6-2017

Quotients of n-fold Hyperspaces

Sergio Macías
Universidad Autonoma del Estado de Mexico, sergiom@matem.unam.mx

Javier Camargo

Quotients of n-fold hyperspaces

Sergio Macías

Joint work with Javier Camargo

June 2017

A *continuum* is a compact, connected metric space.

A *continuum* is a compact, connected metric space.

Given a continuum X, we define its hyperspaces as the following sets:

A *continuum* is a compact, connected metric space.

Given a continuum X, we define its hyperspaces as the following sets:

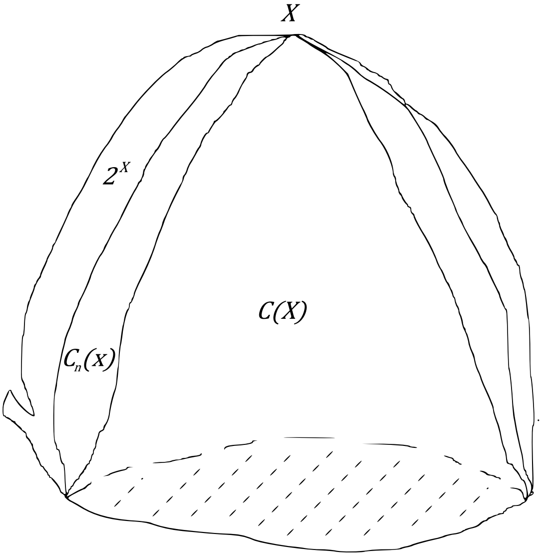
- $2^X = \{A \subset X \mid A \text{ is closed and nonempty}\}$,
- $\mathcal{C}_n(X) = \{A \in 2^X \mid A \text{ has at most } n \text{ components}\}, n \in \mathbb{N}$,
- $\mathcal{F}_n(X) = \{A \in 2^X \mid A \text{ has at most } n \text{ points}\}, n \in \mathbb{N}$.

A *continuum* is a compact, connected metric space.

Given a continuum X, we define its hyperspaces as the following sets:

- $2^X = \{A \subset X \mid A \text{ is closed and nonempty}\}$,
- $\mathcal{C}_n(X) = \{A \in 2^X \mid A \text{ has at most } n \text{ components}\}, n \in \mathbb{N}$,
- $\mathcal{F}_n(X) = \{A \in 2^X \mid A \text{ has at most } n \text{ points}\}, n \in \mathbb{N}$.

They are topologized with the *Vietoris Topology* or the topology generated by the *Hausdorff metric*, $\mathcal{H}$.



X
2^X
C_n(X)
C(X)

2^X is the *hyperspaces of closed subsets* of X.

2^X is the *hyperspaces of closed subsets* of X.

$\mathcal{C}_n(X)$ is the *n-fold hyperspace* of X.

2^X is the *hyperspaces of closed subsets* of X.

$\mathcal{C}_n(X)$ is the *n-fold hyperspace* of X.

$\mathcal{F}_n(X)$ is the *n-fold symmetric product* of X.

2^X is the *hyperspaces of closed subsets* of X.

$\mathcal{C}_n(X)$ is the *n-fold hyperspace* of X.

$\mathcal{F}_n(X)$ is the *n-fold symmetric product* of X.

The *n-fold symmetric product suspension* of X is:

$$\mathcal{SF}_n(X) = \mathcal{F}_n(X)/\mathcal{F}_1(X)$$

2^X is the *hyperspaces of closed subsets* of X.

$\mathcal{C}_n(X)$ is the *n-fold hyperspace* of X.

$\mathcal{F}_n(X)$ is the *n-fold symmetric product* of X.

The *n-fold symmetric product suspension* of X is:

$$\mathcal{SF}_n(X) = \mathcal{F}_n(X)/\mathcal{F}_1(X)$$

Our main object of study is:

2^X is the *hyperspaces of closed subsets* of X.

$\mathcal{C}_n(X)$ is the *n-fold hyperspace* of X.

$\mathcal{F}_n(X)$ is the *n-fold symmetric product* of X.

The *n-fold symmetric product suspension* of X is:

$$\mathcal{SF}_n(X) = \mathcal{F}_n(X)/\mathcal{F}_1(X)$$

Our main object of study is:

$$\mathcal{C}_1^n(X) = \mathcal{C}_n(X)/\mathcal{C}_1(X)$$

both quotient spaces with the quotient topology.

Let $q_X^{n1} \colon \mathcal{C}_n(X) \twoheadrightarrow \mathcal{C}_1^n(X)$ denote the quotient map.

Let $q_X^{n1} \colon \mathcal{C}_n(X) \twoheadrightarrow \mathcal{C}_1^n(X)$ denote the quotient map.

We denote $q_X^{n1}(\mathcal{C}_1(X))$ by C_X^{n1}.

Let $q_X^{n1} \colon \mathcal{C}_n(X) \twoheadrightarrow \mathcal{C}_1^n(X)$ denote the quotient map.

We denote $q_X^{n1}(\mathcal{C}_1(X))$ by C_X^{n1}.

Remark

Note that $q_X^{n1}|_{\mathcal{C}_n(X)\setminus\mathcal{C}_1(X)} \colon \mathcal{C}_n(X) \setminus \mathcal{C}_1(X) \twoheadrightarrow \mathcal{C}_1^n(X) \setminus \{C_X^{n1}\}$ *is a homeomorphism.*

Lemma

Let X be a continuum and let n be a positive integer greater than one. Then $\mathcal{C}_n(X) \setminus \mathcal{C}_1(X)$ is connected.

Lemma

Let X be a continuum and let n be a positive integer greater than one. Then $\mathcal{C}_n(X) \setminus \mathcal{C}_1(X)$ is connected.

A continuum X has *the property of Kelley* provided that for each $\varepsilon > 0$, there exists $\delta > 0$ such that if a and b are two points of X such that $d(a, b) < \delta$, and a belongs to a subcontinuum A of X, then there exists there exists a subcontinuum B of X such that $b \in B$ and $\mathcal{H}(A, B) < \varepsilon$. The number δ is called a *Kelley number* for ε.

Let A and B be two elements of $\mathcal{C}_n(X)$. We say that the pair (B, A) satisfies *property* (OA) provided that $B \subset A$ and each component of A intersects B.

Let A and B be two elements of $\mathcal{C}_n(X)$. We say that the pair (B, A) satisfies *property* (OA) provided that $B \subset A$ and each component of A intersects B.

Let us note that this condition guarantees the existence of an order arc, in $\mathcal{C}_n(X)$, from B to A when $B \neq A$.

Let A and B be two elements of $\mathcal{C}_n(X)$. We say that the pair (B, A) satisfies *property* (OA) provided that $B \subset A$ and each component of A intersects B.

Let us note that this condition guarantees the existence of an order arc, in $\mathcal{C}_n(X)$, from B to A when $B \neq A$.

Let X be a continuum and let n be a positive integer. If $B \in \mathcal{C}_n(X)$, define:

$$\mathcal{C}_n(B, X) = \{A \in \mathcal{C}_n(X) \mid B \subset A\};$$

Let A and B be two elements of $\mathcal{C}_n(X)$. We say that the pair (B, A) satisfies *property* (OA) provided that $B \subset A$ and each component of A intersects B.

Let us note that this condition guarantees the existence of an order arc, in $\mathcal{C}_n(X)$, from B to A when $B \neq A$.

Let X be a continuum and let n be a positive integer. If $B \in \mathcal{C}_n(X)$, define:

$$\mathcal{C}_n(B, X) = \{A \in \mathcal{C}_n(X) \mid B \subset A\};$$

$$\mathcal{OA}_n(B,X) = \{A \in \mathcal{C}_n(X) \mid (B,A) \text{ satisfies property } (OA)\}.$$

$$\mathcal{OA}_n(B,X) = \{A \in \mathcal{C}_n(X) \mid (B,A) \text{ satisfies property } (OA)\}.$$

If $A \in \mathcal{OA}_n(B,X)$, then

$$\mathcal{OA}_n(B,A) = \{D \in \mathcal{OA}_n(B,X) \mid D \subset A\}.$$

$$\mathcal{OA}_n(B,X) = \{A \in \mathcal{C}_n(X) \mid (B,A) \text{ satisfies property } (OA)\}.$$

If $A \in \mathcal{OA}_n(B,X)$, then

$$\mathcal{OA}_n(B,A) = \{D \in \mathcal{OA}_n(B,X) \mid D \subset A\}.$$

It is known that these sets are absolute retracts.

Given a continuum X and a positive integer n, we define the function $\alpha_X^n \colon \mathcal{C}_n(X) \twoheadrightarrow \mathcal{C}_1(\mathcal{C}_n(X))$ by:

Given a continuum X and a positive integer n, we define the function $\alpha_X^n \colon \mathcal{C}_n(X) \twoheadrightarrow \mathcal{C}_1(\mathcal{C}_n(X))$ by:

$$\alpha_X^n(B) = \mathcal{O}\mathcal{A}_n(B, X).$$

Given a continuum X and a positive integer n, we define the function $\alpha_X^n \colon \mathcal{C}_n(X) \twoheadrightarrow \mathcal{C}_1(\mathcal{C}_n(X))$ by:

$$\alpha_X^n(B) = \mathcal{OA}_n(B, X).$$

Theorem

Let X be a continuum and let n be a positive integer. Then the following are equivalent:

- *X has the property of Kelley;*
- *α_X^n is continuous;*
- *$\alpha_X^n|_{\mathcal{F}_1(X)}$ is continuous.*

We now present properties of $\mathcal{C}_1^n(X)$.

We now present properties of $\mathcal{C}_1^n(X)$.

Theorem

Let X be a continuum and let n be an integer greater than one. Then $\mathcal{C}_1^n(X) \setminus \{C_X^{n1}\}$ is connected.

We now present properties of $\mathcal{C}_1^n(X)$.

Theorem

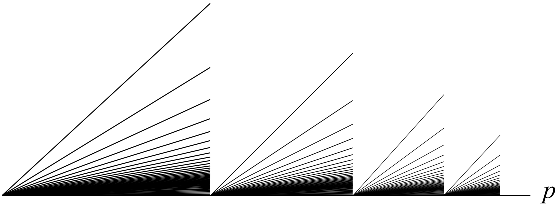
Let X be a continuum and let n be an integer greater than one. Then $\mathcal{C}_1^n(X) \setminus \{C_X^{n1}\}$ is connected.

Theorem

If X is a continuum and n is an integer greater than one, then each map from $\mathcal{C}_1^n(X)$ into the unit circle, $\mathcal{S}^1$, is homotopic to a constant map.

A continuum X is *connected im kleinen at a point* p of X provided that for each open subset U of X, containing p, there exists a subcontinuum W of X such that $p \in Int_X(W) \subset W \subset U$.

A continuum X is *connected im kleinen at a point p* of X provided that for each open subset U of X, containing p, there exists a subcontinuum W of X such that $p \in Int_X(W) \subset W \subset U$.



Theorem

Let X be a continuum with the property of Kelley and let n be an integer greater than one, then $\mathcal{C}_1^n(X)$ is connected im kleinen at C_X^{n1}.

Theorem

Let X be a continuum with the property of Kelley and let n be an integer greater than one, then $\mathcal{C}_1^n(X)$ is connected im kleinen at C_X^{n1}.

Theorem

Let X be a continuum and let n be an integer greater than one. If X has the property of Kelley, then $\mathcal{C}_1^n(X)$ is contractible.

Theorem

Let X be a continuum with the property of Kelley and let n be an integer greater than one, then $\mathcal{C}_1^n(X)$ is connected im kleinen at C_X^{n1}.

Theorem

Let X be a continuum and let n be an integer greater than one. If X has the property of Kelley, then $\mathcal{C}_1^n(X)$ is contractible.

Theorem

Let X be a continuum and let n be an integer greater than one. Then $\dim(\mathcal{C}_n(X)) = \dim(\mathcal{C}_1^n(X))$.

A finite-dimensional continuum X is a *Cantor manifold* if for any subset A of X such that $\dim(A) \leq \dim(X) - 2$, then $X \setminus A$ is connected.

A finite-dimensional continuum X is a *Cantor manifold* if for any subset A of X such that $\dim(A) \leq \dim(X) - 2$, then $X \setminus A$ is connected.

Theorem

Let X be a continuum and let n be an integer greater than one. If $\mathcal{C}_n(X)$ is a finite-dimensional Cantor manifold and $\dim(\mathcal{C}_n(X)) \geq \dim(\mathcal{C}_1(X)) + 2$, then $\mathcal{C}_1^n(X)$is a finite-dimensional Cantor manifold.

A finite-dimensional continuum X is a *Cantor manifold* if for any subset A of X such that $\dim(A) \leq \dim(X) - 2$, then $X \setminus A$ is connected.

Theorem

Let X be a continuum and let n be an integer greater than one. If $\mathcal{C}_n(X)$ is a finite-dimensional Cantor manifold and $\dim(\mathcal{C}_n(X)) \geq \dim(\mathcal{C}_1(X)) + 2$, then $\mathcal{C}_1^n(X)$is a finite-dimensional Cantor manifold.

Theorem

If n is an integer greater than one, then $\mathcal{C}_1^n([0,1])$ and $\mathcal{C}_1^n(\mathcal{S}^1)$ are $2n$-dimensional Cantor manifolds.

Let Z be a metric space and let A be a nonempty subset of Z. Then a map $r\colon Z \twoheadrightarrow A$ is a *retraction* provided that $r(a) = a$ for all $a \in A$. In this case A is a *retract* of Z.

Let Z be a metric space and let A be a nonempty subset of Z. Then a map $r\colon Z \twoheadrightarrow A$ is a *retraction* provided that $r(a) = a$ for all $a \in A$. In this case A is a *retract* of Z.

Let Z be a metric space. By a *deformation* we mean a map $H\colon Z \times [0,1] \twoheadrightarrow Z$ such that $H((z,0)) = z$ for each $z \in Z$. Let $A = \{H((z,1)) \mid z \in Z\}$. If the map $r\colon Z \twoheadrightarrow A$ given by $r(z) = H((z,1))$ is a retraction from Z onto A, then H is a *deformation retraction* from Z onto A. If H is a deformation retraction from Z onto A and for each $a \in A$ and each $t \in [0,1]$, $H((a,t)) = a$, then H is a *strong deformation retraction* from Z onto A. The set A is called a *deformation retract* of Z (*strong deformation retract* of Z, respectively).

Theorem

Let X be a continuum and let n and m be integers greater than one such that $n < m$. Then the following holds:

- *If $\mathcal{C}_n(X)$ is a retract of $\mathcal{C}_m(X)$, then $\mathcal{C}_1^n(X)$ is a retract of $\mathcal{C}_1^m(X)$.*
- *If $\mathcal{C}_n(X)$ is a deformation retract of $\mathcal{C}_m(X)$, then $\mathcal{C}_1^n(X)$ is a deformation retract of $\mathcal{C}_1^m(X)$.*
- *If $\mathcal{C}_n(X)$ is a strong deformation retract of $\mathcal{C}_m(X)$, then $\mathcal{C}_1^n(X)$ is a strong deformation retract of $\mathcal{C}_1^m(X)$.*

A continuum X is *aposyndetic* provided that for each pair of points x_1 and x_2 of X, there exists a subcontinuum W of X such that $x_1 \in Int_X(W) \subset W \subset X \setminus \{x_2\}$.

A continuum X is *aposyndetic* provided that for each pair of points x_1 and x_2 of X, there exists a subcontinuum W of X such that $x_1 \in Int_X(W) \subset W \subset X \setminus \{x_2\}$.

Lemma

Let X be a continuum. Then the following holds:

- *If n is an integer greater than two, then $\mathcal{SF}_n(X)$ is aposyndetic.*
- *If X is an aposyndetic continuum, then $\mathcal{SF}_2(X)$ is aposyndetic.*

Theorem

Let X be a continuum. Then the following holds:

- *If n is an integer greater than two, then $\mathcal{C}_1^n(X)$ is aposyndetic.*
- *If X is an aposyndetic continuum, then $\mathcal{C}_1^2(X)$ is aposyndetic.*

Theorem

Let X be a continuum. Then the following holds:

- *If n is an integer greater than two, then $\mathcal{C}_1^n(X)$ is aposyndetic.*
- *If X is an aposyndetic continuum, then $\mathcal{C}_1^2(X)$ is aposyndetic.*

Lemma

Let X be a continuum and let n be an integer greater than one. Then X is locally connected if and only if $\mathcal{C}_n(X) \setminus \mathcal{C}_1(X)$ is locally connected.

Theorem

Let X be a continuum and let n be an integer greater than one. Then X is locally connected if and only if $\mathcal{C}_1^n(X)$ is locally connected.

Theorem

Let X be a continuum and let n be an integer greater than one. Then X is locally connected if and only if $\mathcal{C}_1^n(X)$ is locally connected.

Theorem

Let X be a locally connected continuum and let n and m be integers greater than one such that $n < m$. Then the following holds:

- $\mathcal{C}_1^n(X)$ *is a retract of* $\mathcal{C}_1^m(X)$.
- $\mathcal{C}_1^n(X)$ *is a deformation retract of* $\mathcal{C}_1^m(X)$.
- $\mathcal{C}_1^n(X)$ *is a strong deformation retract of* $\mathcal{C}_1^m(X)$.

Theorem

Let X be a continuum and let n be an integer greater than one. Then $\mathcal{C}_1^n(X)$ is homeomorphic to the Hilbert cube if and only if X is locally connected and every arc in X has empty interior.

Theorem

Let X be a continuum and let n be an integer greater than one. Then $\mathcal{C}_1^n(X)$ is homeomorphic to the Hilbert cube if and only if X is locally connected and every arc in X has empty interior.

Theorem

Let n be an integer greater than one. If X is a continuum such that $\mathcal{C}_1^n(X)$ is homeomorphic to either $\mathcal{C}_1^n([0,1])$ or $\mathcal{C}_1^n(\mathcal{S}^1)$, then X is homeomorphic to either $[0,1]$ or $\mathcal{S}^1$.

THANK YOU!