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Topology and Order

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Western Kentucky University

Summer Topology Conference
June 2017

33rd Summer Conference on Topology and its Applications

July 17-20, 2018
Western Kentucky University (Bowling Green, KENTUCKY)

Set-theoretic topology

Topology in analysis and topological algebras
(dedicated to W.W. Comfort)

Topological methods in geometric group theory

Dynamical systems and continuum theory

Asymmetric topology

Applications of knot theory to physical sciences

The interplay of topology and materials properties

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* Topologies as orders
Alexandroff topologies as quasiorders

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* Orders on sets of topologies
Lattices of topologies on a (finite) set

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* Orders on topologies on ordered sets
Lattices of convex topologies on a poset

An *Alexandroff topology* is a topology closed under arbitrary intersections of open sets.

Equivalently,
an Alexandroff topology is a topology whose closed sets also form a topology, or

an Alexandroff topology is a topology in which every point x has a smallest neighborhood $N(x)$

P. Alexandroff [*Diskrete Räume*, Mat. Sb. (N.S.) **2** (1937) 501–518].

Also called *principal topologies*.

Some Examples

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$Superset(\{a\})$ is the particular point topology.
$Disjoint(\{a\})$ is the excluded point topology.
$Superset(\emptyset) = Subset(X) = \mathcal{P}(X)$ is the discrete topology.

B is disjoint from A iff $B \subseteq X - A$

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so $Disjoint(A) = Subset(X - A)$.

$$B \text{ is disjoint from } A \quad \text{iff} \quad B \subseteq X - A$$
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A is disjoint from B iff $X - A \supseteq B$

so $\mathit{Disjoint}(A) = \mathit{Subset}(X - A)$.

And the $\mathit{Superset}(B)$-closed set are the sets A disjoint from B,

so the topology of $\mathit{Superset}(B)$-*closed* sets is $\mathit{Disjoint}(B)$.

$\mathcal{T}$	$\mathcal{T}$-closed sets
$Super(S)$	$Disjoint(S)$
$Disjoint(S)$	$Super(S)$
$Sub(S)$	$Super(X - S)$

$$Disjoint(S) = Sub(X - S)$$

$\mathcal{T}$	minimal neighborhoods $N(x)$
$Super(S)$	$N(x) = \{x\} \cup S$
$Disjoint(S)$	$N(x) = \begin{cases} X & \text{if } x \in S \\ \{x\} & \text{if } x \notin S \end{cases}$
$Sub(S)$	$N(x) = \begin{cases} \{x\} & \text{if } x \in S \\ X & \text{if } x \notin S \end{cases}$

Table: $Super(S)$, $Disjoint(S)$, and $Sub(S)$

Another large class of Alexandroff topologies:

Topologies on Finite Sets

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Topologies on Finite Sets

The only Hausdorff topology on a finite set X is the discrete topology.

- Every point is open.
- Every set is open.
- No point is near any other point.
- Every function with domain X is continuous.
- The only convergent sequences are eventually constant.

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Topologies on Finite Sets

Non-Hausdorff topologies can have convergent sequences converging to two or more different limits.

Another large class of Alexandroff topologies:

Topologies on Finite Sets

This is not a strong case for topologies on finite sets.

Who would use non-Hausdorff topological properties such as nearness and convergence on finite sets?

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Quasiorders

For any Alexandroff topology, there is an associated order relation

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which is reflexive and transitive.
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For any Alexandroff topology, there is an associated order relation

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Every quasiorder $\lesssim$ gives an equivalence relation $\approx$ defined by $a \approx b$ iff $a \lesssim b$ and $b \lesssim a$,

and defines a partial order $\leq$ on the equivalence classes by taking $[a] \leq [b] \iff a \lesssim b$.

Conversely, every partial order on equivalence classes of an equivalence relation on X gives a quasiorder on X,

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The increasing hull of A $= i(A) = \uparrow A = \{y \in X : \exists a \in A, y \gtrsim a\}$
The decreasing hull of A $= d(A) = \downarrow A = \{y \in X : \exists a \in A, y \lesssim a\}$

A is an increasing set if $A = i(A)$;
A is a decreasing set if $A = d(A)$;

The link between quasiorders and Alexandroff topologies

	Order Theory			Topology
$x \lesssim y$	$\Longleftrightarrow$	$x \in d(y)$	$\Longleftrightarrow$	$x \in cl\{y\}$
$\Updownarrow$				$\Updownarrow$
$y \gtrsim x$	$\Longleftrightarrow$	$y \in i(x)$	$\Longleftrightarrow$	$y \in N(x)$

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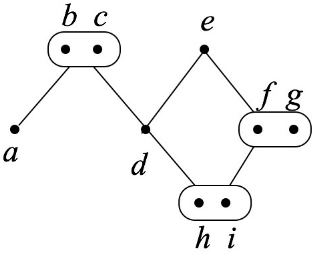
decreasing sets are closed sets

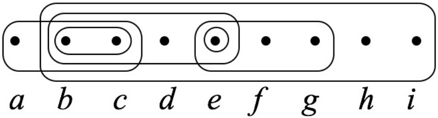
$$d(A) = cl(A)$$

increasing sets are open sets

$$i(\{x\}) = N(x)$$

Example 1



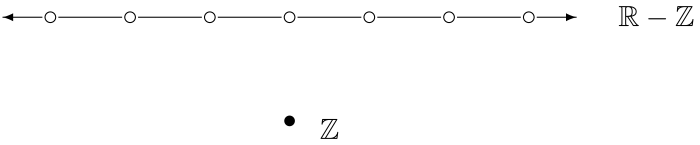


The Hasse diagram for a quasiorder and a basis for its specialization topology

Example 2

Define $\lesssim$ on $\mathbb{R}$ by $x \lesssim y$ if and only if $x = y$ or $x \in \mathbb{Z}$.

The Hasse diagram for the partial order on equivalence classes:



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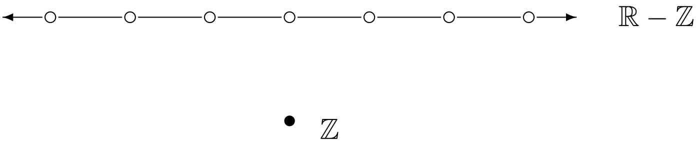

Open sets = increasing sets . . .

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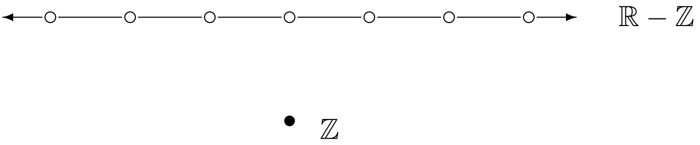

Open sets = increasing sets ...

The associated topology is *Disjoint*$(\mathbb{Z})$.

Aside: Math on Equivalence Classes

- A **quasiorder** on X gives a partial order on equivalence classes, and the increasing sets are a topology on X.

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- A **pseudometric** on X (drop $d(x, y) = 0 \Rightarrow x = y$) gives a metric on equivalence classes from $x \approx y$ iff $d(x, y) = 0$.

Example: $d(f, g) = \int_0^1 |f(x) - g(x)| dx$.

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- **Quotient spaces** correspond to topologies on equivalence classes.

Such situations may arise from *loss of resolution*.

The Lattice of Topologies

Coverings in the lattice of topologies on a finite set

Theorem (see TR 1998)

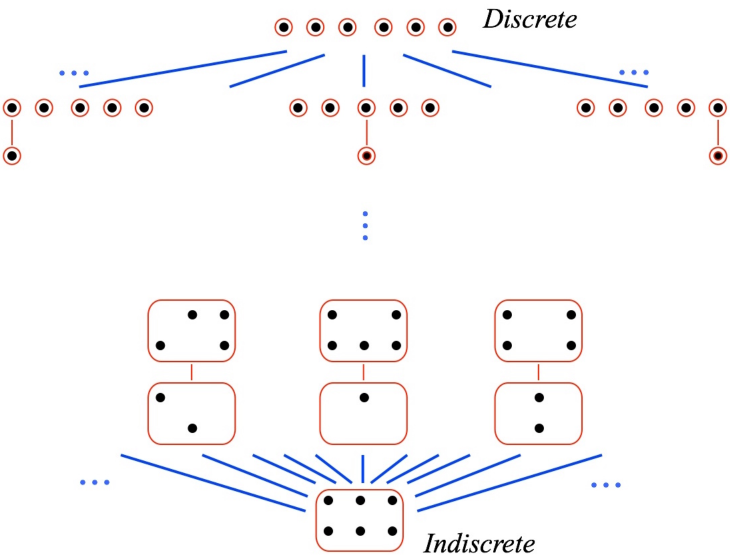
Represent two topologies on a finite set X by $(\mathcal{P}, \leq_{\mathcal{P}})$ and $(\mathcal{Q}, \leq_{\mathcal{Q}})$ where $\mathcal{P}, \mathcal{Q}$ are partitions and $\leq_{\mathcal{P}}, \leq_{\mathcal{Q}}$ are partial orders on the partitions.

If $(\mathcal{P}, \leq_{\mathcal{P}})$ covers $(\mathcal{Q}, \leq_{\mathcal{Q}})$, then either

1. *$\mathcal{P} = \mathcal{Q}$ and $\leq_{\mathcal{Q}}$ contains exactly one more ordered pair than $\leq_{\mathcal{P}}$, or*
2. *$(\mathcal{Q}, \leq_{\mathcal{Q}})$ is obtained by identifying a pair of blocks from $(\mathcal{P}, \leq_{\mathcal{P}})$, one of which covers the other.*

Corollary

The atoms in the lattice of topologies on a finite set X *are of form* $\{A, X - A\}$ *with* $a \lesssim b\ \forall a \in A, b \in X - A$.
The coatoms are discrete partitions with orders of form $\Delta \cup \{(a, b)\}$ *for some distinct* $a, b \in X$.


Discrete
Indiscrete

Complementation

Tops τ and τ' on X are complements iff $\tau \wedge \tau' = \{\emptyset, X\}$ and $\tau \vee \tau' = \mathcal{P}(X)$.

If X if finite, τ and τ' are complements iff $N(x) \cap N'(x) = \{x\}$ and the only sets open in both τ and τ' are $\emptyset$ and X.

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at least 2 complements. (Hartmanis, 1958)

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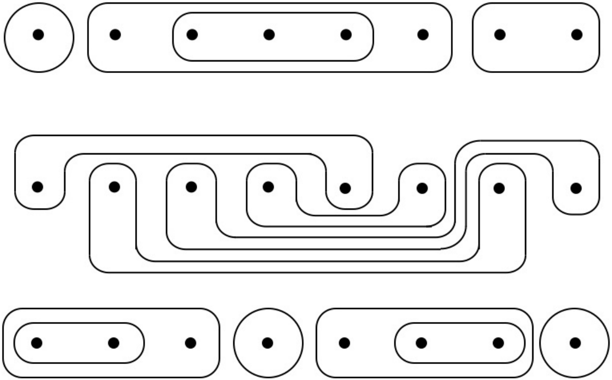
at least n-1 complements. (Schnare, 1968)

at least 2^n complements (except for some special cases). (Brown & Watson, 1996)

Convex complements in finite totally ordered top. spaces

Here are bases for a topology and two of its complements.



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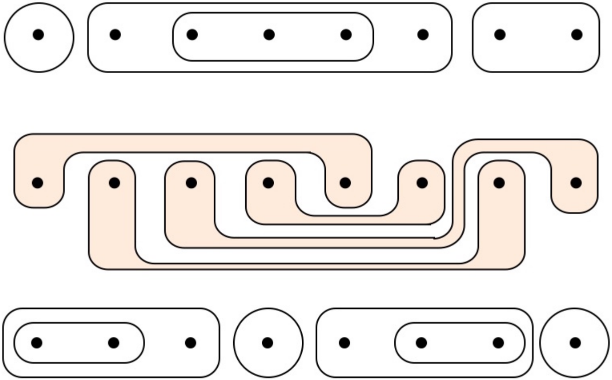

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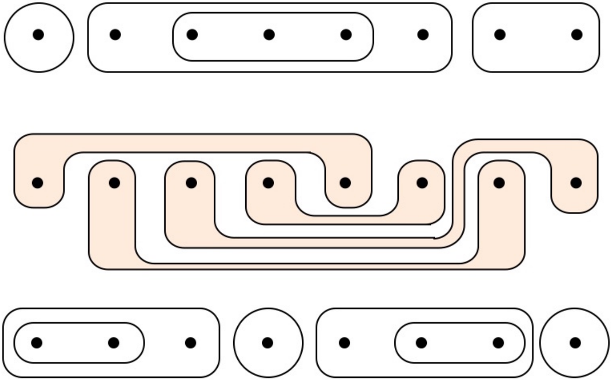

The convex one is nicer.

Convex complements in finite totally ordered top. spaces

To determine convexity, we need an order.

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OPEN QUESTION: Does every convex topology on a poset $(X, \leq)$ have a convex complement?

Theorem (TR 2013)

Every convex topology τ on a totally ordered set has a convex complement τ'.

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The proof is by constructive algorithm using the basis of minimal neighborhoods $N(x)$ for τ and $N'(x)$ for τ'.

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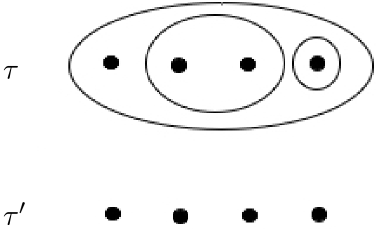
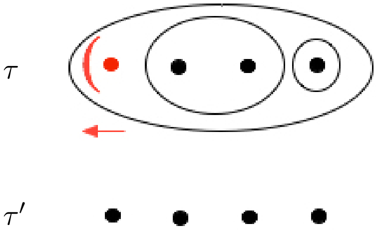
Theorem: Except at the left and right endpoints
τ' "*breaks to the left of x*" iff τ does not, and
τ' "*breaks to the right of x*" iff τ does not.


τ
τ'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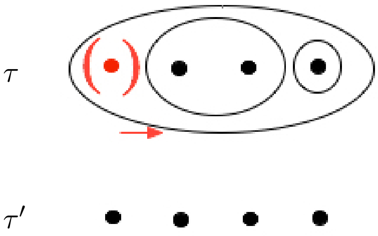
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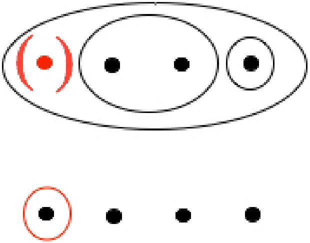
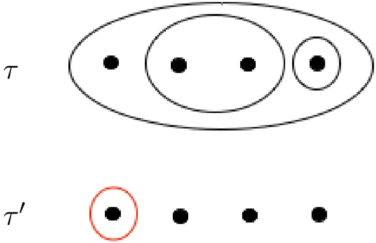
τ'

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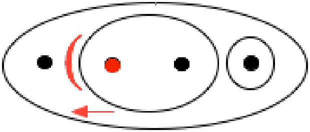
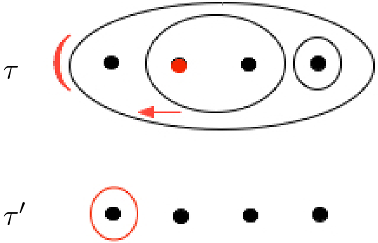
τ



τ'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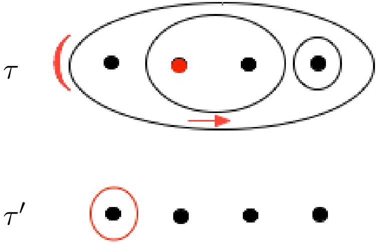

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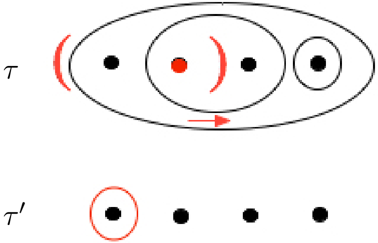
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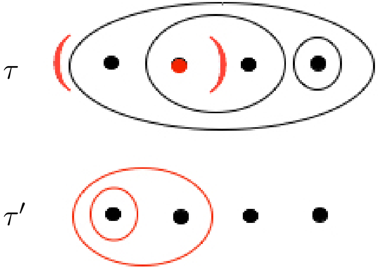
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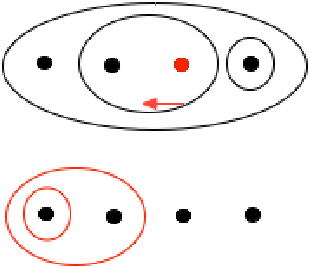

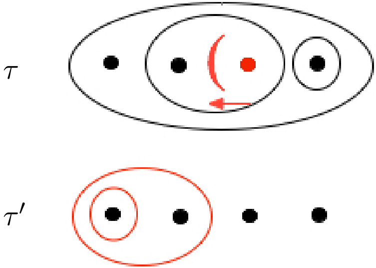
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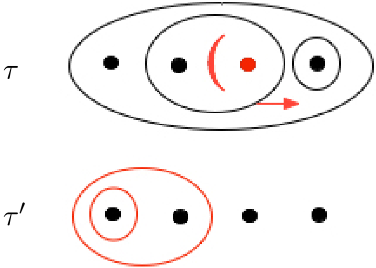
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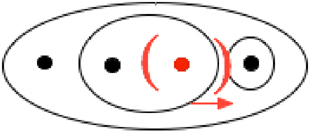

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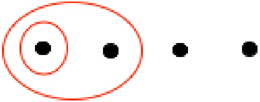

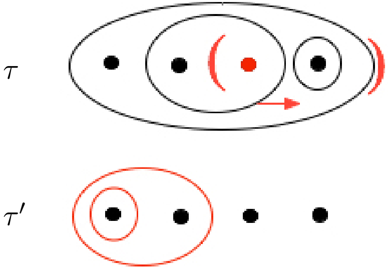
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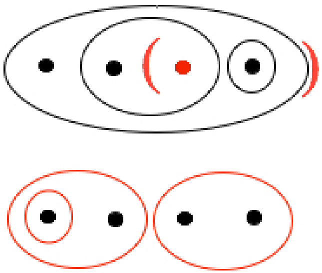

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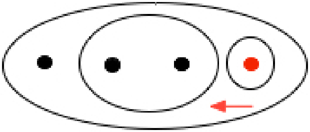

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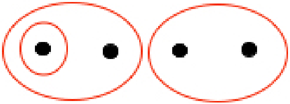

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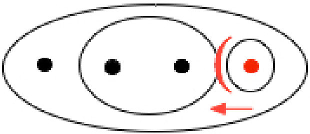

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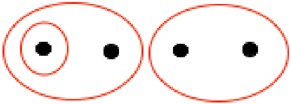

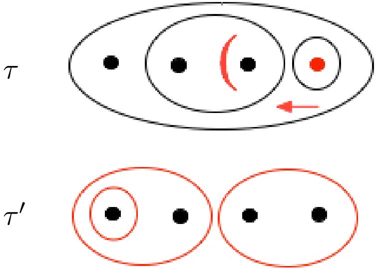

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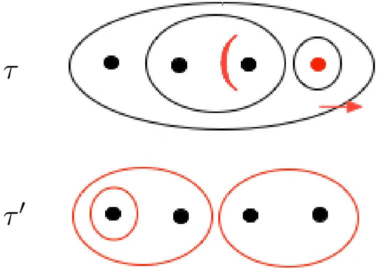
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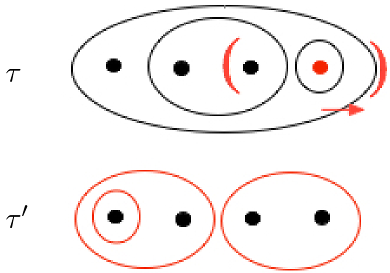
τ'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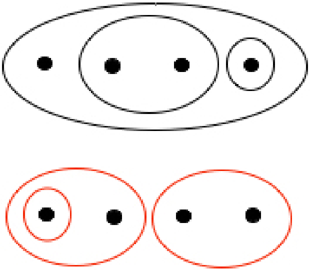

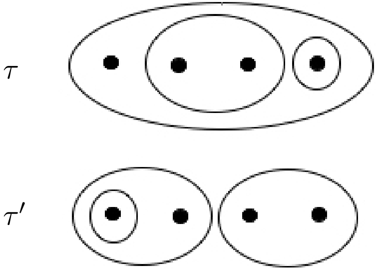

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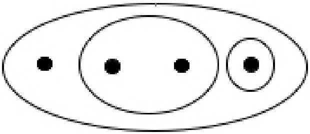
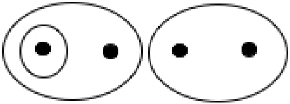
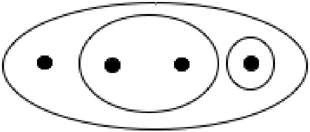


τ'



τ''

τ



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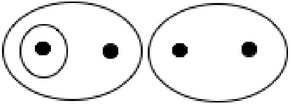

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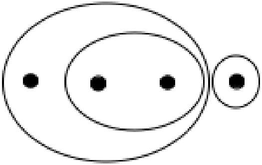
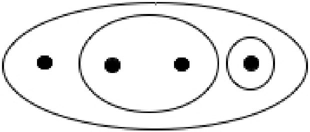


$$\tau \neq \tau''$$

τ



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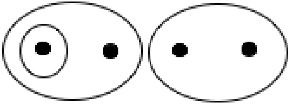

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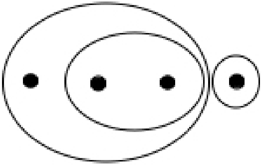
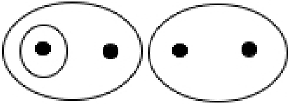
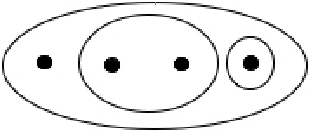


$\tau \neq \tau''$

τ'''



τ



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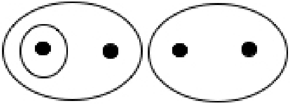

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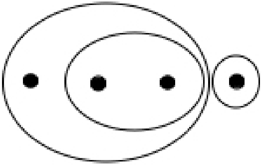

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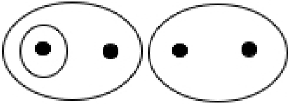
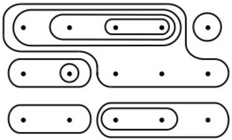


Theorem: $\tau' = \tau'''$

Theorem (TR & Mhemdi, 2017)

Any convex topology on a product of totally ordered sets with the product order $(a, b) \leq (c, d)$ iff $a \leq c$ and $b \leq d$ has a convex complement. *(finite number of finite factors)*

Example:



$\mathcal{T}$

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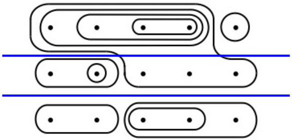

τ

View each row as a totally ordered (sub)space.

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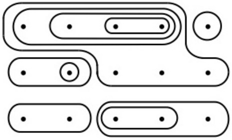
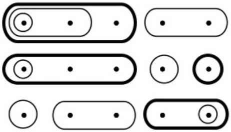


τ



τ^*

Use the algorithm to get a convex complement of each row.

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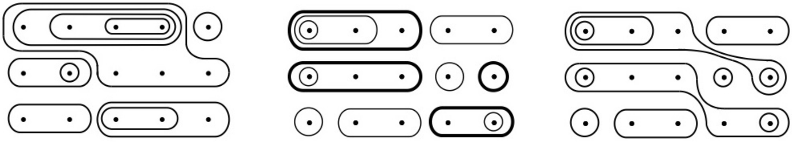

τ τ^* τ'

Link left end of row n to right end of row $n-1$, and link right end of row $n-1$ to left end of row n "as needed"*.

*** See TR & Mhemdi 2017**

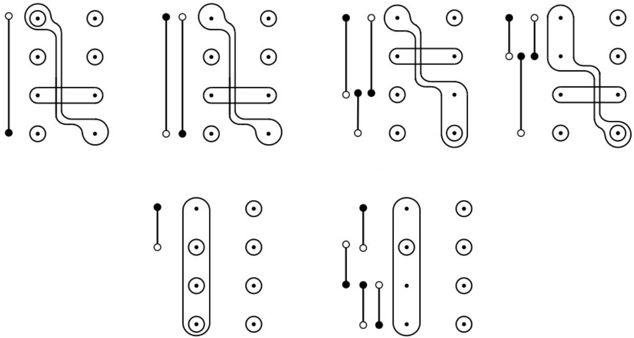

Figure: Link/Break intervals

*** See TR & Mhemdi 2017**

For each j with $1 \leq j \leq n$ we define an interval (j_-, j^+) of rows to be τ'-linked, where

$$j^+ = \begin{cases} \text{the first integer above the first } \overset{|}{\bullet} \geq j & \text{if } \exists \overset{|}{\bullet} \geq j \\ n+1 & \text{otherwise} \end{cases}$$

$$j_- = \begin{cases} \text{the first } \overset{|}{\circ} < j & \text{if } j \in \text{any } \underset{|}{\bullet} \\ \text{the first } \overset{|}{\circ} \text{ below the first } \underset{|}{\bullet} < j & \text{if } j \notin \text{any } \underset{|}{\bullet} \\ 0 & \text{otherwise.} \end{cases}$$

In this algorithm for a convex complement τ' of a convex topology τ on a product of totally ordered spaces,

$$\tau'' \neq \tau.$$

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OPEN CONJECTURE: $\tau''' = \tau'$.

Properties of Alexandroff Tops

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Alexandroff Topology	Specialization quasiorder
T_0	Partial order
$T_1 = T_2$ $= \text{discrete} = \text{metrizable}$	Equality
$T_3 = \text{completely regular}$ $= \text{pseudometrizable}$	Equivalence relation

from
Erné & Stege, *Counting Finite Posets and Topologies*
Order, 1991

Properties of Alexandroff Tops

Alexandroff Topology	Specialization quasiorder
Submaximal	??
Door	??
Resolvable	??
n-resolvable	??

Properties of Alexandroff Tops

(X, τ) is *submaximal* iff every dense set is open
iff $\overline{A} - A$ is closed $\forall A \subseteq X$
iff $A^\circ = \emptyset \Rightarrow A$ is closed.

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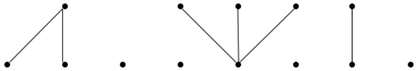

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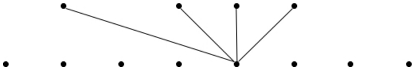
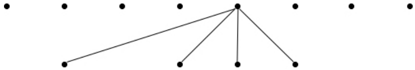


OR



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An Alexandroff space X is a door space iff the graph of the specialization order $\lesssim$ has no chains of length greater than 1, and all chains of length 1 contain a common (maximal or minimal) point.

Proof: $(\Leftarrow)$ Suppose $\lesssim$ has no chains of length > 1 and all chains of length 1 contain a common *minimal* point m.

If A is not closed $=$ not decreasing, A contains $x \gtrsim m$ but $m \notin A$.
$m \notin A \Rightarrow A$ is increasing $=$ open.

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Proof: $(\Rightarrow)$ If the condition on $\lesssim$ fails, either
there exist a chain of 3 distinct points $a \prec b \prec d$
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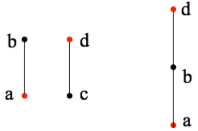

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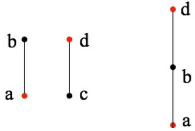

In either case, $\{a, d\}$ is neither increasing nor decreasing
(open nor closed).

Properties of Alexandroff Tops

X is resolvable iff X contains two disjoint dense subsets, is n-resolvable if it contains n mutually disjoint dense subsets, and is exactly n-resolvable if it is n- but not $(n+1)$-resolvable.

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[Hewitt, 1943], [Comfort, García-Ferreira, 1996], [Comfort, Hu, 2012]

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A is dense in X iff $X \subseteq cl(A) = d(A)$
iff $\forall x \in X, \ \exists a \in A$ with $x \lesssim a$
iff A is cofinal in X.

[A.H. Stone, 1968] A quasiordered set has a partition into n mutually disjoint cofinal sets iff each element has at least n successors.

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that is, iff every maximal element $[x]$ in the T_0-reflection $T_0(X)$ arises from a cycle $x = x_1 \prec x_2 \prec \cdots \prec x_{n-1} \prec x_n = x$ with at least n distinct elements x_i.

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An Alexandroff space $(X, \preceq)$ is exactly n-resolvable iff it is n-resolvable and there exists a maximal element $[x]$ in $T_0(X)$ generated from a cycle of length exactly n.

Functionally Alexandroff Spaces

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Any function $f : X \to X$ defines an Alexandroff topology $\mathcal{P}(f)$ on X by taking A to be closed iff $f(A) \subseteq A$ (iff A is f-*invariant*).

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Functionally Alexandroff [Zadeh Shirazi & Golestani 2011]
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In such a space,
$d(x) = cl\{x\} = \{f^n(x) : n \geq 0\} =$ the orbit of x
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Sami Lazaar and his students.

Functionally Alexandroff Spaces

A functionally Alexandroff space X is n-resolvable iff $\forall x \in X$, $|N(x)| \geq n$.

An n-resolvable space X is exactly n-resolvable iff it has at least one periodic point of period n.

The T_0-reflection of X is resolvable iff $N(x)$ is infinite for each $x \in X$.

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[Lazaar, Dahane, Turki, & TR] Suppose X is a finite resolvable functionally Alexandroff space.

The number of CD-sets is $\prod_{i=1}^{n}(2^{p_i} - 2)$ where n is the number of cycles and p_i is the length of the i^{th} cycle.

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Proof: Finite resolvable functionally Alexandroff $\Rightarrow$ every point is periodic.
$cl(CD) = d(CD) = X$ iff CD contains at least one point from each cycle. For its complement to be dense, CD must exclude at least one point from each cycle. Thus, for each cycle of length p_i, CD contains a nonempty, proper subset of those p_i points. There are $2^{p_i} - 2$ such subsets.

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33rd Summer Conference on Topology and its Applications

July 17-20, 2018

Western Kentucky University (Bowling Green, KENTUCKY)

Set-theoretic topology

Topology in analysis and topological algebras (dedicated to W.W. Comfort)

Topological methods in geometric group theory

Dynamical systems and continuum theory

Asymmetric topology

Applications of knot theory to physical sciences

The interplay of topology and materials properties