible middle cliques and all possible triples of components with each middle clique, 'Yes' will be output if and only if G is an intersection graph of subpaths of a 3-star.

Now we will show that Algorithm 2 runs in polynomial time. It may be checked that a graph is chordal in linear time. Thus, to show that the remainder of the algorithm will run in polynomial time, let $G = (V, E)$ be a chordal graph. For a chordal graph, the number of maximal cliques is at most linear with respect to the number of vertices of the graph and these maximal cliques may be found in $O(|V| + |E|)$ time.

Thus, the loop in line 8 occurs a polynomial number of times. In line 9, we may find the set of components of a graph in time proportional to $|V|$. The middle clique partial order may be generated in polynomial time. Since we have a polynomial number of components and interval graphs may be recognized in linear time, line 11 will run a polynomial number of times. There are at most $|V|$ components in Q and the number of triples in Q is bounded by $|Q|^3$, the loop in line 12 runs in polynomial time. Finally, by the remarks after Algorithm 1, we can check the condition in line 13 in polynomial time. Thus our algorithm runs in polynomial time. □

Remark: It should be possible to extend this algorithm to answer the following decision question. Given a graph G, does G have an intersection representation of subpaths of a $k - star$ for a fixed value k. The algorithm will change in the following ways. Instead of searching for three components X, Y, Z that satisfy the intersection criterion, we search for $k - tuples$ of components with the property that every three in this $k - tuple$ have an empty neighborhood intersection. We then use Algorithm 1 to determine if we can cover the partial order with k chains with our $k - tuple$ the tops of these chains. This method will yield a polynomial time solution to that question.

4 A Result on the Boxicity of $n - star$ Graphs

Definition. *The boxicity of a graph G is the minimum n such that G is the intersection graph of n − dimensional axis-parallel boxes.*

The definition of boxicity has an equivalent statement, which we will use in the proof of the next theorem. The boxicity of a graph, G is the minimum n such that there exist interval graphs $I_1, \ldots I_n$ with $V(I_i) = V(G)$ for each i and $E(G) = \bigcap_{i=1}^{n} E(I_i)$.

Theorem 4.1. *If G is an $n - star$ graph, then $box(G) \leq \lceil \frac{n}{2} \rceil$.*

Proof. We will show how to form $\lceil \frac{n}{2} \rceil$ interval graphs I_i that satisfy the definition of boxicity. Let G be an $n - star$ graph and let T be an $n - star$ representation of G with middle clique C. Let $B_1, B_2, \ldots B_n$ be the branches of T. First, for each I_i, let $V(I_i) = V(G)$. Now, form a maximal clique C_i of I_i with vertex set $V(C) \cup V(B_1) \cup V(B_2) \cdots \cup V(B_{2i-2}) \cup V(B_{2i+1}) \cup \cdots \cup V(B_n)$. Note that for each I_i, we keep the maximal cliques in B_{2i} and B_{2i-1} separate from clique C_i. Clearly, using branches B_{2i} and B_{2i-1} with clique C_i between, we immediately have an intersection representation of subpaths of a path for each I_i. Therefore each I_i is an interval graph. We now need only show that $\bigcap_{i=1}^{\lceil \frac{n}{2} \rceil} E(I_i) = E(G)$.

Claim 1: If $uv \in E(G)$, then for all i, $uv \in E(I_i)$.

Let $uv \in E(G)$. Then, there must be some clique of G that contains both u and v. In constructing I_i, we either left cliques unchanged or we combined cliques into a larger clique. In either case, adjacent vertices in G remain adjacent in I_i. Thus, $uv \in E(I_i)$.

Claim 2: If $uv \notin E(G)$, then there exists some i such that $uv \notin E(I_i)$.

Let $u, v \in V(G)$ with $uv \notin E(G)$. Since u and v are non-adjacent they cannot both be in clique C of G. Thus, without loss of generality, assume that the maximal cliques of G containing u are entirely contained in one branch of T. Then, there is some interval graph I_k where $u \notin C_k$. Excluding C_k, all maximal cliques in I_k are maximal cliques of G. Thus, the neighborhood of u in I_k is the same as the neighborhood of u in G and thus $uv \notin E(I_k)$.

□

Definition. *A split graph is a graph where the vertex set can be partitioned into a clique and an independent set.*

Definition. *In a graph, two vertices u, v are called twins if they the have the same set of neighbors.*

If a vertex has a twin, we will call it a *twin vertex.*

Corollary 4.1.1. *Let G be a split graph and let H be the subgraph of G formed by removing twin vertices from the independent set of G until the independent set is twin free. If the size of the independent set of H is n, then G has boxicity at most $\lceil \frac{n}{2} \rceil$.*

Proof. Let G be a split graph with independent set S and clique C. Let H be the subgraph of G described above and suppose the independent set of H has size n. We can see that H is an $n - star$ graph with middle clique C. Each vertex in the independent set of H is in a maximal clique on a distinct branch. It can be seen that G has an $n - star$ representation by noting that the maximal cliques containing twins can be placed consecutively within a branch. Therefore G has boxicity at most $\lceil \frac{n}{2} \rceil$. □

This result slightly improves upon the result by Roberts and Cozzens in [6] in the cases where the independent set is smaller than the clique and the independent set contains multiple twins.

5 Other Results

Definition. *The smallest k-tree is a complete graph on $k + 1$ vertices. A k-tree on $n + 1$ vertices is formed by beginning with a k-tree on n vertices and adding a new vertex to exactly k pairwise adjacent vertices.*

It is known for k-trees that all maximal cliques have size $k+1$ and all minimal separators have size k.

Theorem 5.1. *A k-tree for $k \geq 3$ is a path graph if and only if it is an interval graph.*

Proof. $\Leftarrow$ Let G be a k-tree with $k \geq 3$ and suppose G is an interval graph. Then, G is the intersection of subpaths of a path and is thus a path graph.

$\Rightarrow$ Let G be a k-tree with $k \geq 3$ and suppose that G is not an interval graph; we will show that G is not a path graph. Since G is chordal, it is the intersection graph of subtrees of clique tree T. We want to show that, no matter the choice of T, we can find a vertex of G that is mapped to a subtree of T. Since G is not an interval graph, T cannot be a path. Thus there exists a node $C \in V(T)$ that has degree at least three in T. Let three of the neighbors of C in T be X, Y, Z; recall that each node in T is a maximal clique of G. Since CX, CY, CZ are edges in T, $V(C) \cap V(X), V(C) \cap V(Y), V(C) \cap V(Z)$ are minimal separators of G [7]. Since G is a k-tree, each of these intersections has size k. Now, we want to show that there is a vertex in G that is in each of C, X, Y, Z and is thus mapped to a subtree of T.

Let the vertices in maximal clique C be $\{1, 2, \ldots k+1\}$. Without loss of generality, let those in X be $\{1, 2, 3, \ldots, k, v\}$ where v and $k+1$ are distinct. If 1 is in both of Y and Z, we are done. Thus assume without loss of generality that 1 is not in Y. Then, Y is $\{2, 3, \ldots k+1, u\}$, with u distinct from 1. Since $k \geq 3$, both 2 and 3 are in each of C, X, and Y. Clique Z contains exactly one vertex not in clique C. Therefore, either 2 or 3 must be in Z and thus there is a vertex that is in each of C, X, Y, Z. □

Now, if we use the algorithm from Booth and Leuker [8] for recognition of interval graphs, we have the following result.

Corollary 5.1.1. *$k-trees$ for $k \geq 3$ that are path graphs can be recognized in linear time.*

The property of k-trees we used in proving these results is the relationship between the size of the maximal cliques and the minimal separators of the graph. We note that the size of minimal separators being one smaller than the size of maximal cliques is an unnecessarily strong condition as k grows larger. We believe it is possible to generalize Theorem 5.1 and Corollarry 5.1.1 to a larger class of chordal graphs provided they satisfy some conditions on the size of their maximal cliques and minimal separators.

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