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Compactly Supported Homeomorphisms as Long Direct Limits

Rafael Dahmen
Technische Universitat Darmstadt, dahmen@mathematik.tu-darmstadt.de

Gábor Lukács
lukacs@topgroups.ca

Compactly supported homeomorphisms as long direct limits

Rafael Dahmen

(joint work with Gábor Lukács)

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Outline of the talk

1. Compactly supported homeomorphisms
2. Homeomorphisms of the Real Line
3. Homeomorphisms of the Long Line

① Compactly supported homeomorphisms

Let X be a connected locally compact Hausdorff space

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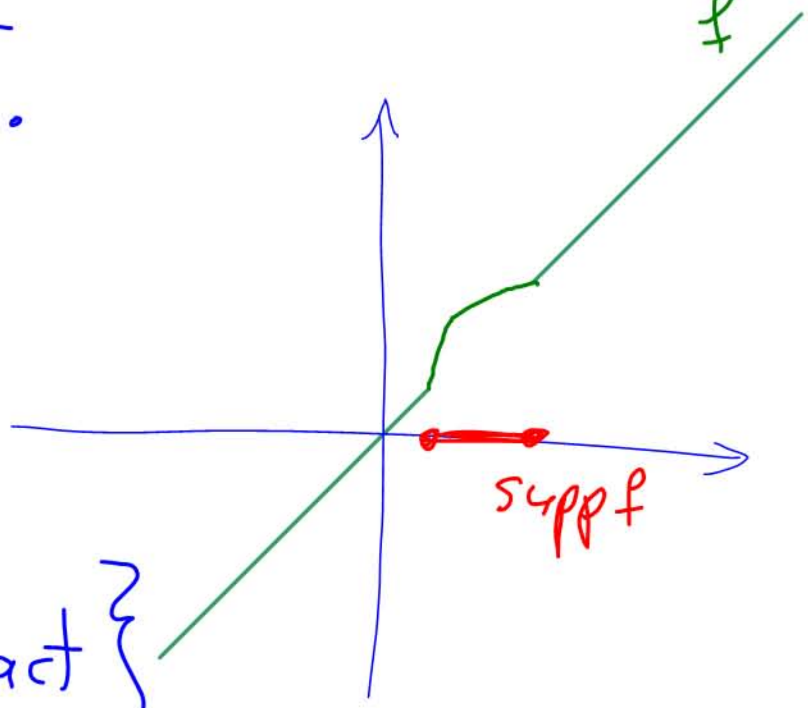
Definition:

- For a homeomorphism $f \in \mathrm{Homeo}(X)$:

$$\operatorname{supp} f := \overline{\{x \in X : f(x) \neq x\}}.$$

(support of f)



$$\mathrm{Homeo}_{cpt}(X) := \{f \in \mathrm{Homeo}(X) : \operatorname{supp} f \text{ compact}\}$$

① Compactly supported homeomorphisms

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- For a compact set $K \subseteq X$

  $\mathrm{Homeo}_K(X) = \{ f \in \mathrm{Homeo}(X) : \mathrm{supp}\, f \subseteq K \}$

1 Compactly supported homeomorphisms

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  $$\mathrm{Homeo}_K(X) = \{f \in \mathrm{Homeo}(X) : \operatorname{supp} f \subseteq K\}$$

- $$\mathrm{Homeo}_{cpt}(X) = \bigcup_{\substack{K \subseteq X \\ \text{compact}}} \mathrm{Homeo}_K(X)$$

Let X be a connected locally compact Hausdorff space

Fact:

$\mathrm{Homeo}(X)$

is a topological group w.r.t. compact open topology

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Remark:

If X is not connected,

inversion may be discontinuous w.r.t. compact open topology

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Question:

How to topologize $\mathrm{Homeo}_{cpt}(X)$?

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(a) subspace topology: $\mathrm{Homeo}_{cpt}(X) \hookrightarrow \mathrm{Homeo}(X)_{c.o.}$

→ topological group

→ topology does not use the directed union $\bigcup_K \mathrm{Homeo}_K(X)$

Let X be a connected locally compact Hausdorff space

Question:

How to topologize $\mathrm{Homeo}_{cpt}(X)$?

(a) subspace topology: $\mathrm{Homeo}_{cpt}(X) \hookrightarrow \mathrm{Homeo}(X)_{c.o.}$

$\rightarrow$ topological group

(b) direct limit in TOP: $\mathrm{Homeo}_{cpt}(X) = \varinjlim_{K}^{TOP} \mathrm{Homeo}_K(X)$

$\rightarrow$ universal property in TOP

$\rightarrow$ inversion continuous

$\rightarrow$ multiplication may be discontinuous

$\Rightarrow$ no topological group!

Let X be a connected locally compact Hausdorff space

Question:

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$\to$ topological group

(b) direct limit in TOP: $\mathrm{Homeo}_{cpt}(X) = \varinjlim_{K}^{TOP} \mathrm{Homeo}_K(X)$

$\to$ universal property in Top

(c) direct limit in TOPGRP: $\mathrm{Homeo}_{cpt}(X) = \varinjlim_{K}^{TOPGRP} \mathrm{Homeo}_K(X)$

$\to$ topological group

$\to$ universal property in TOPGRP

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Fact: On $\mathrm{Homeo}_{cpt}(\mathbb{R})$ the topologies (a), (b), (c) are different!

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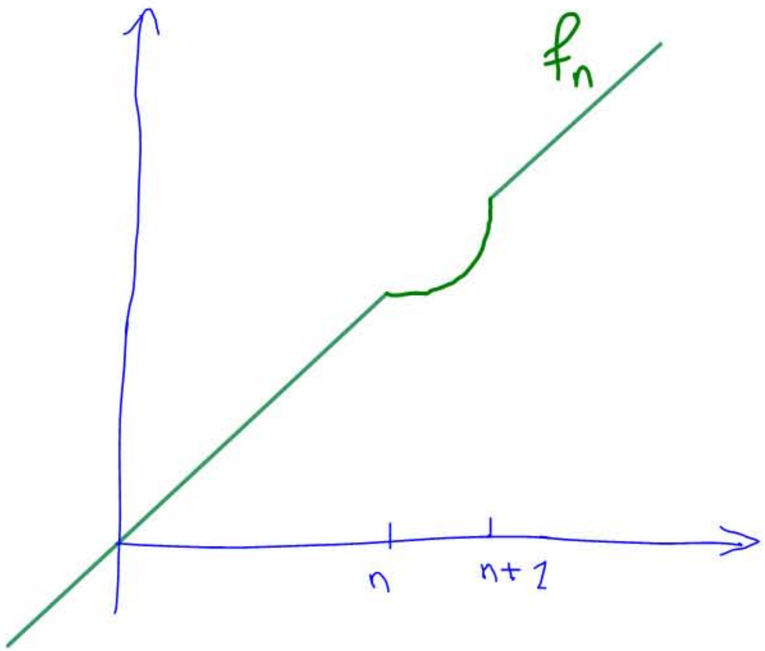

$f_n \longrightarrow \mathrm{id}$ in compact-open-topology (a)

but $(f_n)_n$ is divergent in $\underrightarrow{\mathrm{Lim}}^{\mathrm{TOP}}$ (b)

and in $\underrightarrow{\mathrm{Lim}}^{\mathrm{TOPGRP}}$ (c)

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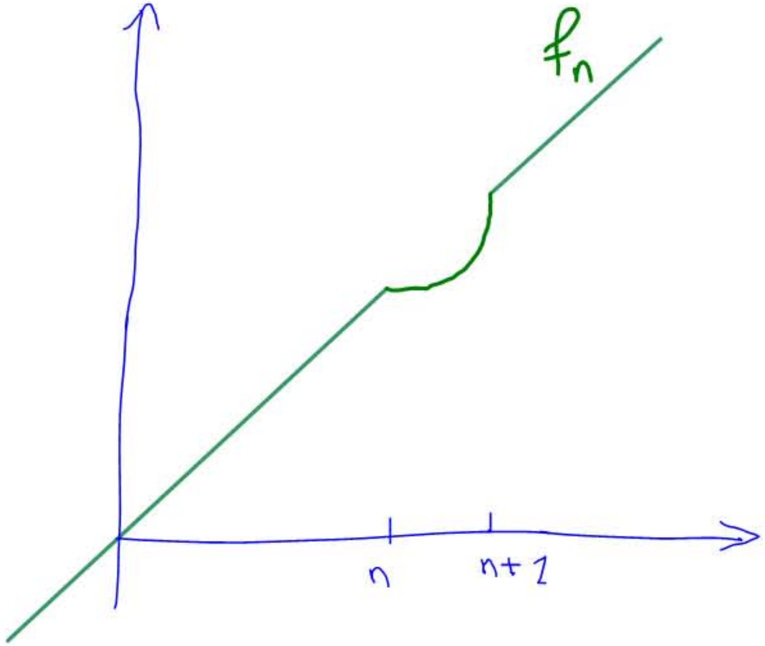

$f_n \longrightarrow id$ in compact-open-topology (a)

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This shows: (a) $\neq$ (b) and (a) $\neq$ (c).

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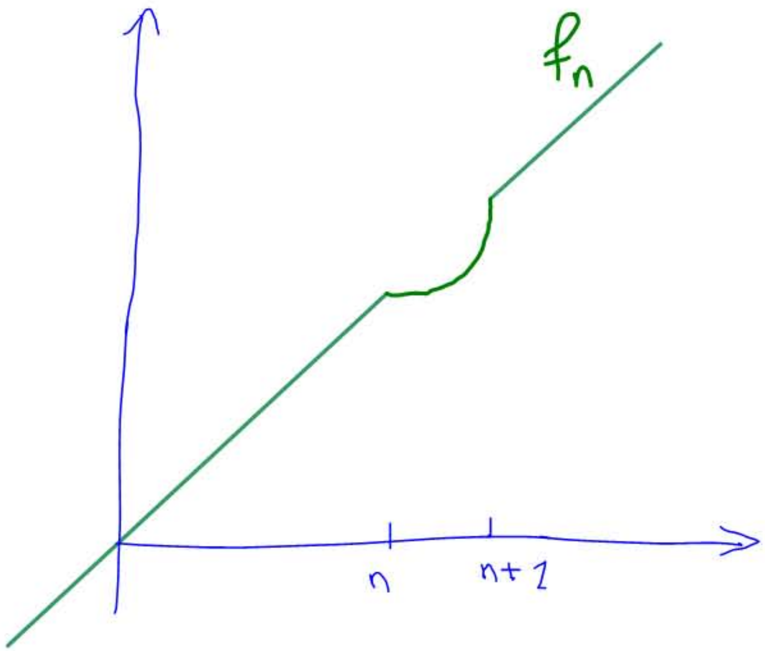

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This shows: (a) $\neq$ (b) and (a) $\neq$ (c).

Why is (b) $\neq$ (c)?

$$G \simeq \varinjlim_{K}^{\text{TOP}} \mathrm{Homeo}_K(\mathbb{R}) \qquad (b)$$

$$G = \varinjlim_{K}^{\mathrm{TOP}} \mathrm{Homeo}_K(\mathbb{R})$$ (b)

$$G = \varinjlim_{n}^{\mathrm{TOP}} G_n \quad \text{with} \quad G_n = \mathrm{Homeo}_{[-n,n]}(\mathbb{R})$$

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<u>countable</u> direct limit in TOP.

$$G_1 \subseteq G_2 \subseteq G_3 \subseteq G_4 \subseteq \cdots \qquad G = \bigcup_{n=1}^{\infty} G_n$$

$$G = \varinjlim_{K}^{TOP} \mathrm{Homeo}_K(\mathbb{R}) \quad (b)$$

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<u>countable</u> direct limit in TOP.

<u>Theorem</u> (Yamasaki 1998)

$G_1 \subseteq G_2 \subseteq G_3 \subseteq \cdots$ increasing sequence of top. groups

- G_k closed but not open in G_{k+1}
- G_k metrizable but not locally compact

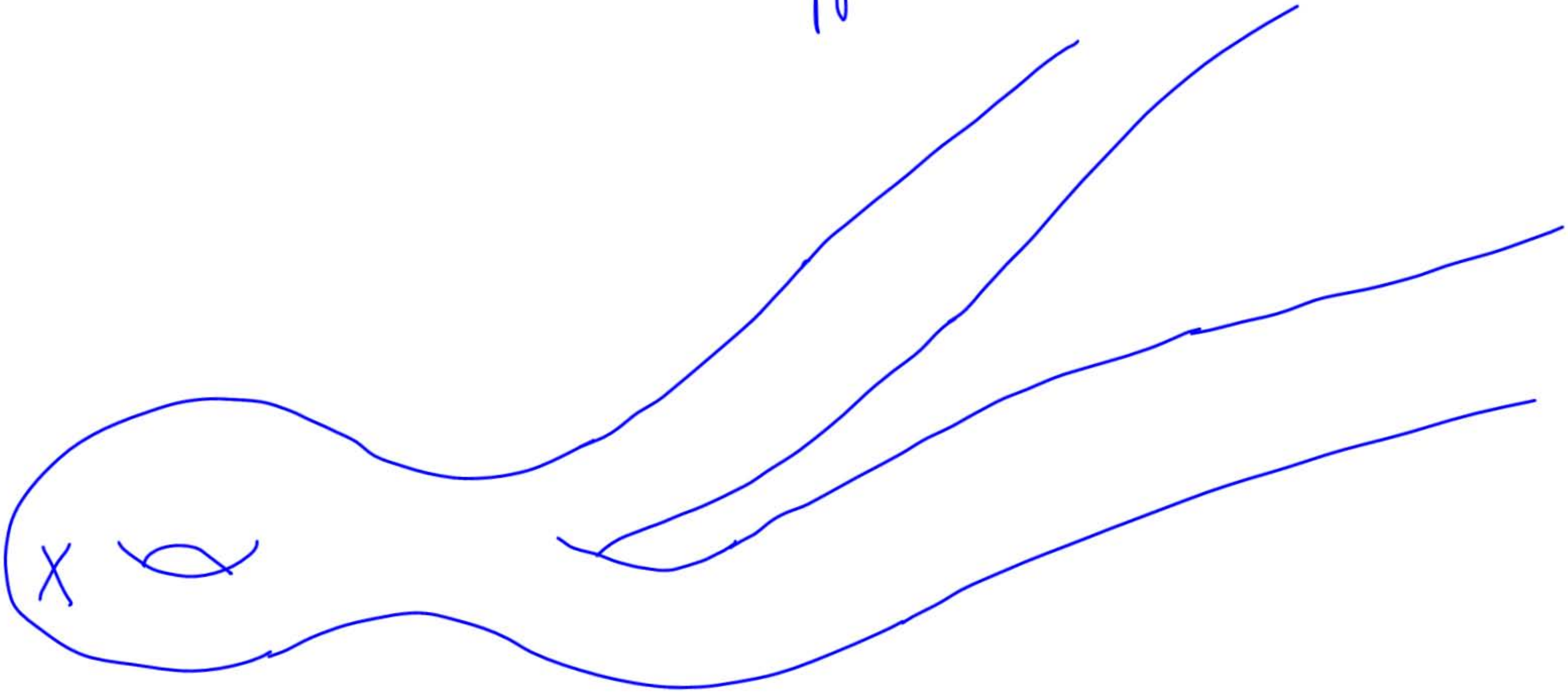
$\Longrightarrow$ multiplication is discontinuous on $\left(\varinjlim_{n}^{TOP} G_n\right) \times \left(\varinjlim_{n}^{TOP} G_n\right)$

$$(b) \ \varinjlim_{n}^{TOP} G_n \neq \varinjlim_{n}^{TOPGRP} G_n \ (c)$$

Similarly,

If X is a connected, hausdorff, σ-compact, non-compact top. manifold

Then the same applies.



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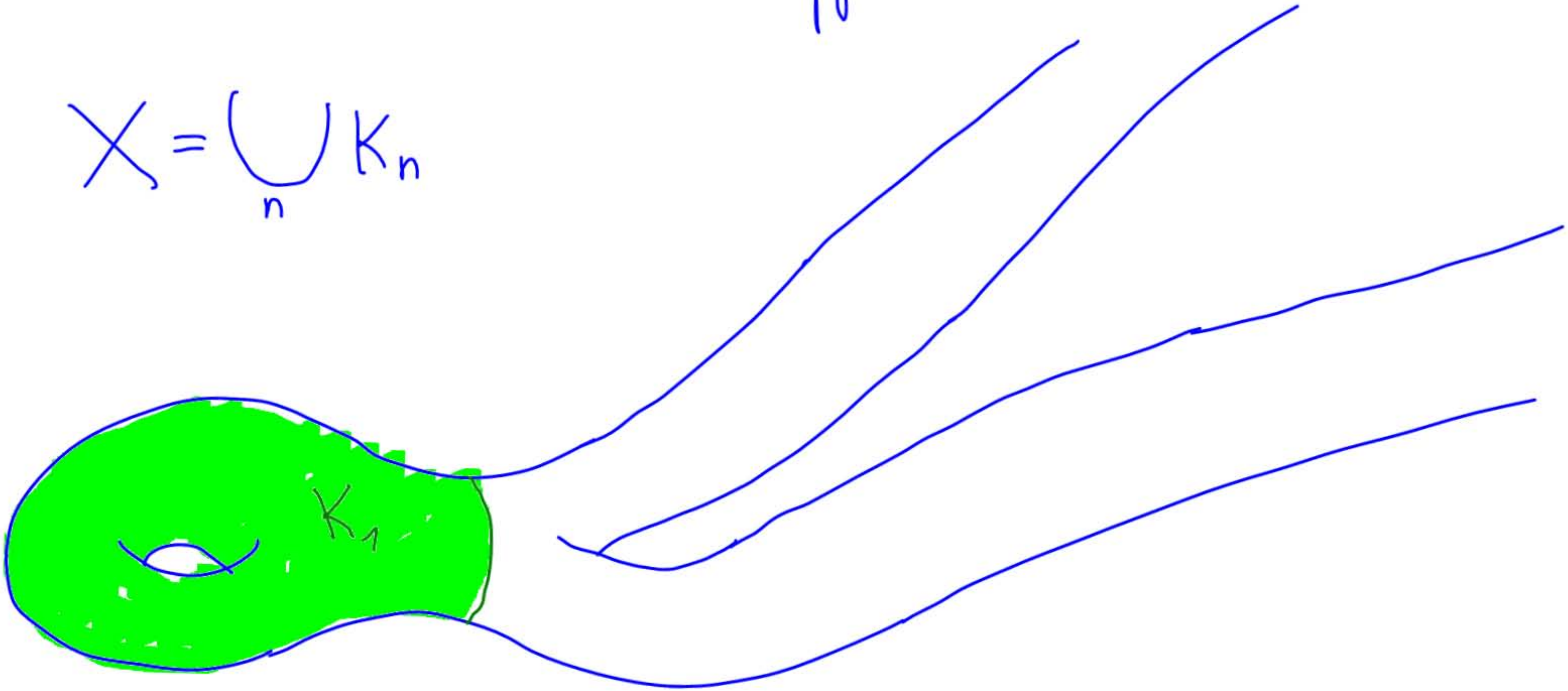

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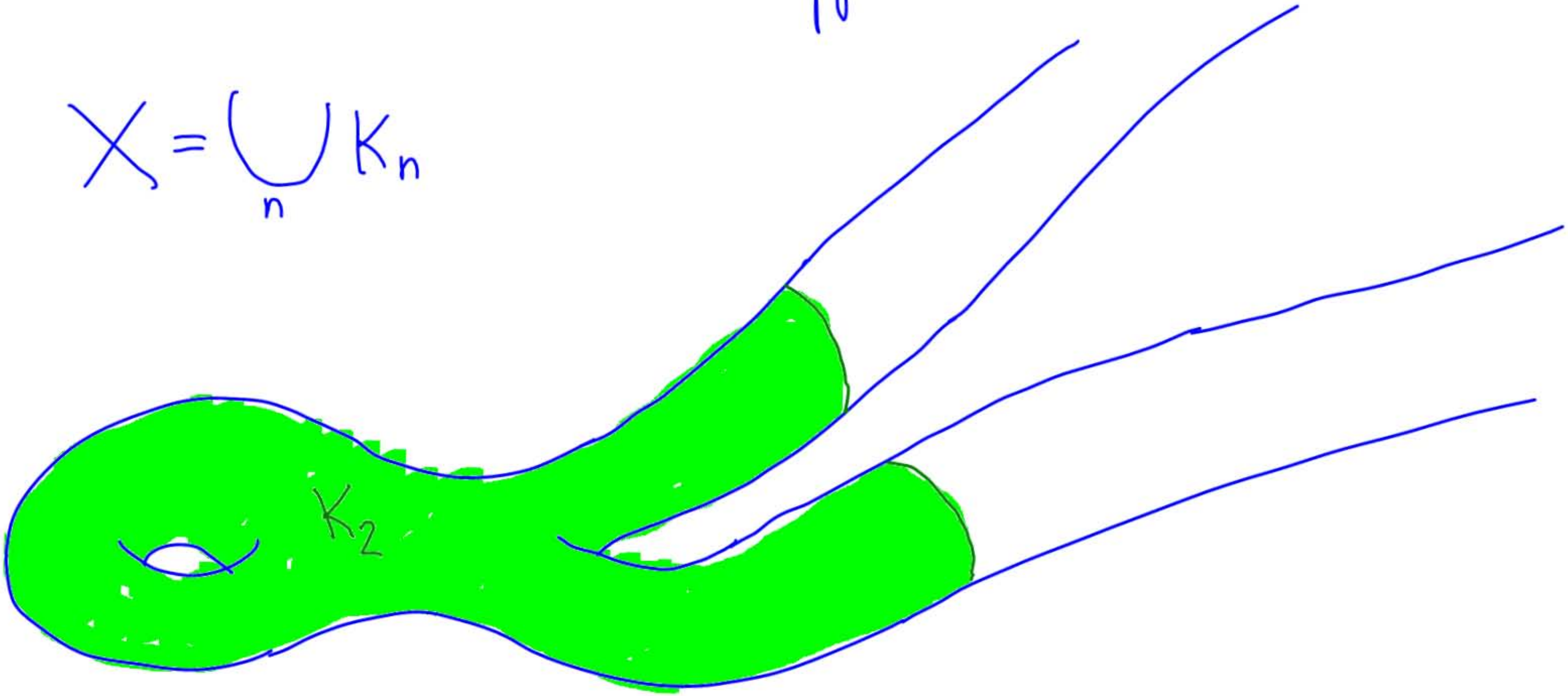

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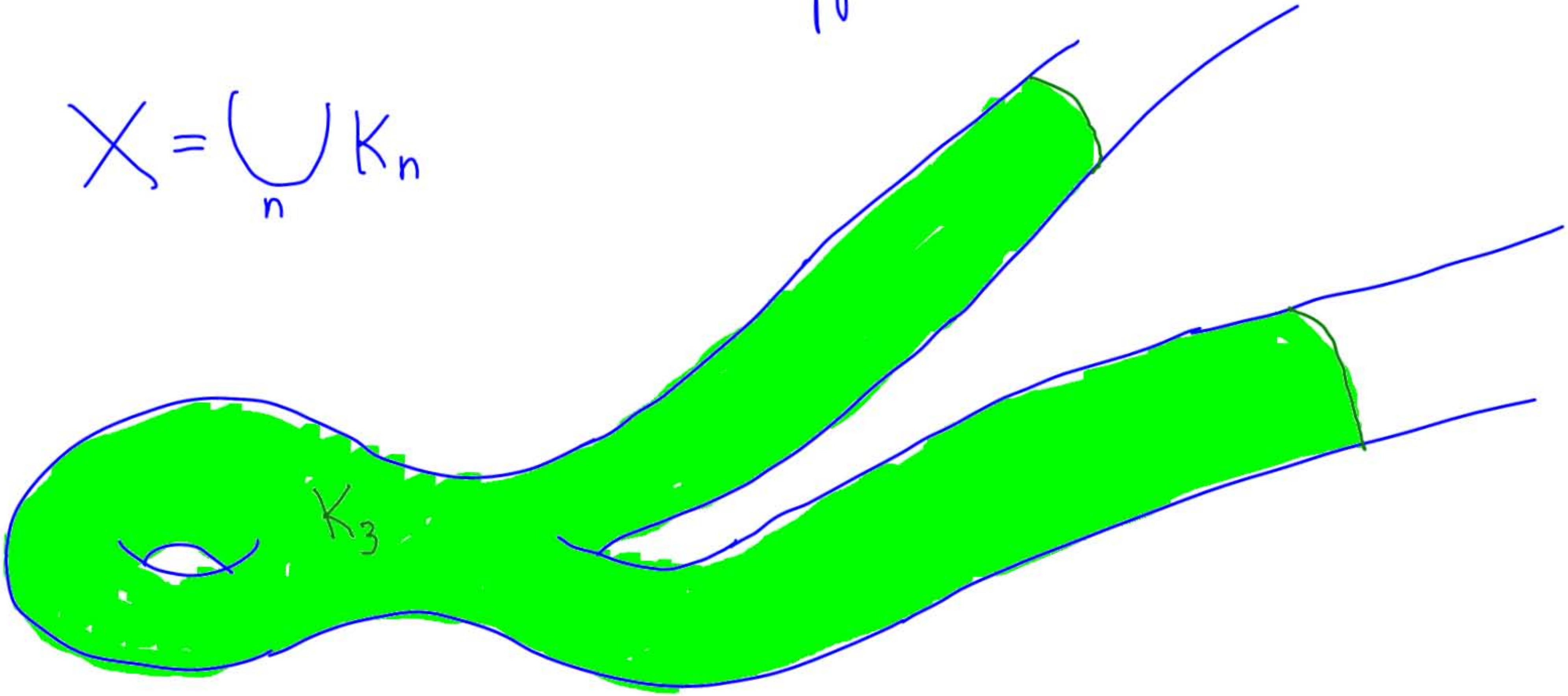

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$$X = \bigcup_n K_n$$

$$\mathrm{Homeo}_{cpt}(X) = \bigcup_{n=1}^{\infty} \mathrm{Homeo}_{K_n}(X)$$

$\Rightarrow$ By Yamasaki $\boxed{(b)\ \varinjlim_n^{TOP} \neq \varinjlim_n^{TOPGRP}\ (c)}$

③ Homeomorphisms of the Long Line

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Recall: • $\mathbb{L}_c^+ = \omega_1 \times [0,1[$ with lexicographic order and the order topology

$$\mathbb{L}_c^+ = \left\{ \alpha + t \ : \ \alpha < \omega_1 \ ; t \in [0,1[\right\} \supseteq \omega_1$$

"Closed Long Ray"

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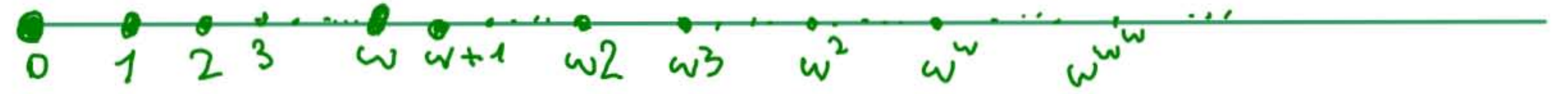
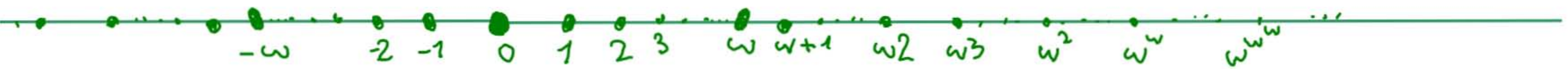


• $\mathbb{L}$: Take two copies of $\mathbb{L}_c^+$ and glue them together at their boundary point.



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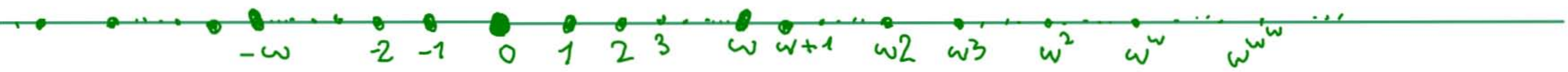

Properties: • $\mathbb{L}_c^+$ connected, 1-dim. top manifold with boundary

• $\mathbb{L}$ connected, 1-dim. top manifold

$\mathbb{L}, \mathbb{L}_c^+$ are NOT metrizable
NOT paracompact
NOT σ-compact

Theorem (D., Lukács 2016)

Let $X = \mathbb{L}$ (Long Line)

or $X = \mathbb{L}_c^+$ (Closed Long Ray)

THEN on $\mathrm{Homeo}_{cpt}(X)$ the topologies

(a) , (b) , (c) coincide

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$$\underset{(a)}{\mathrm{Homeo}_{cpt}(X)_{\text{compact-open}}} = \underset{(b)}{\varinjlim_{K}{}^{TOP} \mathrm{Homeo}_K(X)} = \underset{(c)}{\varinjlim_{K}{}^{TOPGRP} \mathrm{Homeo}_K(X)}$$

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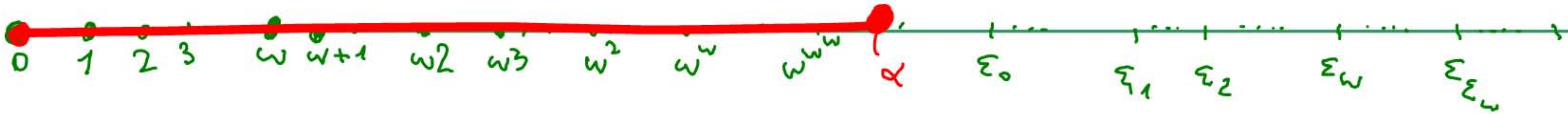
$$\mathrm{Homeo}_{cpt}(X)_{\text{compact-open}} = \varinjlim_{K}{}^{\mathrm{TOP}}\, \mathrm{Homeo}_K(X) = \varinjlim_{K}{}^{\mathrm{TOPGRP}}\, \mathrm{Homeo}_K(X)$$

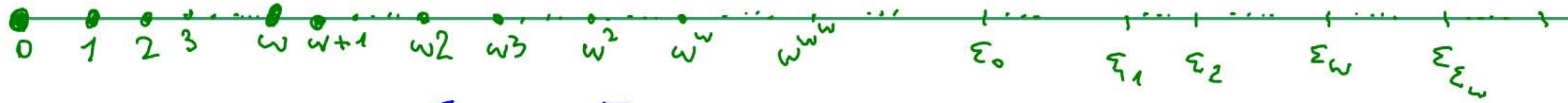
How to show that?

$$\mathbb{L}_c^+ = \bigcup_{\alpha<\omega_1} [0,\alpha]_{\mathbb{L}}$$



$\forall \alpha < \omega_1$

$$\alpha \neq 0 \Rightarrow \quad [0,\alpha]_{\mathbb{L}} \cong [0,1]_{\mathbb{R}}$$



$$\mathbb{L}_c^+ = \bigcup_{\alpha<\omega_1} [0,\alpha]_{\mathbb{L}}$$

$$\underbrace{\mathrm{Homeo}_{cpt}(\mathbb{L}_c^+)}_{G} = \bigcup_{\alpha<\omega_1} \underbrace{\mathrm{Homeo}_{[0,\alpha]_{\mathbb{L}}}(\mathbb{L}_c^+)}_{G_\alpha}$$

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$$G_1 \hookrightarrow G_2 \hookrightarrow G_3 \hookrightarrow \cdots \hookrightarrow G_\omega \hookrightarrow G_{\omega+1} \hookrightarrow \cdots \hookrightarrow G_{\omega^2} \hookrightarrow G_{\omega^\omega} \hookrightarrow$$

$$\boxed{G = \bigcup_{\alpha<\omega_1} G_\alpha}$$

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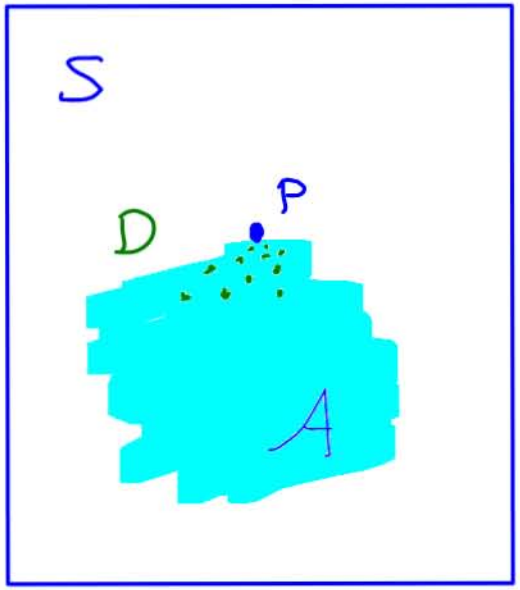
How to prove $\varinjlim^{\mathrm{TOP}} = \varinjlim^{\mathrm{TOPGRP}}$?

Recall: A topological space S is countably tight if

$\forall p \in S$

$\forall A \subseteq S$ with $p \in \overline{A}$

$\exists D \subseteq A$ countable with $p \in \overline{D}$



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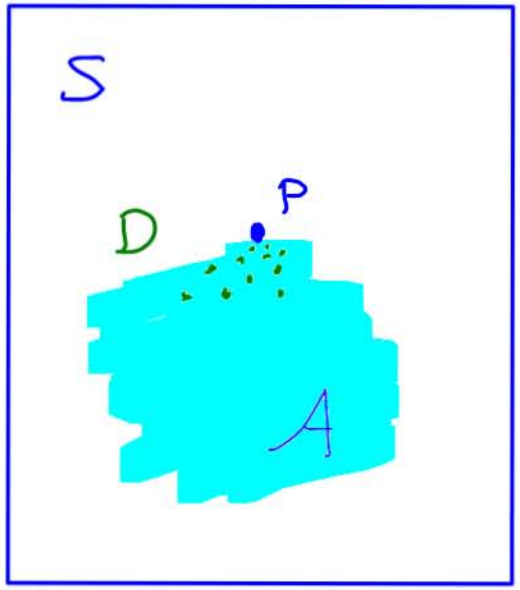

metrizable $\Rightarrow$ first countable

$\Rightarrow$ Fréchet-Urysohn

$\Rightarrow$ sequential

$\Rightarrow$ countably tight

Main Lemma (D., Lukács 2016)

$(S_\alpha, \tau_\alpha)_{\alpha<\omega_1}$ topological spaces

$\forall \alpha \quad (S_\alpha, \tau_\alpha)$ countably tight

$(\forall \alpha \leq \beta) \quad S_\alpha \hookrightarrow S_\beta$ topological embedding

$$S := \bigcup_{\alpha<\omega_1} S_\alpha$$

τ topology on S

THEN

$$\boxed{(S,\tau) = \varinjlim^{\mathrm{Top}} (S_\alpha, \tau_\alpha)} \iff \boxed{\begin{array}{l} \bullet\ (S_\alpha, \tau_\alpha) \hookrightarrow (S,\tau) \text{ top. emb.} \\ \bullet\ (S,\tau) \text{ countably tight} \end{array}}$$

Given $(G_\alpha)_{\alpha<\omega_1}$ topological groups

$\forall \alpha \quad G_\alpha$ countably tight

$(\forall \alpha \leq \beta)\ G_\alpha \hookrightarrow G_\beta$ topological embedding of top. groups

$$G := \bigcup_{\alpha<\omega_1} G_\alpha$$

Given $(G_\alpha)_{\alpha<\omega_1}$ topological groups

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Conjecture (D., 2016)

$$\underset{\alpha<\omega_1}{\underrightarrow{\mathrm{Lim}}}^{\mathrm{TOP}}\, G_\alpha = \underset{\alpha<\omega_1}{\underrightarrow{\mathrm{Lim}}}^{\mathrm{TOPGRP}}\, G_\alpha$$

Given $(G_\alpha)_{\alpha<\omega_1}$ topological groups

$\forall \alpha$ G_α countably tight

$(\forall \alpha \leq \beta)$ $G_\alpha \hookrightarrow G_\beta$ topological embedding of top. groups

$$G := \bigcup_{\alpha<\omega_1} G_\alpha$$

How to show that

$$\boxed{\varinjlim^{TOP} G_\alpha = \varinjlim_{\alpha}^{TOPGRP} G_\alpha} \ ?$$

Given $(G_\alpha)_{\alpha<\omega_1}$ topological groups

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How to show that

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Strategy: • Guess a topology $\mathcal{T}$ on G

Given $(G_\alpha)_{\alpha<\omega_1}$ topological groups

$\forall \alpha$ G_α countably tight

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$$G := \bigcup_{\alpha<\omega_1} G_\alpha$$

How to show that

$$\boxed{\varinjlim^{\mathrm{TOP}} G_\alpha = \varinjlim_{\alpha}^{\mathrm{TOPGRP}} G_\alpha} \; ?$$

Strategy:
- Guess a topology τ on G
- Show: $G_\alpha \longrightarrow (G,\tau)$ top. embedding

Given $(G_\alpha)_{\alpha<\omega_1}$ topological groups
$\forall\alpha$ G_α countably tight

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- Show: (G,τ) countably tight

Given $(G_\alpha)_{\alpha<\omega_1}$ topological groups

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$$G := \bigcup_{\alpha<\omega_1} G_\alpha$$

How to show that

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Strategy:
- Guess a topology τ on G
- Show: $G_\alpha \longrightarrow (G,\tau)$ top. embedding
- Show: (G,τ) countably tight

$\Big\} \Longrightarrow (G,\tau) = \underrightarrow{\operatorname{Lim}}_{\alpha}^{TOP} G_\alpha$

by Main Lemma

Given $(G_\alpha)_{\alpha<\omega_1}$ topological groups

$\forall \alpha$ G_α countably tight

$(\forall \alpha \leq \beta)$ $G_\alpha \hookrightarrow G_\beta$ topological embedding of top. groups

$$G := \bigcup_{\alpha<\omega_1} G_\alpha$$

How to show that

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Strategy:
- Guess a topology τ on G
- Show: $G_\alpha \hookrightarrow (G,\tau)$ top. embedding
- show: (G,τ) countably tight

$\Big\} \Rightarrow (G,\tau) = \varinjlim_{\alpha}^{TOP} G_\alpha$

- Show: group multiplication

$$(G,\tau) \times (G,\tau) \longrightarrow (G,\tau) \text{ continuous}$$

Given $(G_\alpha)_{\alpha<\omega_1}$ topological groups

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- Guess a topology τ on G
- Show: $G_\alpha \longrightarrow (G,\tau)$ top. embedding
- Show: (G,τ) countably tight

$\left.\right\} \Rightarrow (G,\tau) = \varinjlim_{\alpha}^{\mathrm{TOP}} G_\alpha$

- Show: group multiplication $(G,\tau)\times(G,\tau) \longrightarrow (G,\tau)$ continuous

$\Rightarrow (G,\tau) = \varinjlim_{\alpha}^{\mathrm{TOP}} G_\alpha = \varinjlim_{\alpha}^{\mathrm{TOPGRP}} G_\alpha$

Back to

$$G = \mathrm{Homeo}_{cpt}(\mathbb{L}_c^+) \qquad G_\alpha = \mathrm{Homeo}_{[0,\alpha]_{\mathbb{L}}}(\mathbb{L}_c^+)$$

$$G = \bigcup_{\alpha < \omega_1} G_\alpha$$

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$\tau :=$ compact open topology on G, i.e.

$$(G, \tau) \hookrightarrow \mathrm{Homeo}(\mathbb{L}_c^+)_{c.o.} \qquad (a)$$

Back to

$$G = \mathrm{Homeo}_{cpt}(\mathbb{L}_c^+) \qquad G_\alpha = \mathrm{Homeo}_{[0,\alpha]_{\mathbb{L}}}(\mathbb{L}_c^+)_{k..}$$

$$G = \bigcup_{\alpha < \omega_1} G_\alpha$$

τ := compact open topology on G (a)

Strategy:

- Show: $G_\alpha \hookrightarrow (G,\tau)$ top. embedding
- Show: (G,τ) countably tight
- Show: group multiplication
  $(G,\tau) \times (G,\tau) \to (G,\tau)$ continuous

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$$G = \mathrm{Homeo}_{cpt}(\mathbb{L}_c^+) \qquad G_\alpha = \mathrm{Homeo}_{[0,\alpha]_{\mathbb{L}}}(\mathbb{L}_c^+)_{\ldots}$$

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$\tau :=$ compact open topology on G (a)

Strategy:

- Show: $G_\alpha \hookrightarrow (G, \tau)$ top. embedding ✓
- Show: (G, τ) countably tight
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Strategy:

- Show: $G_\alpha \hookrightarrow (G,\tau)$ top. embedding ✓
- Show: (G,τ) countably tight
- Show: group multiplication $(G,\tau)\times(G,\tau) \to (G,\tau)$ continuous ✓

Why is (G,τ) countably tight?

Let $f \in \mathrm{Homeo}_{cpt}(\mathbb{L}_c^+)$.

Let $A \subseteq \mathrm{Homeo}_{cpt}(\mathbb{L}_c^+)$ with $f \in \overline{A}$

to show: $\exists\, D \subseteq A$ countable

$f \in \overline{D}$

$\mathrm{id} \in \mathrm{Homeo}_{cpt}(\mathbb{L}_c^+)$.

Let $A \subseteq \mathrm{Homeo}_{cpt}(\mathbb{L}_c^+)$ with $\mathrm{id} \in \overline{A}$

to show: $\exists\, D \subseteq A$ countable

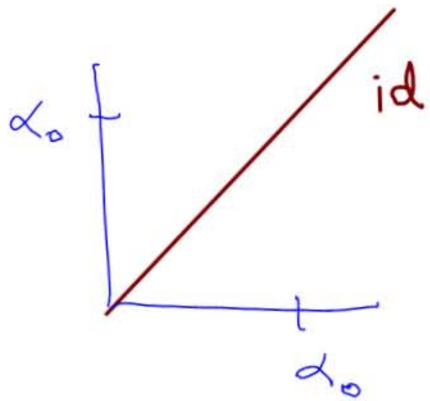
$\mathrm{id} \in \overline{D}$

$\mathrm{id} \in \mathrm{Homeo}_{cpt}(\mathbb{L}_c^+)$.

Let $A \subseteq \mathrm{Homeo}_{cpt}(\mathbb{L}_c^+)$ with $\mathrm{id} \in \overline{A}$

Let $\alpha_0 < \omega_1$.

$[0, \alpha_0] \subseteq \mathbb{L}_c^+$ compact.



$\mathrm{id} \quad \in \mathrm{Homeo}_{cpt}(\mathbb{L}_c^+)$.

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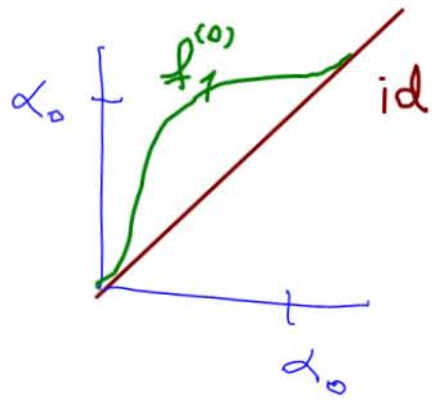

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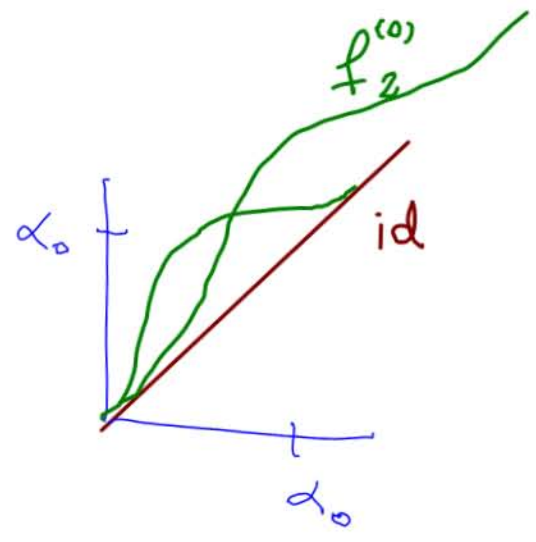

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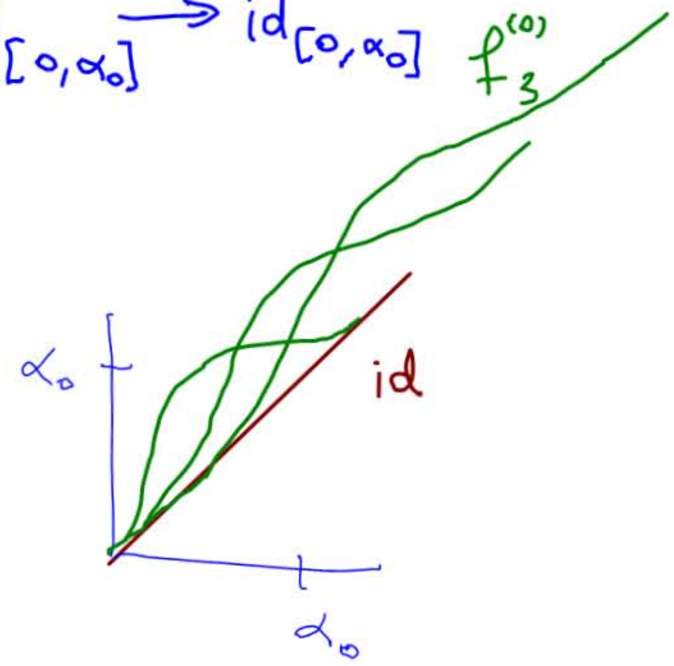

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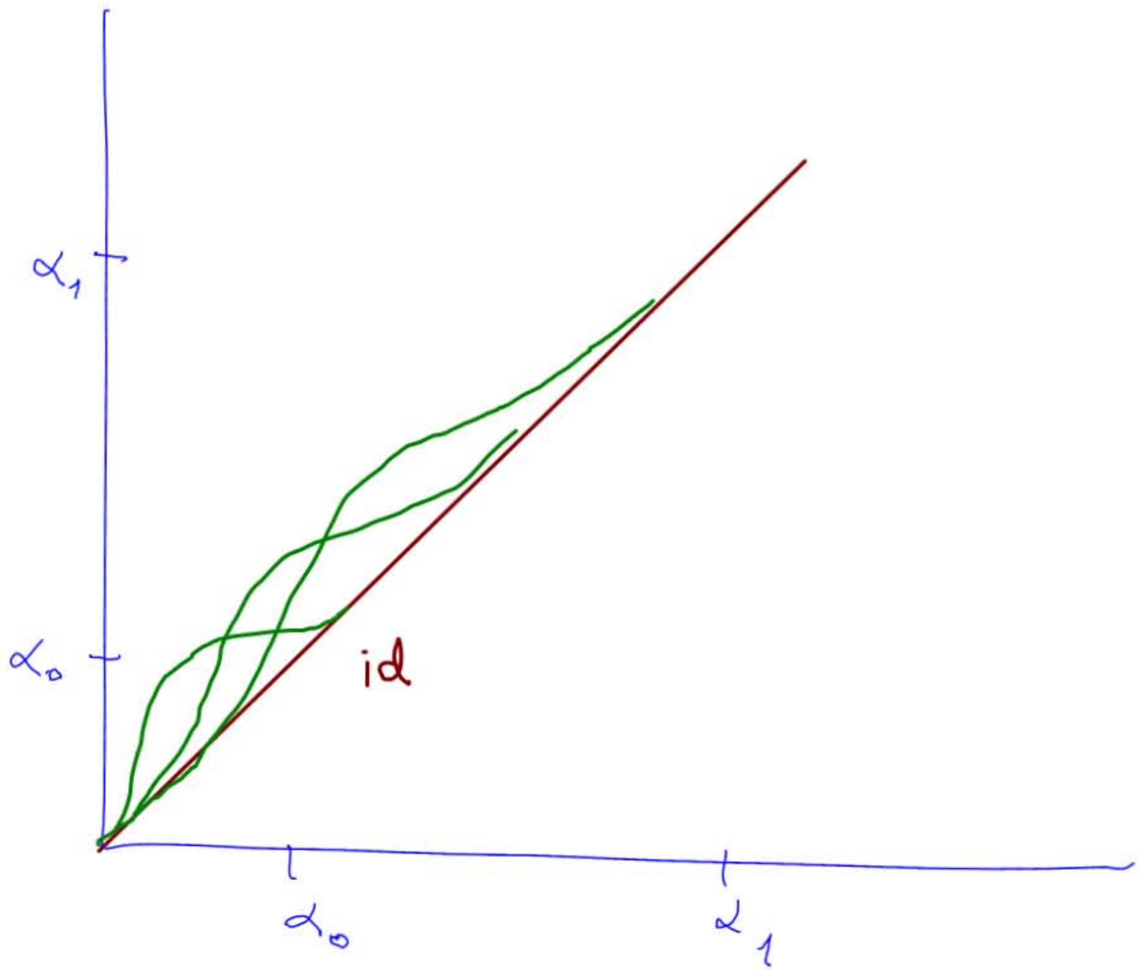

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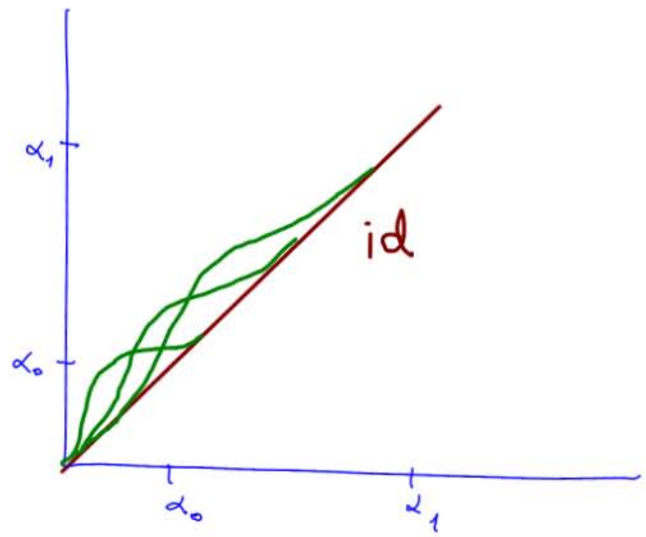

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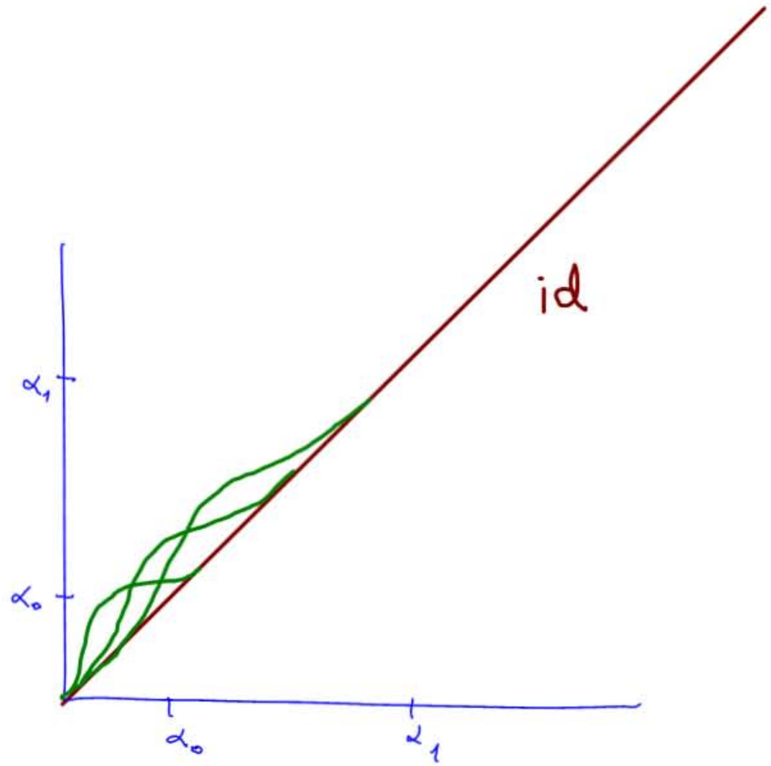

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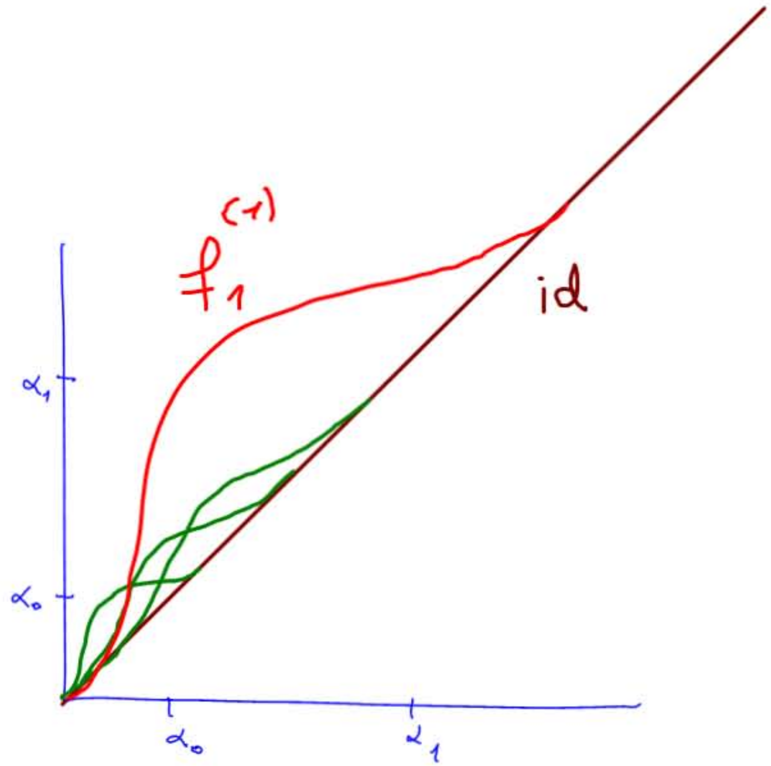

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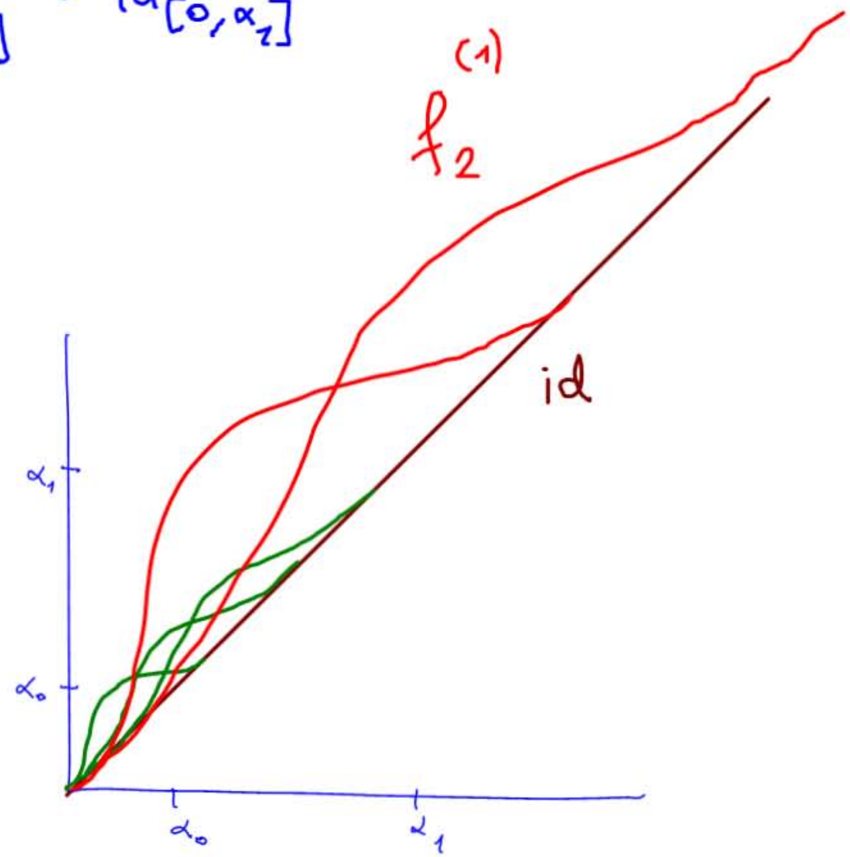

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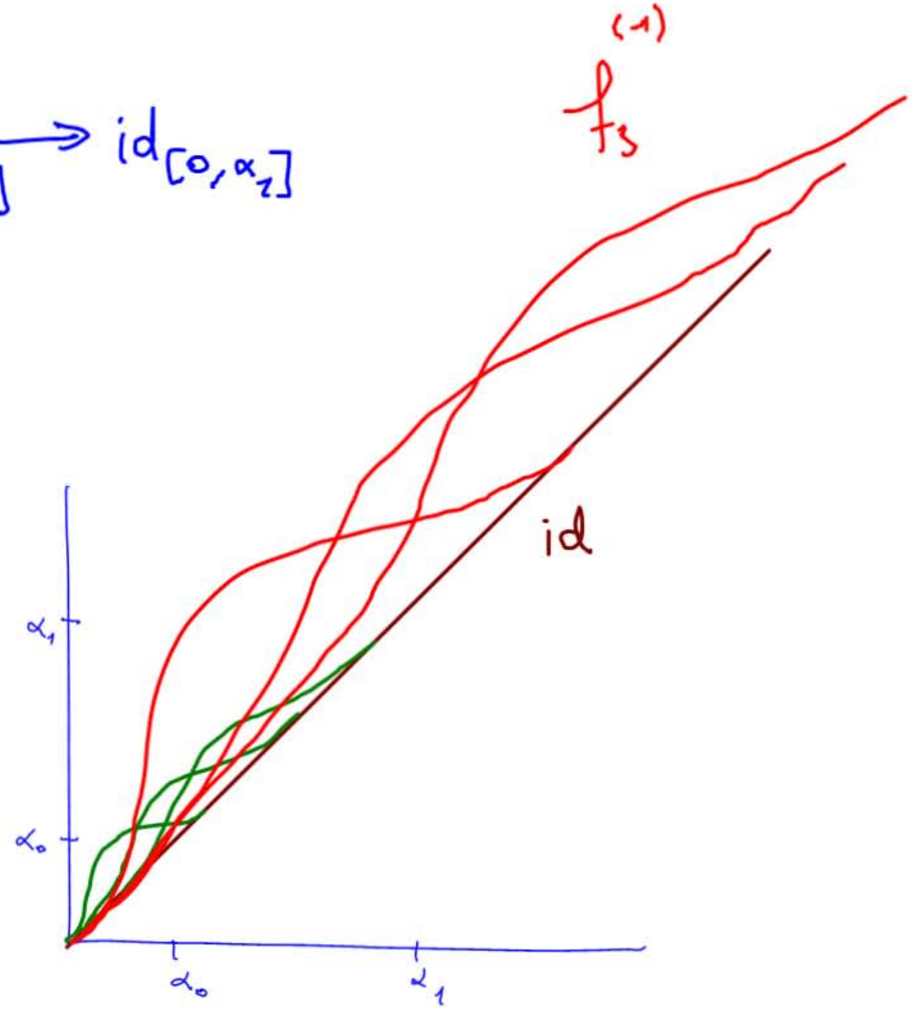

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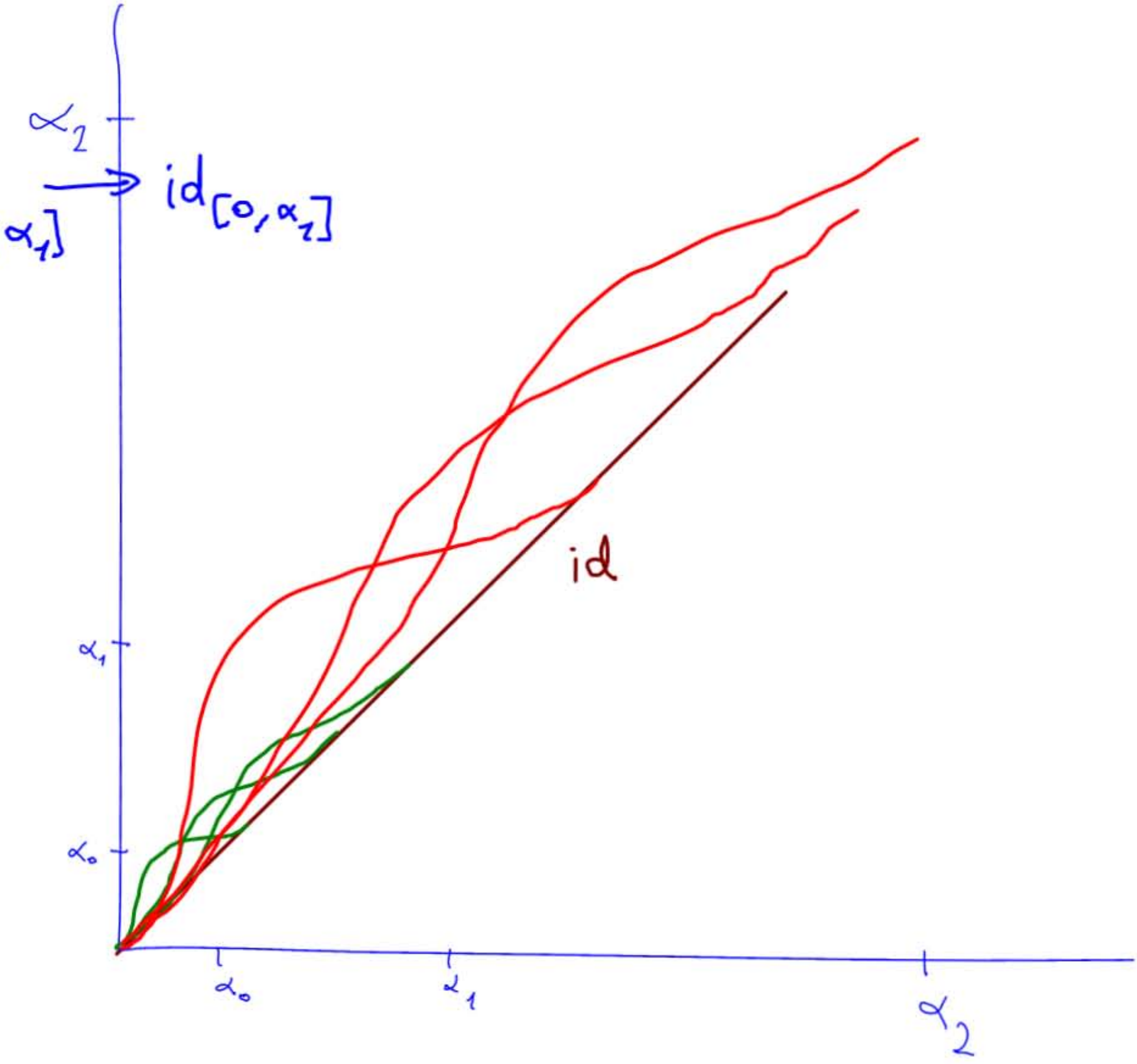
$\bigcup_{k\in\mathbb{N}} \mathrm{supp}\left(f_k^{(0)}\right) \subseteq [0,\alpha_2]$



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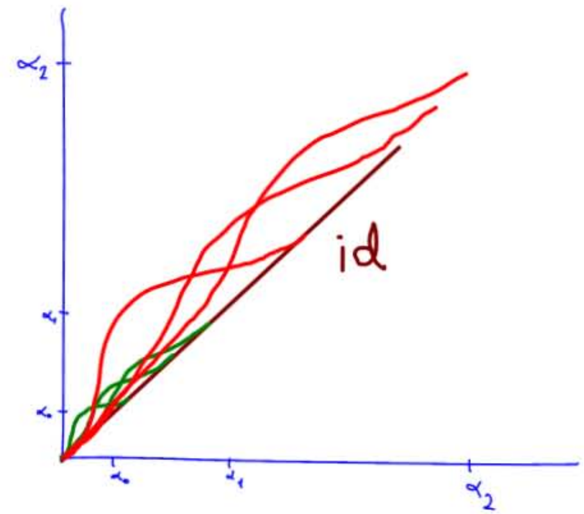

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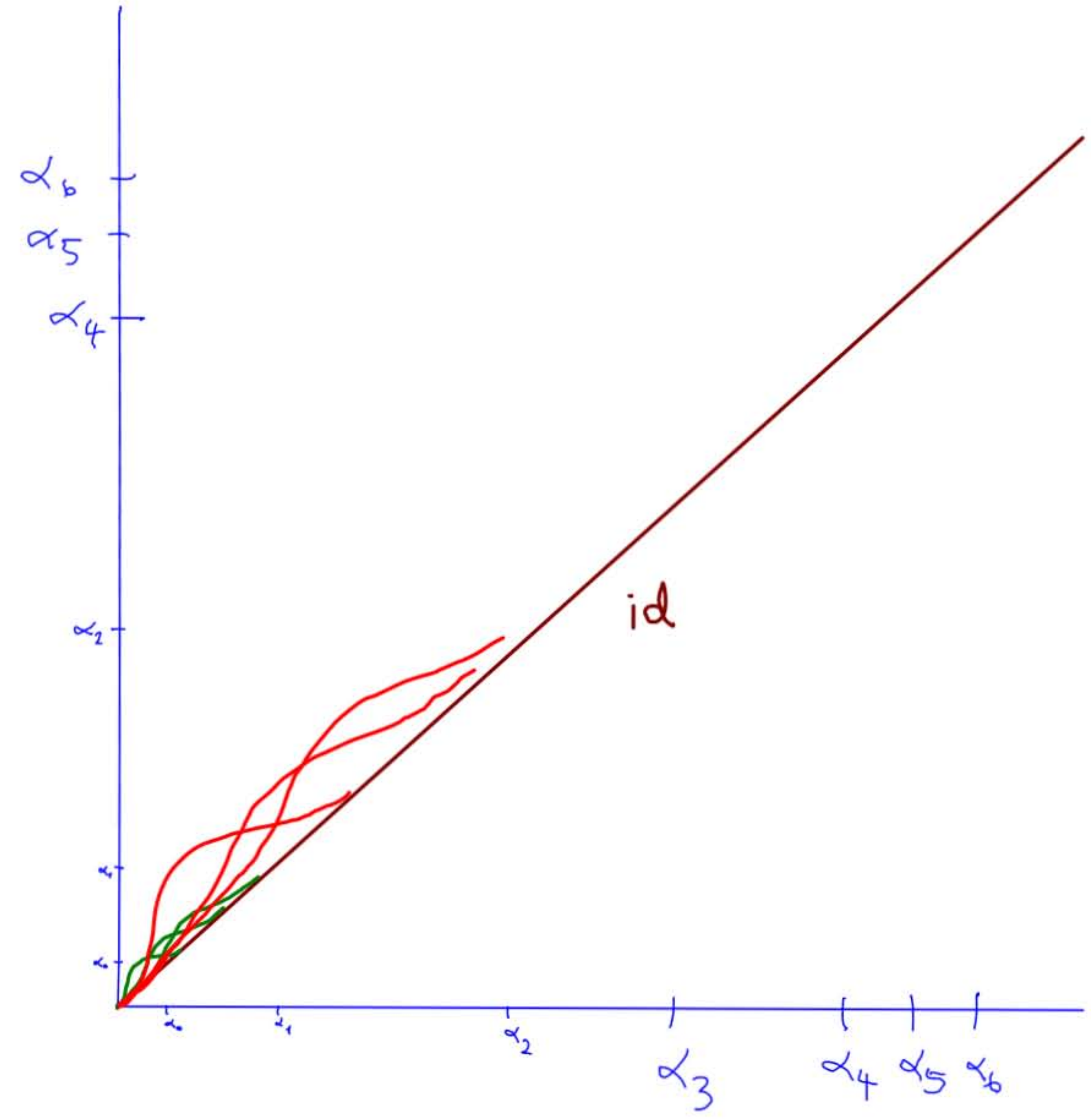
Let $A \subseteq \mathrm{Homeo}_{cpt}(\mathbb{L}_c^+)$ with $\mathrm{id} \in \overline{A}$

Inductively, construct an increasing sequence of ordinals

$\alpha_0 < \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < \cdots$



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Inductively, construct an increasing sequence of ordinals

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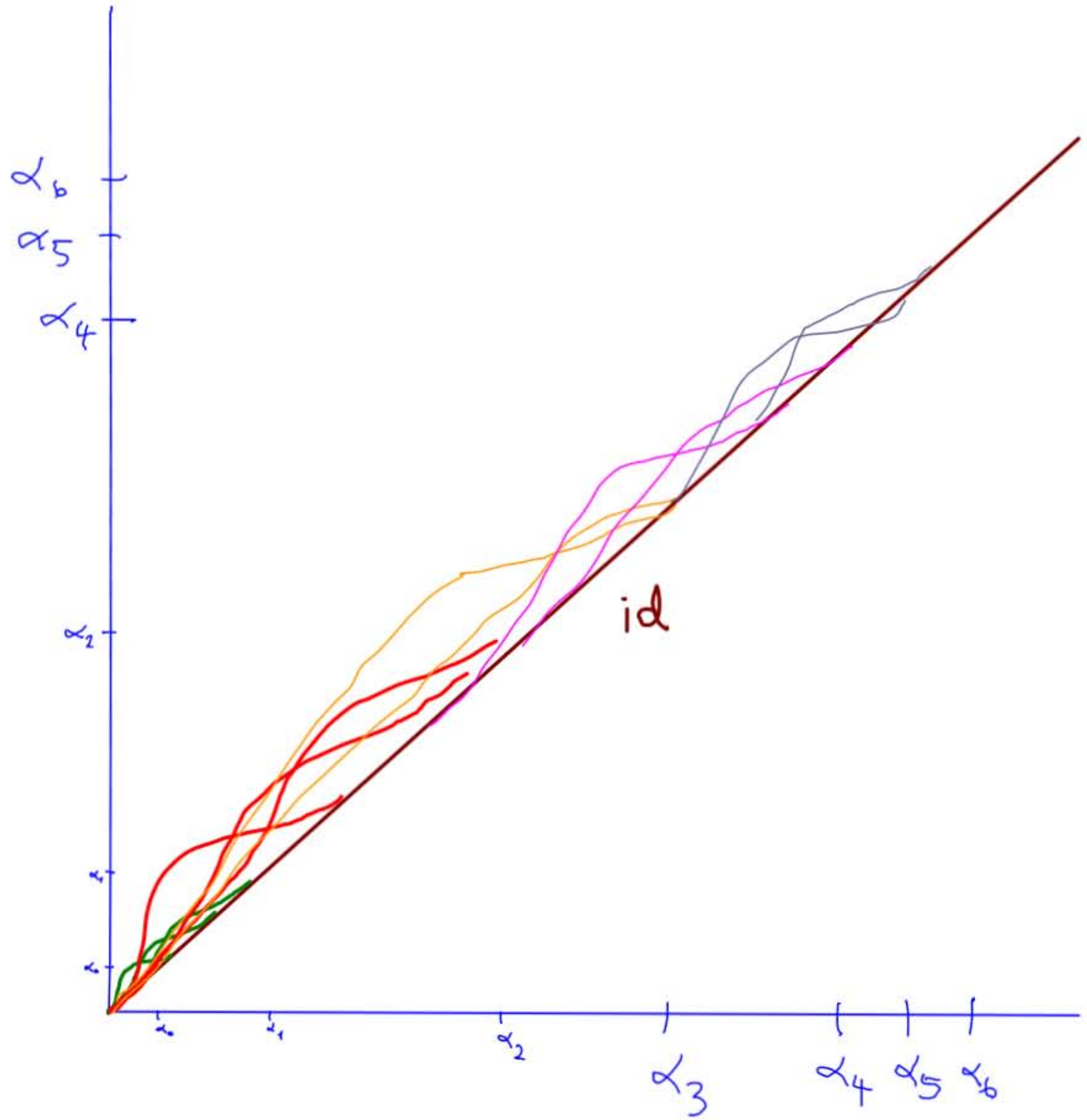
and for each α_n a sequence

$f_1^{(n)}, f_2^{(n)}, f_3^{(n)}, \ldots$ in A



converging to id on $[0, \alpha_n]$ with support in $[0, \alpha_{n+1}]$

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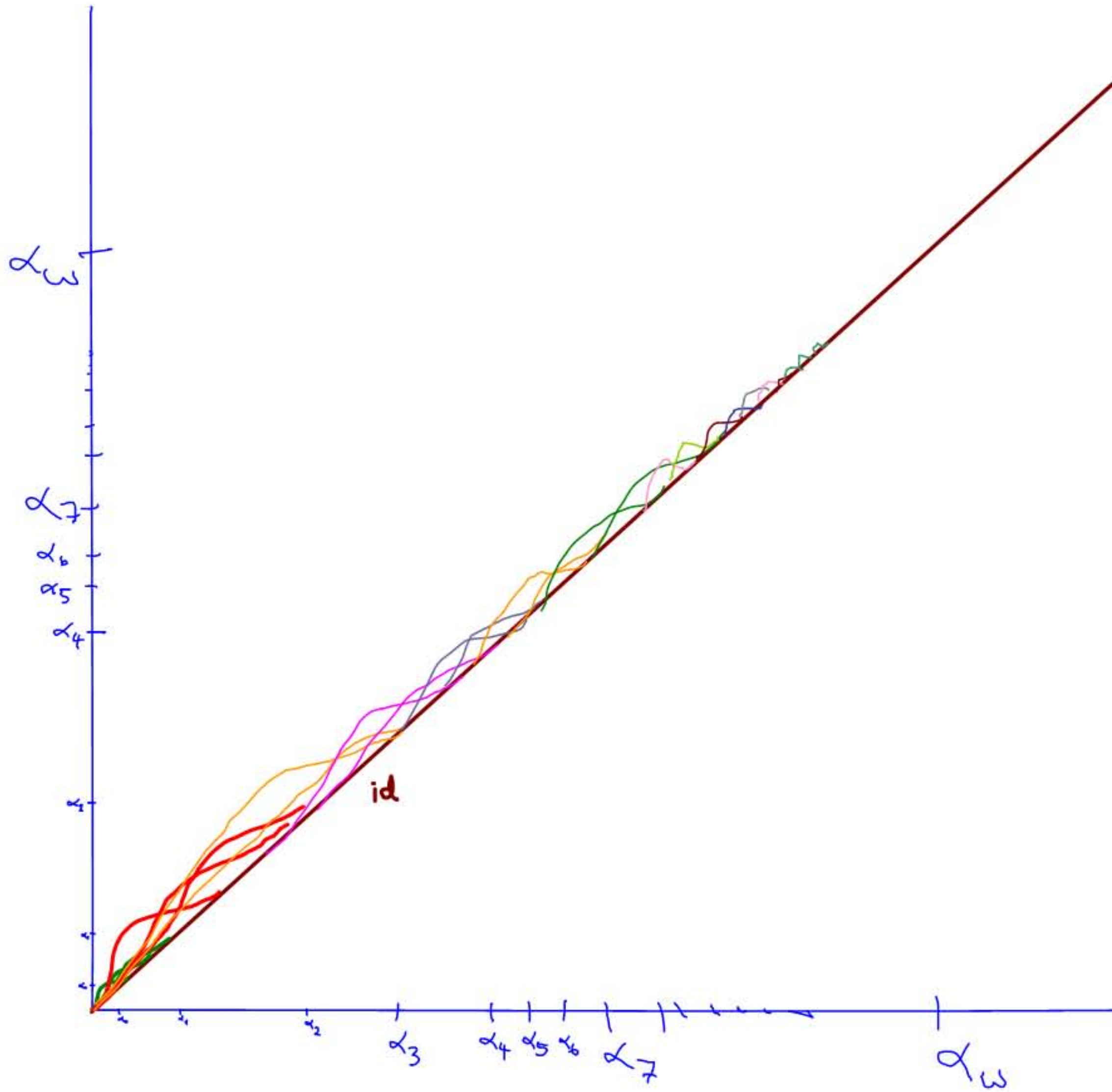
$$\alpha_\omega := \lim_n \alpha_n < \omega_1$$



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converging to id on $[0, \alpha_n]$, with support in $[0, \alpha_{n+1}]$

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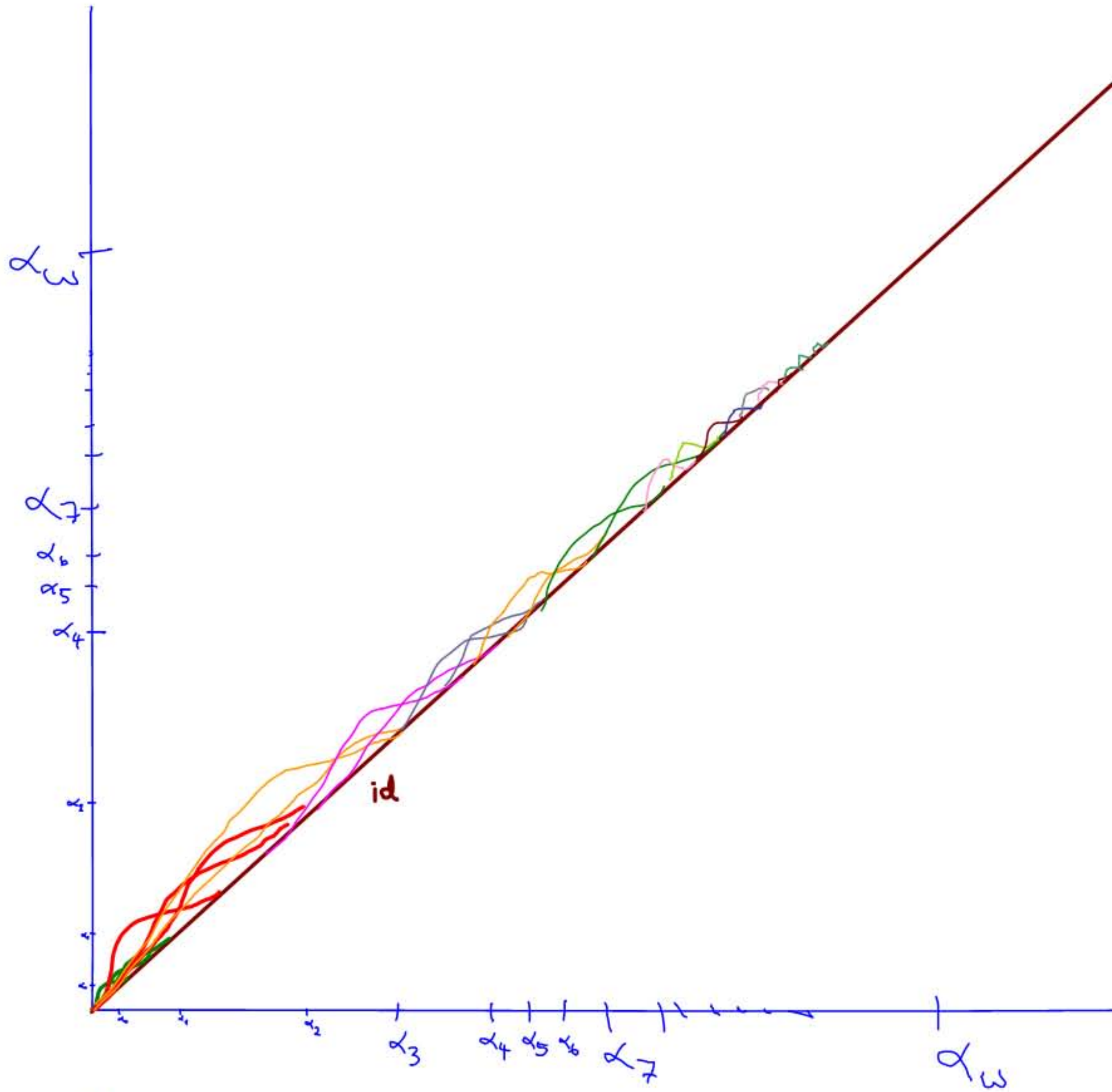
$D = \{ f_k^{(n)} : k, n \in \mathbb{N} \} \subseteq A$



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$\Rightarrow$ $\mathrm{Homeo}_{cpt}(\mathbb{L}_c^+)_{c.o.} = \underset{\alpha}{\underrightarrow{\mathrm{Lim}}}^{\mathrm{Top}} \mathrm{Homeo}_{[0,\alpha]}(\mathbb{L}_c^+)$

Main Lemma

(a) (b)

$$\Longrightarrow \quad \mathrm{Homeo}_{cpt}(\mathbb{L}_c^+)_{c.o.} \text{ is countably tight.}$$

$$\Longrightarrow \quad \underbrace{\mathrm{Homeo}_{cpt}(\mathbb{L}_c^+)_{c.o.}}_{(a)} = \underset{\overrightarrow{\alpha}}{\mathrm{Lim}}^{\mathrm{Top}} \underbrace{\mathrm{Homeo}_{[0,\alpha]}(\mathbb{L}_c^+)}_{(b)}$$

$$\Longrightarrow \quad \underbrace{\mathrm{Homeo}_{cpt}(\mathbb{L}_c^+)_{c.o.}}_{(a)} = \underset{\overrightarrow{\alpha}}{\mathrm{Lim}}^{\mathrm{Top}} \underbrace{\mathrm{Homeo}_{[0,\alpha]}(\mathbb{L}_c^+)}_{(b)} = \underset{\overrightarrow{\alpha}}{\mathrm{Lim}}^{\mathrm{TopGRP}} \underbrace{\mathrm{Homeo}_{[0,\alpha]}(\mathbb{L}_c^+)}_{(c)}$$

since composition is continuous w.r.t. (a)

$$\Longrightarrow \quad \mathrm{Homeo}_{cpt}(\mathbb{L}_c^+)_{c.o.} \text{ is countably tight.}$$

$$\Longrightarrow \quad \underset{(a)}{\mathrm{Homeo}_{cpt}(\mathbb{L}_c^+)_{c.o.}} = \varinjlim_{\alpha}{}^{\mathrm{Top}} \underset{(b)}{\mathrm{Homeo}_{[0,\alpha]}}(\mathbb{L}_c^+)$$

$$\Longrightarrow \quad \underset{(a)}{\mathrm{Homeo}_{cpt}(\mathbb{L}_c^+)_{c.o.}} = \varinjlim_{\alpha}{}^{\mathrm{Top}} \underset{(b)}{\mathrm{Homeo}_{[0,\alpha]}}(\mathbb{L}_c^+) = \varinjlim_{\alpha}{}^{\mathrm{TopGrp}} \underset{(c)}{\mathrm{Homeo}_{[0,\alpha]}}(\mathbb{L}_c^+)$$

$$\blacksquare$$

THANK YOU!