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Relationships between Hereditary Sobriety, Sobriety, TD, T1, and Locally Hausdorff

Stephen Rodabaugh
Youngstown State University, rodabaug@math.ysu.edu

Jeffrey T. Denniston
Kent State University, jdennist@kent.edu

Austin Melton
Kent State University, amelton@kent.edu

Jamal K. Tartir
Youngstown State University, jktartir@ysu.edu

Relationships Between Hereditary Sobriety, Sobriety, T_D, T_1, and Locally Hausdorff

Jeffrey T. Denniston, Austin Melton, Stephen E. Rodabaugh*, Jamal K. Tartir

Department of Mathematical Sciences
Kent State University, Kent, OH, USA, 44242, jdennist@kent.edu

Departments of Computer Science and Mathematical Sciences
Kent State University, Kent, OH, USA, 44242, amelton@kent.edu

Institute for Applied Topology and Topological Structures
Youngstown State University, Youngstown, OH, USA 44555-3347
rodabaug@math.ysu.edu, serodabaugh@ysu.edu

Institute for Applied Topology and Topological Structures
Youngstown State University, Youngstown, OH, USA 44555-1812, jktartir@ysu.edu

*Presenter

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Contents

Basic Notions

Defn. Closed subset of space $(X, \Im)$ is *irreducible* if it is nonempty and not the union of two nonempty, proper closed subsets.

Defn. Space $(X, \Im)$ is *sober* if each irreducible closed subset is the closure of unique singleton.

Note: T_0 equivalent to each irreducible closed subset being closure of at most one singleton.

Defn. Space $(X, \Im)$ is *quasi-sober*, or S_0, if each irreducible closed subset is closure of at least one singleton.

Note: sober $\Leftrightarrow T_0 + S_0$.

Defn. Space $(X, \Im)$ is *hereditarily sober* [*hereditarily* S_0] if each subspace is sober [S_0].

Defn. Space $(X, \Im)$ is T_D if each $\{x\}'$ closed (equiv., each $\{x\}$ locally closed).

HEREDITARY SOBRIETY, T_D, LOCALLY HAUSDORFF

Basic Result. $T_2 \Rightarrow$ sober $\Rightarrow T_0$; $T_2 \Rightarrow T_1 \Rightarrow T_D \Rightarrow T_0$; no other implications except by transitivity.

Questions. How do the following fit in?
(1) locally Hausdorff
(2) hereditary sobriety

Hereditary Sobriety and T_D

Lemma. Sobriety is weakly hereditary, i.e., each closed subspace of sober space is sober.

Theorem. Space $(X, \mathfrak{I})$ is hereditarily sober $\Leftrightarrow$ it is sober and T_D.

Comments on Proof.

Necessity. Hereditarily sober $\Rightarrow$ T_D is established by series of results from S. F. Barger [QM, 1997].

Sufficiency. Two types of proof known to us: point-set proof; spectrum proof

Point-Set Proof of Sufficiency.

Let $Y \subset X$, E irreduc. closed subset of Y. <u>Show</u> E closure of some singleton of Y. Deny. Consider $\overline{E}^X$.

<u>Claim</u> $\overline{E}^X$ not closure of any singleton in X.

Deny Claim; get contradiction to previous denial using X is T_D. So Claim holds. Apply sobriety of X: $\overline{E}^X$ reducible in X and hence in $\overline{E}^X$. But Lemma says $\overline{E}^X$ is sober.

Hence $\exists$ nonempty, proper closed $E_1, E_2 \subset \overline{E}^X$, $\overline{E}^X = E_1 \cup E_2$. Can show $E_1 \not\subseteq E$ and $E_2 \not\subseteq E$, so that $E = (E_1 \cap Y) \cup (E_2 \cap Y)$ reduces E in Y.

Contradiction. So Y is S_0. $\square$

HEREDITARY SOBRIETY, T_D, LOCALLY HAUSDORFF

Spectrum Proof of Sufficiency. Given space $(Z,\mathcal{W})$, have $\Psi_Z : Z \to pt(\mathcal{W})$ by

$$\Psi_Z(z) : \mathcal{W} \to \mathbf{2} \quad \text{by} \quad \Psi_Z(z)(U) = \chi_U(z)$$

Have Ψ_Z inj. iff T_0, surj. iff S_0. Show X hereditarily S_0. Let $(Y, \mathfrak{I}_Y)$ be subspace; show Ψ_Y surj. Let $p \in pt(\mathfrak{I}_Y)$. Put

$$\varphi : \mathfrak{I} \to \mathfrak{I}_Y \quad \text{by} \quad \varphi(U) = U \cap Y$$

Then $p \circ \varphi \in pt(\mathfrak{I})$. Since X is S_0, $\exists\, x_p \in X$, $\Psi_X(x_p) = p \circ \varphi$. Since X is T_D, there is $U_p \in \mathfrak{I}$, $x_p \in U_p$ and $\{x_p\}$ closed in U_p as subspace of X.

Claim $x_p \in Y$.

Note: Claim implies $\Psi_Y(x_p) = p$, so that Y is S_0. Two possible cases:

Case A $U_p \cap Y = \varnothing$. Denial of Claim implies Case A impossible.

Case B $U_p \cap Y \neq \varnothing$. Denial of Claim implies Case B impossible.

So Claim true. $\square$

HEREDITARY SOBRIETY, T_D, LOCALLY HAUSDORFF

Corollary. Sober + $T_1 \Rightarrow$ hereditary sobriety.

Example. Sobriety $\not\Rightarrow$ hereditary sobriety. Put $Y = (\mathbb{N}, \mathfrak{I}_{cof})$. Y not sober. Put $X = Y \cup \{\omega\}$. For the topology $\mathfrak{I}$ on X, do following: open nbhds of ω are cofinite subsets of X; an open set of $n \in Y$ is of form $U \cup \{\omega\}$, where $n \in U \in \mathfrak{I}_{cof}$, and throw in the empty set. It follows that X is sober—$X = \overline{\{\omega\}}$; and $\mathfrak{I}_Y = \mathfrak{I}_{cof}$, so Y as a subspace is not sober. So X is not hereditarily sober. So X is sober and not T_D and hence sober and not T_1. Also the case X is T_0 and not T_D.

Hereditary Sobriety and Locally Hausdorff

Theorem. Locally T_2 space $(X, \mathfrak{I})$ is (hered.) sober + T_1. Hence each manifold (including non-Hausdorff) is hereditarily sober + T_1.

Comments on Proof.

For T_1. Let $x \neq y$, have open T_2 nbhd U of x. If y not in U, then done. Assume y in U. $\exists$ disjoint, open $V, W \subset U$, $x \in V$, $y \in W$. So X is T_1.

For quasi-sober. Let closed $E \subset X$, $|E| \geq 2$. Let $x \in E$. If $E \setminus \{x\}$ closed, then done, since $E = (E \setminus \{x\}) \cup \{x\}$ reduces E. Suppose $E \setminus \{x\}$ not closed—this forces $x \in \overline{E \setminus \{x\}}$. Let U be open T_2 nbhd of x, let $y \in U \cap E$ with $y \neq x$. Then $\exists$ disjoint, open $V, W \subset U$, $x \in V$, $y \in W$. Then

$$E = E \setminus (V \cap W) = (E \setminus U) \cup (E \setminus W)$$

reduces E. Hence no non-singleton closed subset is irreducible. Since X is T_1, irreducible closed subsets are precisely closures of singletons; so X is quasi-sober. □

Counter-Example (hereditarily sober + T_1, not locally Hausdorff). Put

$$X = [(0,\infty)\times\{0\}] \cup [\mathbb{N}\times\{1\}] \cup \{(1,2)\}$$

Let $\mathfrak{I}_{\mathbb{R}}$ be usual topology on $\mathbb{R}$. The basis of a topology on X is given by:

$$\text{for } (r,0)\in(0,\infty)\times\{0\},\ \text{put } \mathcal{B}_{(r,0)} = \{U\times\{0\} : r\in U\in\mathfrak{I}_{\mathbb{R}}\};$$

$$\text{for } (n,1)\in\mathbb{N}\times\{1\},\ \text{put } \mathcal{B}_{(n,1)} = \{\,[\,\{\,(n,1)\,\}\cup(U\times\{0\})\setminus\{\,(n,0)\,\}\,] : n\in U\in\mathfrak{I}_{\mathbb{R}}\};$$

$$\text{and for } (1,2),\ \text{put } \mathcal{B}_{(1,2)} = \{\,\{\,(1,2)\}\cup(\,[\,(r,\infty)\times\{0,1\}\,]\cap X\,) : r\in(0,\infty)\}.$$

Let $\mathfrak{I}$ be topology on X generated by $\bigcup_{x\in X}\mathcal{B}_x$ as basis. Observe $X\setminus\{(1,2)\}$ is open and manifold, X is T_1. The point $(1,2)$ has no Hausdorff nbhd, so X not locally Hausdorff. And each closed subset E with $|E|\geq 2$ is reducible, so X quasi-sober, hence hereditarily sober.

Counter-Example (hereditarily sober + T_1, not locally Hausdorff). Let $\{A_n\}_{n\in\mathbb{N}}$ be countable, pairwise disjoint family of countably infinite sets. Choose two, one-to-one sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ such that

$$(\{x_n\}_{n=1}^{\infty} \cap \{y_n\}_{n=1}^{\infty}) = \varnothing, \text{ and}$$

$$\forall i \in \mathbb{N},\ A_i \cap (\{x_n\}_{n=1}^{\infty} \cup \{y_n\}_{n=1}^{\infty}) = \varnothing.$$

$\forall n \in \mathbb{N}$, put $Y_n = A_n \cup \{x_n, y_n\}$, choose $z \notin \bigcup_{n=1}^{\infty} Y_n$, put $X = \bigcup_{n=1}^{\infty} Y_n \cup \{z\}$. The basis of a topology on X is given by:

$$\text{for } x \in \bigcup_{n\in\mathbb{N}} A_n,\ \text{put } \mathcal{B}_x = \{\{x\}\};$$

$$\text{for } n \in \mathbb{N},\ \text{put } \mathcal{B}_{x_n} = \{\{x_n\} \cup (A_n \setminus F) : F \subset A_n,\ |F| < \aleph_0\};$$

$$\text{for } n \in \mathbb{N},\ \text{put } \mathcal{B}_{y_n} = \{\{y_n\} \cup (A_n \setminus F) : F \subset A_n,\ |F| < \aleph_0\};$$

$$\text{and for } z,\ \text{put } \mathcal{B}_z = \left\{X \setminus \bigcup_{n\in\mathbb{N}} Y_n : n \in \mathbb{N}\right\}.$$

HEREDITARY SOBRIETY, T_D, LOCALLY HAUSDORFF

Let $\mathfrak{I}$ be topology on X generated by $\bigcup_{x \in X} \mathcal{B}_x$ as basis. Note each Y_n is modified Fort space, hence is locally T_2, but not T_2; it follows X is T_1 and also not locally T_2 (z has no Hausdorff nbhd).

Each closed subset E with $|E| \geq 2$ is reducible, so X is quasi-sober, hence hereditarily sober: this uses that X is T_1, that each $\bigcup_{i=1}^{m} Y_i$ is clopen in X, and following cases:

<u>Case A</u> E is finite. Choose $x \in E$ and write $E = \{x\} \cup (E \setminus \{x\})$.

<u>Case B</u> $E \cap \left(\bigcup_{n \in \mathbb{N}} A_n \right) \neq \varnothing$. Choose $x \in E$ and write $E = \{x\} \cup (E \setminus \{x\})$.

<u>Case C</u> E is infinite and $E \cap \left(\bigcup_{n \in \mathbb{N}} A_n \right) = \varnothing$. $\exists\, m \in \mathbb{N}$, $x_m \in E$ or $y_m \in E$. Write

$$E = [(\cup_{i=1}^{m} Y_i) \cap E] \cup [E \setminus \cup_{i=1}^{m} Y_i].$$

So each closed subset E with $|E| \geq 2$ is reducible. $\square$

Sobriety and T_1

Example (Xu-Yuan (2009)). Let $\mathfrak{I}_{\mathbb{R}}$ be usual topology on $\mathbb{R}$. Put

$$\mathfrak{I}_d = \{U \in \mathfrak{I}_{\mathbb{R}} : U \text{ dense}\} \cup \{\varnothing\}.$$

Then $(\mathbb{R}, \mathfrak{I}_d)$ is T_1 but not T_2. Xu-Yuan (2009) claim $(\mathbb{R}, \mathfrak{I}_d)$ is sober (so it is sober + T_1 but not Hausdorff). This claim is now examined.

Lemma. Let X be any topological space, U any open dense subset, and D any dense subset. Then $U \cap D$ is dense.

Lemma. Let X be any topological space such that each nonempty open subset is dense. Then X is irreducible closed set.

Theorem. Let X be any nonempty T_1 topological space such that each nonempty open subset is dense. Then X is sober if and only if $|X| = 1$. In particular, if $|X| \geq 2$, then X is infinite and non-sober.

Comments. Following hold:

(1) Šierpinski space is sober but not T_1. It can also be shown that Šierpinski space is T_D. Hence this space is hereditarily sober—or sober + T_D—and not T_1. It is also not locally Hausdorff (previous section).

(2) For infinite X, the space $(X, \mathfrak{I}_{cof})$ is T_1 but not sober.

(3) $(\mathbb{R}, \mathfrak{I}_d)$ of Xu-Yuan (2009) is T_1 but not sober. Their claim of sobriety is false.

Summary

See Hasse diagrams on later slides.

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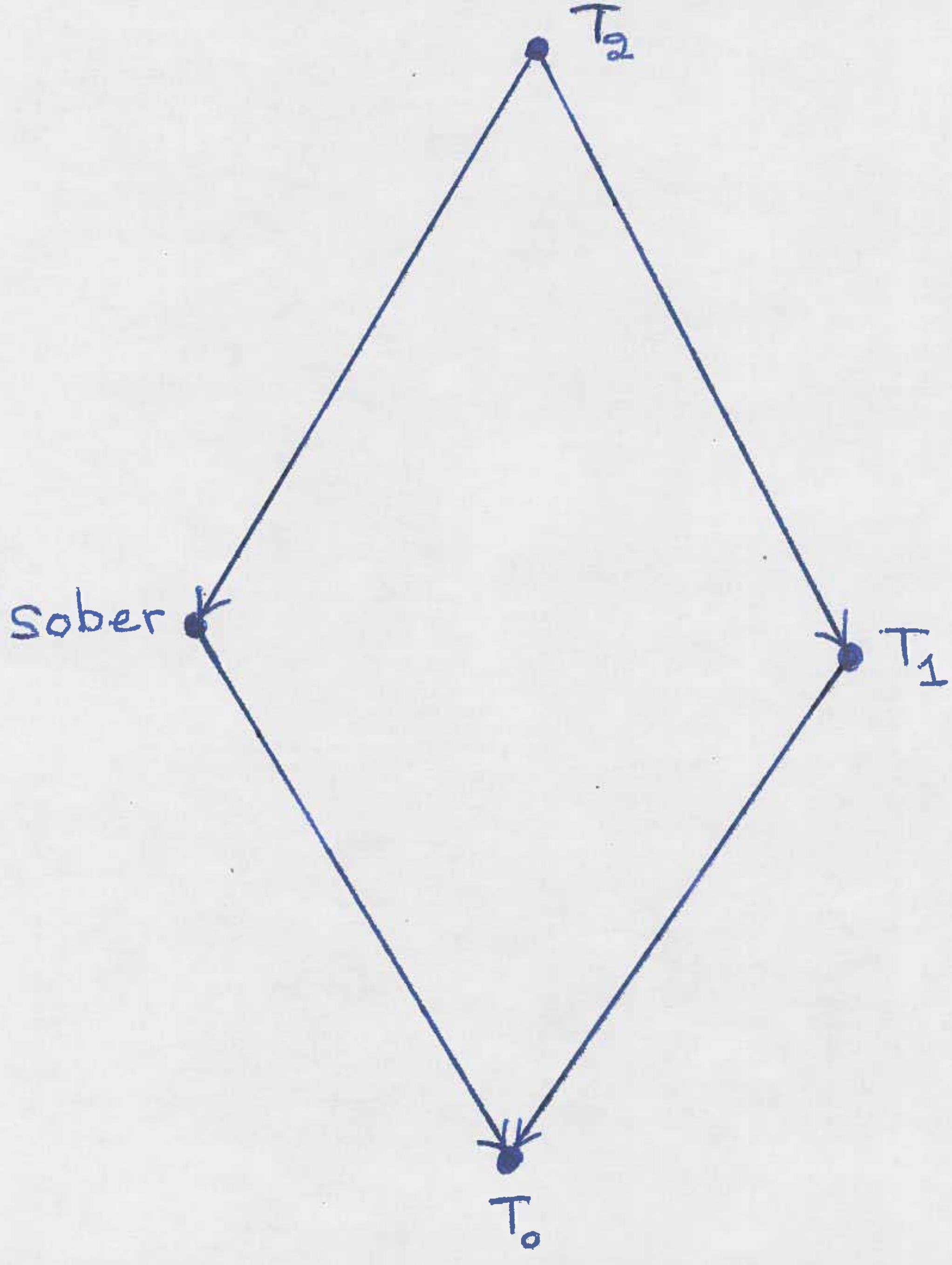
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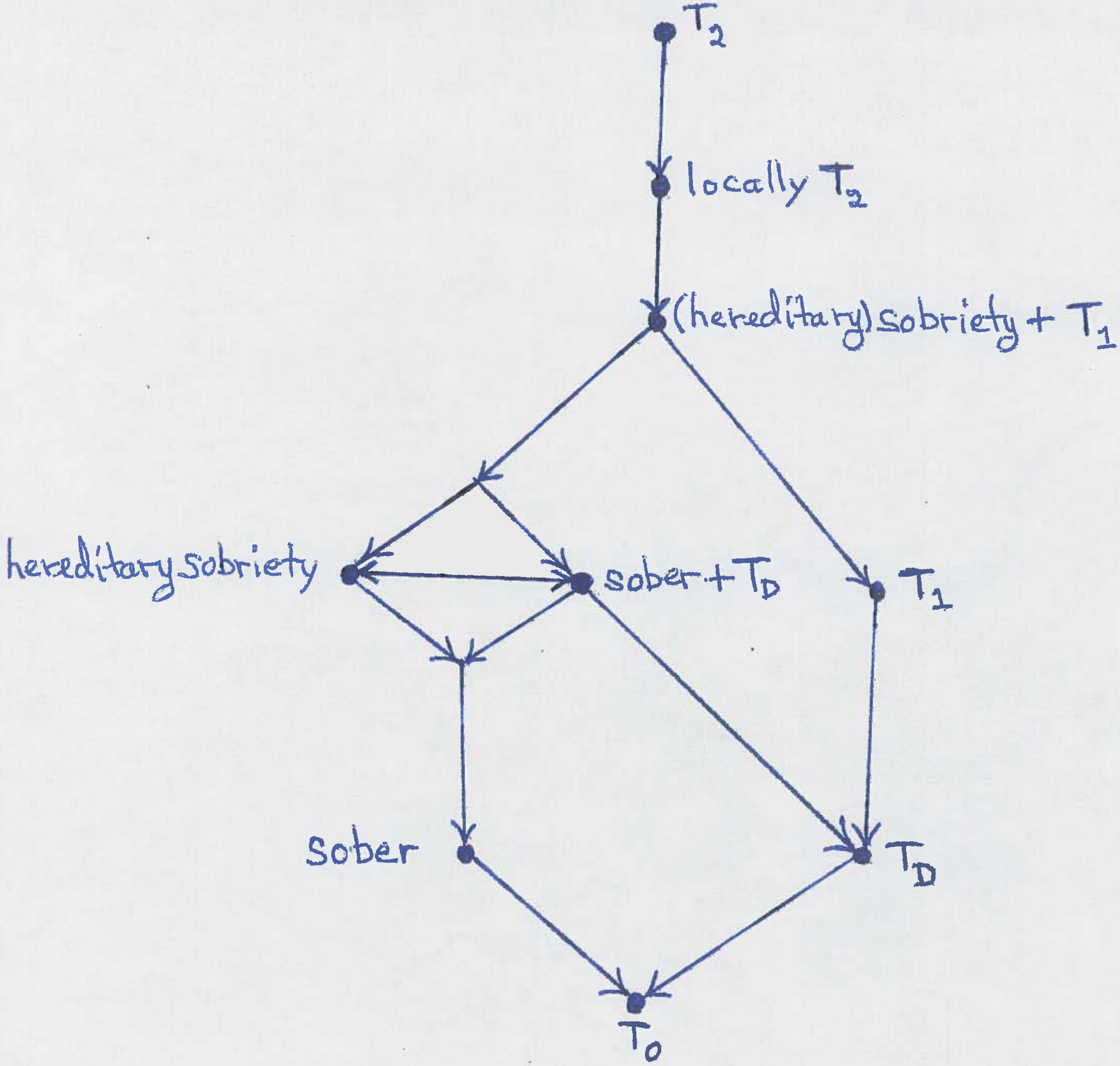
Sober
T1
T0


T_2
locally T_2
(hereditary) sobriety + T_1
hereditary sobriety
sober + T_D
T_1
sober
T_D
T_0