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Enriched Topology and Asymmetry

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ENRICHED TOPOLOGY AND ASYMMETRY

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Motivations for Asymmetry

First Motivation: Quasimetric Spaces

Consider possible conditions for (X, d) , where $d : X \times X \rightarrow \mathbb{R}$:

(M1) $\forall x, y \in X, d(x, y) \geq 0$ (non-negativity)

(M2) $\forall x, y \in X, [d(x, y) > 0 \text{ or } d(y, x) > 0] \Leftrightarrow x \neq y$ (weak pos. definiteness)

(M2a) $\forall x, y \in X, d(x, y) = 0$ if $x = y$ (zero-distance on diagonal)

(M2b) $\forall x, y \in X, [d(x, y) > 0 \text{ and } d(y, x) > 0] \Leftrightarrow x \neq y$ (strong pos. def.)

(M3) $\forall x, y \in X, d(x, y) = d(y, x)$ (symmetry)

(M4) $\forall x, y, z \in X, d(x, z) \leq d(x, y) + d(y, z)$

Note: $M2b \Rightarrow M2$, $M3 \Rightarrow (M2 \Leftrightarrow M2b)$, $M1 \Rightarrow (M2 \Rightarrow M2a, M2b \Rightarrow M2a)$,

$M2a \Leftrightarrow s(x, y) = 1 - d(x, y)$ reflexive

Defn. (X, d) is: *metric space* if M1,M2,M3,M4;

pseudometric space if M1,M2a,M3,M4

quasimetric space if M1,M2b,M4

hemimetric space if M1,M2a,M4

For hemimetric space (X, d) , have topology \mathcal{T}_d generated by ε -balls. Then for hemimetric d , (X, \mathcal{T}_d) is: $T_0 \Leftrightarrow M2$; $T_1 \Leftrightarrow M2b$; T_2 if $M2$ and $M3$.

Second Motivation: Specialization preorders. Let (X, \mathcal{T}) be topological space; put

$$x \leq_{\mathcal{T}} y \Leftrightarrow y \in \overline{\{x\}}$$

This (or dual) is *specialization* preorder. It is antisymmetric iff (X, \mathcal{T}) is T_0 ; it is antisymmetric and symmetric (and hence equality =) iff (X, \mathcal{T}) is T_1 .

Working with sober, non- T_1 spaces means working with asymmetric specialization preorders.

Defn. Topological space (X, \mathcal{T}) *asymmetric* if specialization order not symmetric.

In practice, asymmetry means T_0 but not T_1 spaces.

Order-Theoretics: PO-Groupoids, IIA Operators/Involutions

Groupoid/magma: (X, \otimes) , $\otimes: X \times X \rightarrow X$ ("flat") *semitensor*

unital: (X, \otimes, e) , two-sided identity e

monoid: (X, \otimes, e) unital semigroup

semiquantale: (L, \leq, \otimes) with (L, \leq) complete lattice, (L, \otimes) groupoid

Po-groupoid: (L, \leq, \otimes) with (L, \leq) poset, (L, \otimes) groupoid, \otimes isotone both var.

IIA (or involutive) po-groupoid: $(L, \leq, \otimes, *)$, (L, \leq, \otimes) po-groupoid, $*$: $L \rightarrow L$
 involutive, isotone, anti-automorphism in these senses:

involutive ($a^{**} = a$), isotone ($a \leq b \Rightarrow a^* \leq b^*$),

interchanges with \otimes ($(a \otimes b)^* = b^* \otimes a^*$)

left[right]-residuated po-groupoid: (L, \leq, \otimes) with $\searrow [\swarrow] : L \times L \rightarrow L$ s.t.

$$a \searrow b \geq c \Leftrightarrow a \otimes c \leq b \quad [b \swarrow a \geq c \Leftrightarrow c \otimes a \leq b]$$

po-monoid: (L, \leq, \otimes) unital po-groupoid with \otimes associative

complete po-groupoid: ordered semiquantale, i.e., (L, \leq) complete lattice

semiframe/semilocal: $\otimes = \wedge$ (binary), integral

complete po-semigroup: \otimes associative

quantale: \otimes distributes on both sides across arbitrary \vee , \otimes assoc.

frame/locale: $\otimes = \wedge$ (binary), integral

Semiquantale morphisms: preserve \vee, \otimes

Comment. Given (L, \leq, \otimes) , an IIA-operator/involution may be chosen to be $* = id_L$ iff \otimes is commutative. Allowable choices of IIA-operators/involutions for \otimes roughly gauge deviation of \otimes from being commutative.

Non-commutative \otimes and associated IIA-operators/involutions $*$ tied to possible symmetries of L -valued specialization orders for L -topological spaces.

Example. $(L, \leq, ')$ DeMorgan algebra (complete). Construct $(S(L), \leq, \circ, *)$ by:

$$S(L) = \{f : L \rightarrow L \mid f \text{ preserves } \bigvee\},$$

\leq taken pointwise,

◦ composition of functions,

$$f^*(a) = (f^\vdash(a'))',$$

where $f^\vdash : L \rightarrow L$ is right adjoint of f guaranteed by AFT(\bigvee) and given by

$$f^\vdash(b) = \bigvee_{f(a) \leq b} a.$$

Then $(S(L), \leq, \circ, *)$ is unital, IIA quantale; and it is both non-integral and non-commutative iff $|L| \geq 3$.

Powerset monads and topology. Fix X, Y sets, L semiquantale, $f : X \rightarrow Y$; have $f_L^\rightarrow : L^X \rightarrow L^Y$, $f_L^\leftarrow : L^X \leftarrow L^Y$, $f_{L \rightarrow} : L^X \rightarrow L^Y$, given respectively by

$$f_L^\rightarrow(a)(y) = \bigvee_{f(x)=y} a(x), \quad f_L^\leftarrow(b) = b \circ f, \quad f_{L \rightarrow}(a) = \bigwedge_{a \leq f_L^\leftarrow(b)} b.$$

Fact $f_L^{\rightarrow} \dashv f_L^{\leftarrow} \dashv f_{L \rightarrow}$

(X, τ) is L -topological space if $\tau \subset L^X$ closed under \bigvee, \otimes, \perp . $f: (X, \tau) \rightarrow (Y, \sigma)$

L -continuous if $\forall v \in \sigma, f_L^{\leftarrow}(v) \in \tau$.

Fact L -**Top** topological construct.

Sets Enriched by PO-Monoids

Enriched category \mathcal{C} over a monoidal category $(\mathcal{M}, \otimes, I, a, \lambda, \rho)$ is class of objects with data C0, C1, and C2 subject to axioms D1, D2, and D3:

C0: $\forall a, b \in \mathcal{C}, \exists ! \mathcal{C}(a, b) \in |\mathcal{M}|$ (existence of hom-objects)

C1: $\forall a \in \mathcal{C}, \exists id_a : I \rightarrow \mathcal{C}(a, a)$ (existence of identities)

C2: $\forall a, b, c \in \mathcal{C}, \exists ! \circ_{abc} : \mathcal{C}(b, c) \otimes \mathcal{C}(a, b) \rightarrow \mathcal{C}(a, c)$ (comp. of hom-objects)

D1: $\forall a, b, c, d \in \mathcal{C}, (\circ_{abd}) \circ (\circ_{bcd} \otimes \mathbf{1}_{\mathcal{C}(a,b)}) = (\circ_{acd}) \circ (\mathbf{1}_{\mathcal{C}(c,d)} \otimes \circ_{abc}) \circ a$

D2: $\forall a, b \in \mathcal{C}, \lambda = (\circ_{abb}) \circ (id_b \otimes \mathbf{1}_{\mathcal{C}(a,b)})$

D3: $\forall a, b \in \mathcal{C}, \rho = (\circ_{aab}) \circ (\mathbf{1}_{\mathcal{C}(a,b)} \otimes id_a)$

Comment. Po-monoid $(L, \leq \otimes)$, taken as a preordered category, is a monoidal category in which \otimes is the categorical tensor product, I is the unit e , and the associator a and the unitors λ, ρ are all identities..

Prop. If $(L, \leq \otimes)$ is po-monoid, then set X replacing \mathcal{C} is L -enriched category iff there is *enrichment relation*, or L -(valued) *preorder* P on X s.t.:

E0: $P : X \times X \rightarrow L$ is a mapping (*degrees of comparison/precedence*)

E1: $\forall x \in X, e \leq P(x, x)$ (*reflexivity*)

E2: $\forall x, y, z \in X, P(y, z) \otimes P(x, y) \leq P(x, z)$ (*transitivity*)

Defn. For $(L, \leq \otimes)$ a unital po-groupoid, (X, P) is an L -preordered set, or (X, L, P) is a *preordered set*, if:

P0: $P : X \times X \rightarrow L$ is a mapping (*degrees of comparison/precedence*)

P1: $\forall x \in X, e \leq P(x, x)$ (*reflexivity*)

P2: $\forall x, y, z \in X, P(x, y) \otimes P(y, z) \leq P(x, z)$ (*transitivity*)

Defn. Given \mathcal{M} -enriched categories \mathcal{C}, \mathcal{D} , $F : \mathcal{C} \rightarrow \mathcal{D}$ is \mathcal{M} -enriched functor if:

F1: $\forall a \in \mathcal{C}, \exists! F(a) \in \mathcal{D}$

F2: $\forall a, b \in \mathcal{C}, \exists! F_{ab} \in \mathcal{M}(\mathcal{C}(a, b), \mathcal{D}(F(a), F(b)))$

F3: $\forall a \in \mathcal{C}, F_{aa} \circ id_a = id_{F(a)}$ (in \mathcal{M})

F4: $\forall a, b, c \in \mathcal{C}, F_{ac} \circ (\circ_{abc}) = (\circ_{F(a)F(b)F(c)}) \circ (F_{bc} \otimes F_{ab})$ (in \mathcal{M})

Prop. Let L be po-monoid. Given L -preordered sets (X, P) , (Y, Q) taken as L -enriched categories, then $f : (X, P) \rightarrow (Y, Q)$ is L -enriched functor iff $f : X \rightarrow Y$ is mapping such that

$$P(x, y) \leq Q(f(x), f(y)).$$

Defn. For $(L, \leq \otimes)$ a unital po-groupoid, an L -isotone map $f : (X, P) \rightarrow (Y, Q)$ is a mapping $f : X \rightarrow Y$ such that $P(x, y) \leq Q(f(x), f(y))$.

Defn (fixed-basis). For unital po-groupoid L , the category L -**PreSet** comprises L -preordered sets and L -isotone mappings together with the composition and identities of **Set**.

Theorem (fixed-basis). For each complete unital po-groupoid, the category L -**PreSet** is a topological category over **Set** w.r.t. expected forgetful functor; i.e., each L -**PreSet** is a topological construct. Hence so is **PreSet**.

Comment. Variable-basis enriched functors can be defined for enriched category theory, with corresponding variable-basis morphisms between preordered sets (X, L, P) . Topologicity quite delicate; see DMR FSS 2014.

Many-Valued Enriched Topological Systems & Their Extent Spaces

Question. Suppose bitstring x precedes bitstring y to some degree α , and bitstring y satisfies predicate a to some degree β . How should bitstring x satisfy predicate a to at least some degree related to both α and β ? What if conjunction of predicates is non-commutative? Potential applications in data-mining and pattern-matching.

Complete po-groupoid based topological systems. Let A, L be complete po-groupoids. An L -topological system (X, A, \vDash) comprises: set X , and many-valued *satisfaction relation* $\vDash : X \times A \rightarrow L$ which respectively satisfies *join, pretensor, top interchange laws*:

$$\forall x \in X, \forall \{a_\gamma\}_{\gamma \in \Gamma} \subset A, \vDash \left(x, \bigvee_{\gamma \in \Gamma} a_\gamma \right) = \bigvee_{\gamma \in \Gamma} \vDash (x, a_\gamma),$$

$$\forall x \in X, \forall a, b \in A, \vDash (x, a \otimes b) = \vDash (x, a) \otimes \vDash (x, b),$$

$$\forall x \in X, \vDash (x, \top) = \top.$$

Note. $\forall x \in X, \vDash (x, \perp) = \perp$; i.e., each bitstring never satisfies **false**.

Extents of predicates. Let (X, A, \vDash) be L -topological system. The L -extent operator $ext_L : A \rightarrow L^X$ given by

$$ext_L(a)(x) = \vDash(x, a)$$

Then ext_L preserves arbitrary \bigvee, \otimes, \top (to $\underline{\top}$) from A to L^X . So *extent space*

$$(X, (ext_L)^{\rightarrow}(A))$$

is L -topological space as defined above.

Notes.

- (1) *Each* object of $L\text{-Top}$ is produced in this way, i.e., is extent space.
- (2) Information (Chu) systems named by the character of their extent spaces; above justifies name " L -topological systems".

PO-Ringoids. $(L, \leq, \otimes, \otimes^*)$ is *po-ringoid* means:

R1: (L, \leq, \otimes) po-groupoid with pretensor \otimes

R2: (L, \leq, \otimes^*) po-groupoid with multiplication \otimes^*

R3: *left-partial distributive law* of \otimes over \otimes^* holds:

$$a \otimes (b \otimes^* c) \leq (a \otimes b) \otimes^* (a \otimes c)$$

Idea. Want to add topological system structure or topological structure to preset; (L, \leq, \otimes) with completeness and unit undergirds order, (L, \leq, \otimes^*) with completeness undergirds topology; left-partial distributive law ties together.

Note. May add conditions to po-ringoids. *Groupoidal quantale*: unital complete po-ringoid $(L, \leq, \otimes, \otimes^*, e)$ with (L, \leq, \otimes, e) unital quantale and \otimes distributes partially over \otimes^* from both left and right.

Some Examples.

(1) Complete lattice $(L, \leq, \vee, \wedge, \perp)$ unital complete po-ringoid; if also distributive, then $(L, \leq, \wedge, \vee, \top)$ integral complete po-ringoid.

(2) $([0, 1], \leq, t_{\mathbb{L}}, \wedge, 1)$ integral groupoidal quantale.

(3) $(S(L), \leq, \circ, \otimes_1, id_L), (S(L), \leq, \circ, \otimes_2, id_L)$ non-integ. groupoidal quantales^{note}.

Enriched topological systems. Let $(L, \leq, \otimes, \otimes^*, e)$ be a unital complete po-ringoid and (A, \leq, \otimes) be a complete po-groupoid. An *L-enriched topological system* (X, P, A, \vDash) comprises the following:

ES1: *L*-preordered set (X, P) using unital po-groupoid (L, \leq, \otimes)

ES2: *L*-topological system (X, A, \vDash) , where satisfaction relation $\vDash : X \times A \rightarrow L$ respectively satisfies join, *pretensor-multiplication*, top interchange laws using complete po-groupoids (L, \leq, \otimes^*) and (A, \leq, \otimes) —

$$\forall x \in X, \forall \{a_\gamma\}_{\gamma \in \Gamma} \subset A, \vDash \left(x, \bigvee_{\gamma \in \Gamma} a_\gamma \right) = \bigvee_{\gamma \in \Gamma} \vDash (x, a_\gamma),$$

$$\forall x \in X, \forall a, b \in A, \vDash (x, a \otimes b) = \vDash (x, a) \otimes^* \vDash (x, b),$$

$$\forall x \in X, \vDash (x, \top) = \top.$$

ES3: *P* and \vDash together satisfy *compatibility/enrichment axiom*:

$$\forall x, y \in X, \forall a \in A, P(x, y) \otimes \vDash (y, a) \leq \vDash (x, a).$$

Comment. This definition formulates programming question given above.

Examples. Later.

Enriched extents of predicates. $(L, \leq, \otimes, \oplus, e)$ unital complete po-ringoid, (A, \leq, \otimes) complete po-groupoid, (X, P, A, \vDash) L -enriched topological system. Put $ext_L : (A, \leq, \otimes) \rightarrow (L, \leq, \oplus)^X$

$$ext_L(a) : X \rightarrow L \quad \text{by} \quad ext_L(a)(x) = \vDash(x, a)$$

Then ext_L preserves arbitrary \bigvee , preserves \otimes of A to pointwise lifted \oplus of L , preserves \top_A to $\underline{\top}_L$. **BUT** ES3 above implies new property for ext_L :

$$\forall a \in A, \forall x, y \in X, P(x, y) \otimes ext_L(a)(y) \leq ext_L(a)(x)$$

Have *extent space* $(X, (ext_L)^{\rightarrow}(A))$, where $(ext_L)^{\rightarrow}(A) \subset L^X$ satisfies: closed under arbitrary \bigvee , closed under \oplus (lifted pointwise), contains $\underline{\top}_L$, and satisfies this compatibility/enrichment condition:

$$\forall u \in (ext_L)^{\rightarrow}(A), \forall x, y \in X, P(x, y) \otimes u(y) \leq u(x)$$

Enriched Topology (spaces). Let $(L, \leq, \otimes, \ast, e)$ be unital complete po-ringoid. Then (X, P, τ) is *L-enriched/preordered topological space* if (X, P) is (L, \leq, \otimes) -preordered set and $\tau \subset L^X$ such that τ is:

SP1: closed under arbitrary \bigvee

SP2: closed under \ast

SP3: contains $\underline{\top}_L$

SP4: $\forall u \in \tau, \forall x, y \in X, P(x, y) \otimes u(y) \leq u(x)$ (compatibility/enrichment)

Enriched Topology (fixed-basis). Let $(L, \leq, \otimes, \ast, e)$ be unital complete left-residuated po-ringoid. Then **L-EnrTop** comprises all spaces (X, P, τ) from above together with all morphisms of the form $f: (X, P, \tau) \rightarrow (Y, Q, \sigma)$ such that $f: (X, P) \rightarrow (Y, Q)$ in **L-PreSet** and $f: (X, \tau) \rightarrow (Y, \sigma)$ in **L-Top**.

Theorem. Let $(L, \leq, \otimes, \ast, e)$ be unital complete left-residuated po-ringoid. Then **L-EnrTop** is topological over each of **L-PreSet** and **Set** w.r.t. the expected forgetful functors.

Examples. Later.

Crisp/Many-Valued Specialization Preorders of Many-Valued Spaces

Recall. For (X, \mathcal{T}) be topological space: $x \leq_{\mathcal{T}} y \Leftrightarrow y \in \overline{\{x\}}$. Equivalent to say:

$$x \leq_{\mathcal{T}} y \Leftrightarrow \forall U \in \mathcal{T}, y \in U \Rightarrow x \in U$$

Defn: L -Specialization Order. Let L be complete po-groupoid and (X, τ) be L -topological space. Put: $x \leq_{\tau} y \Leftrightarrow \forall u \in \tau, u(y) \leq u(x)$. Also dual order.

Defn: L -Valued Specialization Orders. Let L be right-residuated complete po-monoid and (X, τ) be L -topological space. Put $P_{\tau} : X \times X \rightarrow L$ by

$$P_{\tau}(x, y) = \bigwedge_{u \in \tau} (u(x) \leftarrow u(y))$$

If L left-residuated complete po-monoid, then put "dual" $Q_{\tau} : X \times X \rightarrow L$ by

$$Q_{\tau}(x, y) = \bigwedge_{u \in \tau} (u(x) \searrow u(y))$$

Note. \leq_{τ} and dual are (crisp) preorders; P_{τ} and Q_{τ} are L -preorders.

Theorem. Crisp orders induced by L -valued specialization orders are precisely the crisp L -specialization orders.

Examples (spaces to enriched spaces to enriched systems). Let $(L, \leq, \otimes, \otimes, e)$ be a complete right-residuated po-ringoid such that (L, \leq, \otimes) is a monoid; let (X, τ) be (L, \leq, \otimes) -topological space.

- (1) Consider L -valued specialization order $P_\tau : X \times X \rightarrow (L, \leq, \otimes)$. Then P_τ is compatible with (X, τ) , i.e., (X, P_τ, τ) is enriched L -topological space.
- (2) Continuing from (1), put $\vDash_\tau : X \times \tau \rightarrow L$ by

$$\vDash_\tau (x, u) = u(x)$$

Then $(X, P_\tau, \tau, \vDash_\tau)$ is enriched (L, \leq, \otimes) -topological system.

- (3) "Dualize" (1,2) using Q_τ if L left-residuated equipped with right partial distrib. law, order of pretensorands in compatibility axioms reversed.

Examples. Let $(L, \leq, \otimes, \otimes, e)$ be unital complete po-ringoid, (A, \leq, \otimes) be complete po-groupoid, (X, P, A, \vDash) be enriched L -topological system. Then $(X, P, \text{ext}_L^\rightarrow(A))$ is enriched L -topological space using satisfaction relation from (2) above.

Examples (systems to enriched systems). Let $(L, \leq, \otimes, \otimes, e)$ be complete right-residuated po-ringoid such that (L, \leq, \otimes) monoid, (A, \leq, \otimes) complete po-groupoid; let (X, A, \vDash) be L -topological system. Then $(X, P_{ext_L^{\rightarrow}(A)}, A, \vDash)$ is enriched L -topological system. "Dualize" using $Q_{ext_L^{\rightarrow}(A)}$ and related concepts and satisfacton relation from (2) above.

Examples (spectra to enriched systems/spaces). Let $(L, \leq, \otimes, \otimes, e)$ be complete right-residuated po-ringoid such that (L, \leq, \otimes) monoid, (A, \leq, \otimes) complete po-groupoid. Put:

$$Lpt(A) = \{p : A \rightarrow L \mid p \text{ preserves } \bigvee, \otimes \text{ to } \otimes, \top\},$$

$$P_A : Lpt(A) \times Lpt(A) \rightarrow L \text{ by } P_A(p, q) = \bigwedge_{a \in A} (p(a) \swarrow q(a)),$$

$$\vDash_A : Lpt(A) \times A \rightarrow L \text{ by } \vDash_A(p, a) = p(a)$$

Then $(Lpt(A), P_A, A, \vDash_A)$ is enriched L -topological system. For corresponding enriched L -topological extent space $(Lpt(A), P_A, ext_L^{\rightarrow}(A))$, $P_A = P_{ext_L^{\rightarrow}(A)}$.

"Dualize" using $Q_A, Q_{ext_L^{\rightarrow}(A)}$ and related concepts.

Examples ((bit)strings from alphabets). Let:

Σ be set with $|\Sigma| \geq 2$, viewed as "alphabet"

$\Sigma^{*\omega} =: \{\text{countable strings of letters from } \Sigma\}$

$\mathbf{B} =: \mathbf{2}^\omega = \{\text{countably infinite strings of letters from } \mathbf{2}\}$ —complete Bool.
alg. with $\otimes = \circledast = \wedge$

Put $P : \Sigma^{*\omega} \times \Sigma^{*\omega} \rightarrow \mathbf{B}$ by—for $n \in \mathbb{N}$ —

$$P(\sigma_1, \sigma_2)(n) = \begin{cases} 1, & \sigma_1(n), \sigma_2(n) \text{ exist and } \sigma_1(n) = \sigma_2(n) \\ 0, & \text{otherwise} \end{cases}$$

Each $P(\sigma_1, \sigma_2)$ is *comparison bitstring*. It follows that $(\Sigma^{*\omega}, P)$ is \mathbf{B} -preordered set.

Now for $a \in \Sigma$, put $p^a : \Sigma^{*\omega} \rightarrow \mathbf{B}$ by

$$p^a(\sigma)(n) = \begin{cases} 1, & \sigma(n) \text{ exists and } \sigma(n) = a \\ 0, & \text{otherwise} \end{cases}$$

Each $p^a(\sigma)$ is *indicator string* and member of $\mathbf{B}^{\Sigma^{*\omega}}$. Now put

$$\mathbf{Q} = \langle\langle \{p^a : a \in \Sigma\} \rangle\rangle \subset \mathbf{B}^{\Sigma^{*\omega}},$$

the \mathbf{B} -topology having subbasis the indicator strings. It can be shown that \mathbf{Q} is compatible with P ; hence $(\Sigma^{*\omega}, P, \mathbf{Q})$ is \mathbf{B} -enriched topological space.

Finally, $(\Sigma^{*\omega}, P, \mathbf{Q}, \vDash_{\mathbf{Q}})$ is \mathbf{B} -enriched topological system, where $\vDash_{\mathbf{Q}} : X \times \mathbf{Q} \rightarrow L$ by

$$\vDash_{\mathbf{Q}}(x, u) = u(x)$$

"String" spaces encountered again below w.r.t. many-valued T_1 separation.

Antisymmetry & L - T_0 Many-Valued Topological Spaces

Defn (antisymmetry and partial orders). Let L be a unital po-groupoid. Then $P : X \times X \rightarrow L$ on X is (L -)antisymmetric if

$$\forall x, y \in X, P(x, y) \geq e, P(y, x) \geq e \Rightarrow x = y;$$

and L -partial order is antisymmetric L -preorder; and L -**Poset** is full subcategory of L -**PreSet** comprising all L -posets.

Comment. Above definition justified in several directions: skeleton of each L -**PreSet**; quotients of L -presets (construction requires L -antisymmetry, IIA operators/involutions, and L -symmetry—latter below with L - T_1 issues); furnishes right-adjoint of each L -**PreSet**; generalizes classical result that **Poset** is monotopological construct; characterizes fundamental L - T_0 axiom in many-valued topology.

Theorem. For each complete unital po-groupoid L , L -**Poset** is monotopological over **Set** w.r.t. expected forgetful functor; so **Poset** is monotopological construct.

Defn (T_0 separation). Let L be complete po-groupoid and (X, τ) be L -topological space. Then (X, τ) is L - T_0 if

$$\forall x, y \in X, x \neq y \Rightarrow \exists u \in \tau, u(x) \neq u(y)$$

The L - T_0 axiom has different formulations (e.g., injectivity of the L -Stone second comparison maps), and is well-established via representation theorems and compactification reflectors, including two successful forms of sobriety. Examples include spectra and the fuzzy real lines and unit intervals. More justification below.

Main Theorem. Assume: L complete po-groupoid, L -topological space (X, τ) .

- (1) \leq_τ (and dual) is antisymmetric (and po) iff (X, τ) is $L-T_0$.
- (2) Further assume L right-residuated complete po-monoid. Then $P_\tau : X \times X \rightarrow L$ is antisymmetric (and po) iff (X, τ) is $L-T_0$.
- (3) Further assume L left-residuated complete po-monoid. Then $Q_\tau : X \times X \rightarrow L$ is antisymmetric (and po) iff (X, τ) is $L-T_0$.
- (4) For L a right[left]-residuated complete po-monoid, \leq_τ (and dual) is antisymmetric iff P_τ [Q_τ] is antisymmetric. Hence for L unital quantale, \leq_τ (and dual) is antisymmetric iff P_τ is antisymmetric iff Q_τ is antisymmetric.
- (5) For L DeMorgan frame with antitone involution $' : L \rightarrow L$, the L -valued hemimetric $P'_\tau : X \times X \rightarrow L$ induced by P_τ satisfies the following positive definiteness condition—

$$[\forall x, y \in X, P'_\tau(x, y) = P'_\tau(y, x) = \perp \Leftrightarrow x = y]$$

if and only if (X, τ) is $L-T_0$.

Notes.

- (1) L -antisymmetry essentially "same as" traditional antisymmetry, so appropriate generalization.
- (2) To check L -antisymmetry of P_τ , suffices to check antisymmetry of \leq_τ .
But latter always included in former (for right-residuated complete po-monoid case) since $\chi_{\leq_\tau}^e \leq P_\tau$, where $\chi_{\leq_\tau}^e \equiv \chi_{\leq_\tau} \wedge \underline{e} : X \times X \rightarrow \{\perp, e\} \subset L$.

Symmetry & $L-T_1(1)$ / $L-T_1(2)$ Separation in Many-Valued Topology

Recall. For L unital groupoid, P is L -valued preorder on X if:

P0: $P : X \times X \rightarrow L$ is mapping (degrees of precedence)

P1: $\forall x \in X, e \leq P(x, x)$ (reflexivity)

P2: $\forall x, y, z \in X, P(x, y) \otimes P(y, z) \leq P(x, z)$ (transitivity)

Can also consider:

P3: $\forall x, y \in X, P(x, y) \geq e, P(y, x) \geq e \Rightarrow x = y$ (antisymmetry)

Defn (symmetry). Let X be set and $(L, \leq, \otimes, e, *)$ be unital IIA po-groupoid. Then P is a symmetric L -valued relation on X if P satisfies P0 and

P4: $\forall x, y \in X, P(x, y) = P^*(y, x)$ (symmetry),

$*$: $L \rightarrow L$ is IIA oper./invol. ($a^{**} = a, a \leq b \Rightarrow a^* \leq b^*, (a \otimes b)^* = b^* \otimes a^*$).

Remark. If \otimes is commutative and $*$ is chosen as id_L , then P4 becomes

$$\forall x, y \in X, P(x, y) = P(y, x)$$

Note. Many-valued symmetry above central to many-valued antisymmetry capturing quotients of L -presets as L -posets and to the latter comprising the right adjoint of the former. Important justification of many-valued symmetry.

Strategy. Know antisymmetry characterizes T_0 for traditional topology, symmetry characterizes T_1 for traditional topology, and antisymmetry (crisp or many-valued) characterizes L - T_0 for many-valued topology. Propose to *define* L - T_1 for many-valued topology using above many-valued symmetry. Accordingly, propose to define *asymmetry* for many-valued topology as the denial of above many-valued symmetry.

Defn. For L a right-residuated complete po-monoid, an L -topological space (X, τ) or its topology τ is *asymmetric* if \leq_τ or P_τ is not symmetric.

Typically, asymmetric space is also L - T_0 .

Standing Assumption. Until stated otherwise, L in sequel is unital IIA quantale.

IIA induced spaces. Let (X, τ) be L -topological space. Put:

$$\tau^* = \{u^* : u \in \tau\}, \quad T = \tau \vee \tau^*$$

Then (X, τ^*) , (X, T) are L -topological spaces; and T is smallest $*$ -invariant topology containing τ (so $T^* = T$).

Example. Suppose \otimes non-commutative with $*$ $\neq id_L$; $\exists a \in L, a^* \neq a$. Put

$$\tau = \{\perp, \underline{a}, \perp\}$$

for some set X . Then (X, τ) is L -topological space and $\tau \subsetneq T$.

Lemma. Let X be set and $x, y \in X$.

$$(1) P_\tau^* : X \times X \rightarrow L \text{ by } P_\tau^*(x, y) = \bigwedge_{u \in \tau} (u^*(y) \searrow u^*(x)).$$

$$(2) P_\tau^*(x, y) = Q_{\tau^*}(y, x).$$

(3) P_τ symmetric iff $P_\tau = Q_{\tau^*}$; Q_τ symmetric iff $Q_\tau = P_{\tau^*}$; P_τ symmetric iff Q_τ symmetric.

(4) P_T symmetric iff $P_T = Q_T$ iff Q_T symmetric.

Theorem (L -specialization *vis-a-vis* L -valued specialization w.r.t. symmetry).

- (1) Let P_τ be symmetric. Then:
 - (a) \leq_τ symmetric;
 - (b) \leq_{τ^*} coincides with \leq_τ and hence symmetric;
 - (c) \leq_T coincides with \leq_τ and hence symmetric.
- (2) Converse to (1)(a) fails, even when \leq_τ additionally assumed antisymmetric, i.e., (X, τ) also $L-T_0$.

Examples.

- (1) Let $X = \{x, y\}$, $L = \{\perp, a, b, \top\}$: $\otimes = \wedge$, $\searrow = \swarrow = \rightarrow$, $*$ = id_L , so $P_\tau^*(y, x) = P_\tau(y, x)$. Put: $u(x) = \perp, u(y) = a$; $v(x) = b, v(y) = \top$; $o(x) = b, o(y) = a$; $\tau = \{\underline{\perp}, \underline{u}, \underline{v}, \underline{o}, \underline{\top}\}$. Then (X, τ) is L -topological space which is $L-T_0$ and for which \leq_τ is both antisymmetric and symmetric (and hence trivial), and P_τ is L -antisymmetric but not L -symmetric.
- (2) As in (1), but not include \underline{o} . Then (X, τ) is L -topological space which is $L-T_0$ and for which \leq_τ is antisymmetric but not symmetric, and P_τ is L -antisymmetric but not L -symmetric.

Defn (L - T_1 separation axioms). Let (X, τ) be L -topological space.

- (1) Suppose L semiquantale. (X, τ) is L - T_1 in *first sense*, or L - $T_1(1)$, if \leq_τ antisymmetric and symmetric.
- (2) Suppose L unital IIA quantale. (X, τ) is L - T_1 in *second sense*, or L - $T_1(2)$, if P_τ antisymmetric and symmetric. So, (X, τ) is L - $T_1(2)$ if P_τ satisfies:
 - P1: $\forall x \in X, P_\tau(x, x) \geq e$;
 - P2: $\forall x, y, z \in X, P_\tau(x, y) \otimes P_\tau(y, z) \leq P_\tau(x, z)$;
 - P3: $\forall x, y \in X, P_\tau(x, y) \geq e, P_\tau(y, x) \geq e \Rightarrow x = y$;
 - P4: $\forall x, y \in X, P(x, y) = P^*(y, x)$.

Corollary. Let L be unital IIA quantale and (X, τ) be L -topological space.

- (1) L - $T_1(1)$ implies L - T_0 , but not conversely.
- (2) L - $T_1(2)$ implies L - T_0 , but not conversely.
- (3) L - $T_1(2)$ implies L - $T_1(1)$, but not conversely.

Comment. When evaluating an L -topological space for asymmetry, it merely suffices to determine if \leq_τ is not symmetric; and if the space is L - T_0 , then it suffices to determine that the space is not L - $T_1(1)$.

Reconciliation with Kubiak (1995) $L-T_1$ axiom. Let L be semiquantale and (X, τ) be L -topological space.

(1) (X, τ) is $L-T_1(K)$ if:

$$\forall x, y \in X, x \neq y \Rightarrow \exists u, v \in \tau, u(y) \not\leq u(x) \text{ and } v(x) \not\leq v(y)$$

(2) Rewrite $L-T_1(K)$:

$$\forall x, y \in X, x \neq y \Rightarrow x \not\leq_\tau y \text{ and } y \not\leq_\tau x. \quad (K1)$$

(3) Rewrite $L-T_1(1)$:

$$\forall x, y \in X, x \leq_\tau y \Leftrightarrow y \leq_\tau x. \quad (K2)$$

(4) Assume $L-T_1(K)$, suppose $x \leq_\tau y$. Then (K1) gives $x = y$, reflexivity of \leq_τ gives $y \leq_\tau x$. So $L-T_1(1)$ holds.

(5) Assume $L-T_1(1)$, suppose $x \neq y$. Antisymmetry of \leq_τ yields $x \not\leq_\tau y$ or $y \not\leq_\tau x$. But (K2) in each case yields $x \not\leq_\tau y$ and $y \not\leq_\tau x$. By (K1), $L-T_1(K)$ holds.

(6) So $L-T_1(1) \Leftrightarrow L-T_1(K)$. Different motivations: $L-T_1(K)$ motivated by giving symmetric version of $L-T_0$; $L-T_1(1)$ motivated by symmetry of L -specialization order \leq_τ .

Spectra Related Examples. Let **SQuant** be category of all semiquantales and all morphisms being those maps preserving arbitrary \bigvee and \otimes ; and let **Squant** $_{\top}$ be subcategory of all semiquantales and those semiquantale morphisms which also present \top .

Fix semiquantale L , and let A be any semiquantale. Put:

$$Lpt(A) = \mathbf{SQuant}_{\top}(A, L) = \{p : A \rightarrow L \mid p \text{ preserves } \bigvee, \otimes, \top\},$$

$$\Phi_L : A \rightarrow L^{Lpt(A)} \text{ by } \Phi_L(a) : Lpt(A) \rightarrow L \text{ by } \Phi_L(a)(p) = p(a).$$

Then $LPt(A) = (Lpt(A), (\Phi_L)^{\rightarrow}(A))$ is L -topological space, the L -spectrum of A .

Defn. Let A be semiquantale.

- (1) $c \in A - \{\top\}$ is \otimes -prime if: $\forall a, b \in A, a \otimes b \leq c \Leftrightarrow a \leq c \text{ or } b \leq c$. $Pr_{\otimes}(A)$ set of all \otimes -primes of A .
- (2) A has two related (\otimes -)primes if $\exists a, b \in Pr_{\otimes}(A), a \leq b$ and $a \neq b$.

Lemma. $LPt(A)$ is $L-T_1(1)$ iff $\forall p, q \in Lpt(A), q \leq p \Leftrightarrow p \leq q$ (in $Lpt(A)$).

Theorem. Assume L integral, \perp annihilator for \otimes in L, A . Then L consistent & A has two related primes $\Rightarrow LPt(A)$ not $L-T_1(1)$. "Iff" if $L = \mathbf{2}$.

Example. Assume L integral & consistent, \perp annihilator for \otimes in L, A , and \mathcal{T}_{cof} cofinite topology of \mathbb{R} . Then $L\text{Pt}(\mathcal{T}_{cof})$ not $L\text{-}T_1(1)$.

Examples. Let L be consistent, complete DeMorgan algebra; let $\mathbb{R}(L), \mathbb{R}_l(L), \mathbb{R}_r(L), \mathbb{I}(L), \mathbb{I}_l(L), \mathbb{I}_r(L)$ be L -fuzzy real line, L -fuzzy left-handed real line, L -fuzzy right-handed real line, L -fuzzy unit interval, L -fuzzy left-handed unit interval, L -fuzzy right-handed unit interval.

- (1) All are $L\text{-}T_0$.
- (2) $\mathbb{R}(L)$ and $\mathbb{I}(L)$ are $L\text{-}T_1(1)$; the others are not $L\text{-}T_1(1)$.
- (3) Recall $\mathbb{R}^*(L)$ and $\mathbb{I}^*(L)$, so-called "alternative" fuzzy real line and unit interval, are defined to be $L\text{Pt}(\mathcal{T})$, where \mathcal{T} is standard topology on \mathbb{R} and \mathbb{I} , respectively. For L a complete Boolean algebra, $\mathbb{R}^*(L)$ and $\mathbb{I}^*(L)$ are both L -sober and $L\text{-}T_1(1)$.
- (4) For L a complete Boolean algebra, $\mathbb{R}(L)$ and $\mathbb{I}(L)$ are both L -sober and $L\text{-}T_1(1)$. (This follows from (3)).

Examples. Recall \mathbf{B} -enriched topological spaces $(\Sigma^{*\omega}, P, \mathbf{Q})$ described above.

- (1) P is reflexive, transitive, compatible, antisymmetric, symmetric.
- (2) \mathbf{B} -valued specialization order $P_{\mathbf{Q}} = P$. Hence $(\Sigma^{*\omega}, P, \mathbf{Q})$ is $L\text{-}T_1(2)$.

Examples (behavior w.r.t. $M_L \dashv G_\chi$). For L - T_0 and L - $T_1(1)$ issues, L is semiquantale; for L - $T_1(2)$ issues, L is integral IIA quantale.

- (1) $(X, \mathcal{T}) T_0$ iff $G_\chi(X, \mathcal{T}) L$ - T_0 ; $(X, \mathcal{T}) T_1$ iff $G_\chi(X, \mathcal{T}) L$ - $T_1(1)$.
- (2) M_L reflects T_0 [T_1] to L - T_0 [L - $T_1(1)$].
- (3) G_χ preserves T_1 to L - $T_1(2)$, and reflects L - $T_1(2)$ to T_1 . Hence G_χ preserves asymmetry.
- (4) M_L reflects T_1 to L - $T_1(2)$, and hence preserves L -asymmetry.

Examples (behavior w.r.t. $\omega_L \dashv \iota_L$). Generally, L is semiquantale.

- (1) Let $(X, \tau) \in |L\text{-Top}|$. Then (X, τ) is L - $T_1(1)$ iff $\iota_L(X, \tau)$ is T_1 . Hence ι_L reflects and preserves L -asymmetry.
- (2) Let $(X, \mathcal{T}) \in |\mathbf{Top}|$. Then (X, \mathcal{T}) is T_1 implies $\omega_L(X, \mathcal{T})$ is L - $T_1(1)$, in which case ω_L reflects L -asymmetry; and the converse holds if L is completely distributive ($\otimes = \wedge$), in which case ω_L preserves asymmetry.
- (3) Continuing (2), if L admits an endomorphism other than the identity, then for (X, \mathcal{T}) being T_1 , $\omega_L(X, \mathcal{T})$ is both L - $T_1(1)$ and not L -sober.

Other Separation Issues.

- (1) Strong L - T_2 (Höhle) implies weak L - T_2 (Kubiak) implies L - $T_1(1)$. But neither Hausdorff axiom implies L -sober: for any L complete DeMorgan frame which is non Boolean, $\mathbb{R}(L)$ and $\mathbb{I}(L)$ are strong L - T_2 but not L -sober.
- (2) For L semiquantale, have: ι_L -Hausdorff implies L - $T_1(1)$ implies L - T_0 , and ι_L -Hausdorff implies ι_L -sober implies L - T_0 , and L - $T_1(1)$ and ι_L -sober unrelated. For L completely distributive DeMorgan algebra, $\mathbb{R}(L)$ and $\mathbb{I}(L)$ are ι_L -Hausdorff and ι_L -sober and L - $T_1(1)$; but if L is also non-Boolean, $\mathbb{R}(L)$ and $\mathbb{I}(L)$ are not L -sober. So ι_L -Hausdorff $\not\Rightarrow$ L -sobriety.

Example (non-commutative conjunctions in programming / web searches).

Two numerical variables x, y related to website.

- (1) y counts how many times website has been visited.
- (2) Whenever y is used in expression, website is accessed before value in y is read, so y always updated.
- (3) Each time y is read, its value increases by 1.
- (4) Predicates related to website take on traditional truth values **T** or **F**. The conjunction of predicates φ, ψ written $\varphi \sqcap \psi$, which is **T** if and only if each of φ, ψ is **T**.
- (5) Website not accessed when conjunctions of associated predicates are formed.
- (6) Conjunction $\varphi \sqcap \psi$ read left-to-right, first φ , then ψ .
- (7) Assume:

current value of x is 9, current value of y is 8

Consider predicates $P : [x = y]$, $Q : [y \geq 10]$.

- (8) Truth value of $P \sqcap Q$ is **T**: reading P first updates y to $y = 9$; reading Q second updates y yet again, giving $y = 10$; so each of P, Q is **T**.
- (9) Truth value of $Q \sqcap P$ is **F**: reading Q first updates y to $y = 9$; reading P second updates y yet again, giving $y = 10$; so each of P, Q is **F**.
- (10) Note also that $P \sqcup Q$ is **T**, while $Q \sqcup P$ is **F**.
- (11) Note if Q is replaced by $\hat{Q} : [y \geq 9]$, then:

$$P \sqcap \hat{Q} \text{ is } \mathbf{T}, \quad \hat{Q} \sqcap P \text{ is } \mathbf{F}$$

$$P \sqcup \hat{Q} \text{ is } \mathbf{T}, \quad \hat{Q} \sqcup P \text{ is } \mathbf{T}$$

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