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Pseudo-Contractibility

Felix Capulín
Universidad Autonoma del Estado de Mexico, fcapulin@gmail.com

Leonardo Juarez-Villa
Universidad Autonoma del Estado de Mexico

Fernando Orozco
Universidad Autonoma del Estado de Mexico

PSEUDO-CONTRACTIBILITY

Félix Capulín Pérez
Leonardo Juarez-Villa
Fernando Orozco-Zitli

Facultad de Ciencias
Universidad Autónoma del Estado de México

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Definition

Let X and Y be topological spaces and let $f,\ g : X \to Y$ be mappings. We say that f is homotopic to g (or f and g are homotopic), written by $f \simeq g$, if there exists a mapping (called homotopy) $H : X \times I \to Y$ satisfying $H(x, 0) = f(x)$ y $H(x, 1) = g(x)$ for each $x \in X$.

Definition

Let X and Y be topological spaces and let $f,\ g : X \to Y$ be mappings. We say that f is pseudo-homotopic to g (or f and g are pseudo-homotopic) if there are a continuum C, points $a,\ b \in C$ and a mapping $H : X \times C \to Y$ such that $H(x, a) = f(x)$ and $H(x, b) = g(x)$ for each $x \in X$. We write $f \simeq_C g$ to say that f is pseudo-homotopic to g with factor space C. The mapping H is called a pseudo-homotopy between f and g with factor space C.

pseudo-homotopic

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Definition

A topological space X is said to be contractible if its identity mapping is homotopic to a constant mapping x_0 in X, i.e., if there exists a mapping $H : X \times [0,1] \to X$ satisfying $H(x,0) = x$ and $H(x,1) = x_0$, for each $x \in X$.

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Notice that X is pseudo-contractible if and only if each mapping $f : X \to X$ is pseudo-homotopic (homotopic) to a constant mapping.

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Notice that X is pseudo-contractible if and only if each mapping $f : X \to X$ is pseudo-homotopic (homotopic) to a constant mapping.

R. H. Bing introduced the notion of pseudo-contractibility. However W. Kuperberg was the first to prove that the notions of pseudo-contractibility and contractibility are different; he asked whether or not the $\sin\frac{1}{x}$ curve is pseudo-contractible. H. Katsuura proves (1992) that the $\sin\frac{1}{x}$ curve is not pseudo-contractible with factor space itself. In the same paper H. Katsuura proves that if Y is a nondegenerate indecomposable continuum such that each one of their composants is arcwise connected and X is a continuum having a proper nondegenerate arc component, then X is not pseudo-contractible with factor Y. Other questions related with this topic are the following:
Question 1.(H. Katsuura, 1992) Is the $\sin\frac{1}{x}$ curve pseudo-contractible relative to the pseudoarc?
Question 2.(Continuum theory problems, 1983) Is the pseudoarc pseudo-contractible with factor pseudoarc?

General comments

W. Debski proves (1994) that the sin $\frac{1}{x}$ curve is not pseudo-contractible. On the other hand, M. Sobolewsky (2007) shows that the only chainable continuum that is pseudo-contractible is the arc, particularly the pseudoarc is another example of a continuum that is not pseudo-contractible.
In this talk we are going to give general facts and results about pseudo-homotopies and pseudo-contractibility. As a consequence of this study we find several conditions that obstruct pseudo-contractibility and we present more examples of pseudo-contractible and non pseudo-contractible continua.

pseudo-homotopic

Let f, g in $C(X, Y)$. We say that f is related to g, written $f \simeq_* g$, if and only if there is a continuum K, such that $f \simeq_K g$

Theorem

The relation $\simeq_*$ is an equivalence relation in $C(X, Y)$.

The equivalence classes in $C(X, Y)$ under the relation $\simeq_*$ are called pseudo-homotopy classes.

Corollary

Let X and Y compact, metric spaces. Then every pseudo-homotopy class is continuumwise connected.

Corollary

Let X, Y be compact metric spaces. Every pair of mappings $f, g : X \to Y$ are pseudo-homotopic if and only if the space $C(X, Y)$ is continuumwise connected.

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Kuperberg's example

pseudo-contractibility

Theorem

Let X be a topological space, the following are equivalent:

1. X *is pseudo-contractible for any factor continuum.*
2. X *is pseudo-contractible for any factor locally connected continuum* C.
3. X *is pseudo-contractible for some factor locally connected continuum* C.
4. X *is pseudo-contractible for some arcwise-connected factor space.*
5. X *is pseudo-contractible for any arcwise-connected factor space.*
6. X *is pseudo-contractible for some factor space* C *such that* a *and* b *can be joined with an arc in* C, *where* C, a *and* b *are as in Definition 4.*
7. X *is contractible.*

observation

Note that the Kuperberg's example cannot be pseudo-contractible with an arcwise-connected continuum factor because it is not contractible.

Definition

Let X be a topological space and let A be a closed subset of X. A retraction from X onto A is a mapping $r : X \to A$ such that $r(a) = a$ for each $a \in A$. The set A is called a retract of X .

We will see that pseudo-contractibility (as well as contractibility) is preserved under retractions.

Theorem

Let X be a pseudo-contractible space. If A is a retract of X, then A is pseudo-contractible.

Proof.

Suppose X is pseudo-contractible. Then there exist a continuum C, points a, $b \in C$, $x_0 \in X$ and a mapping $H : X \times C \to X$ satisfying $H(x, a) = x$ and $H(x, b) = x_0$ for each $x \in X$.
Since A is a retract of X, there exists a mapping $r : X \to A$ such that $r(y) = y$ for each $y \in A$. Let $a_0 = r(x_0) \in A$
Consider the mapping $i : A \times C \to X \times C$ given by $i(y, c) = (y, c)$.
To show that A is pseudo-contractible consider the mapping $G : A \times C \to A$ defined by $G(y, c) = (r \circ H \circ i)(y, c)$. The function G is a pseudo-homotopy between the identity and the constant mapping a_0. □

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Theorem

Pseudo-contractibility is a topological property.

Theorem

Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of topological spaces. The space X_n is pseudo-contractible for all $n \in \mathbb{N}$ if and only if the product space $\prod_{n\in\mathbb{N}} X_n$ is pseudo-contractible.

Corollary

Let X be a topological spaces. The following five statements are equivalent:

1. *X is pseudo-contractible.*
2. *X^n is pseudo-contractible for each $n \in N$.*
3. *X^n is pseudo-contractible for some $n \in N$.*
4. *The cylinder $X \times [0,1]$ is pseudo-contractible.*
5. *$\prod_{n\in\mathbb{N}} X_n$ is pseudo-contractible, where $X_n = X$ for each $n \in \mathbb{N}$.*

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Pseudo-contractibility with respect to Y

Definition

Let X and Y be a topological spaces. We say that X is pseudo-contractible (contractible) with respect to Y if each mapping $f : X \to Y$ is pseudo-homotopic (homotopic) to a constant mapping.

Definition

A subspace Z of X is said to be pseudo-contractible (contractible) in X if the inclusion mapping is pseudo-homotopic (homotopic) to a constant mapping in X.

Note that if $Z \subset X$ and Z is pseudo-contractible (contractible) with respect to X, then Z is pseudo-contractible (contractible) in X

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Theorem

Let X be a continuum, the following sentences are equivalent.

1. *X is pseudo-contractible.*
2. *For each compact metric space Y, X is pseudo-contractible with respect to Y.*
3. *For each compact metric space Z, Z is pseudo-contractible with respect to X.*
4. *$C(X,X)$ is continuumwise connected.*

Proof.

(1) $\Rightarrow$ (2). Let Y be a compact metric space and let $f : X \to Y$ be a mapping. Since X is pseudo-contractible, then the identity mapping is pseudo-homotopic to a constant mapping. The function $f = f \circ id_X$ is pseudo-homotopic to a constant mapping, then X is pseudo-contractible with respect to Y. □

Proof.

(1) $\Rightarrow$ (3). Let Z be a compact metric space and let $g : Z \to X$ be a mapping. Since X is pseudo-contractible then the identity mapping is pseudo-homotopic to a constant mapping. The function $g = id_X \circ g$ is pseudo-homotopic to a constant, then Z is pseudo-contractible with respect to X.

The implications (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1) are trivial.

Since X is a continuum, (1) if and only if (4). $\square$

Pseudo-homotopy equivalent continua and pseudo-contractibility.

Definition

Let X and Y be topological spaces. It said that X and Y are homotopy equivalent (or have the same homotopy type), written $X \approx^E Y$, if there exist two mappings $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$.

Definition

Let X and Y be topological spaces. It said that X and Y are pseudo–homotopy equivalent (or have the same pseudo-homotopy type), written $X \approx^E_P Y$, if there exist two mappings $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq_K id_Y$ and $g \circ f \simeq_C id_X$.

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Pseudo-homotopy equivalent continua and pseudo-contractibility.

Corollary

Let X and Y compact, metric spaces. If $X \approx_P^E Y$ and one of them is pseudo-contractible, then the other one is pseudo-contractible.

The following theorem is easy to prove.

Theorem

Let X be a topological space. X is pseudo-contractible if and only if X has the same pseudo-homotopy type as a point p.

When the factor spaces C and K are equal to the interval $I = [0, 1]$, $Y \approx_P^E X$ is equal to $Y \approx^E X$.

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Definition

A compact metric space K, is called an absolute neighborhood retract, written ANR, provided that whenever K is embedded in a metric space Y, the embedded copy K' of K is a retract of some neighborhood of K' in Y.

Definition

Let X be a continuum. We say that X has trivial shape provided that each mapping from X into an ANR space is homotopic to a constant mapping.

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Definition

Let X be a continuum. We say that X has trivial shape provided that each mapping from X into an ANR space is homotopic to a constant mapping.

It is well known the following result (see [**?**]).

Theorem

Let X be a continuum, the following sentences are equivalents:

1. *X has trivial shape.*
2. *X can be written as $X = \bigcap_{n\in\mathbb{N}} X_n$, where X_n is a contractible continuum for every $n \in \mathbb{N}$.*
3. *X can be written as an inverse limit of contractible continua.*
4. *For all $\varepsilon > 0$ there exists a contractible continuum Y_ε and an ε-map f_ε from X onto Y_ε.*

It is known the following proposition.

Proposition

Let X be a compact metric space and let Y be an ANR space. If $f, g : X \to Y$ are pseudo-homotopic, then they are homotopic.

Proposition

Let X be a compact metric space and let Y be an ANR space. The space X is pseudo-contractible with respect to Y if and only if X contractible with respect to Y.

Proof.

Suppose that X is pseudo-contractible with respect to Y. Let $f : X \to Y$ be a mapping, then by Definition, f is pseudo-homotopic to a constant y_0. Since Y is an ANR space, f is homotopic to a constant y_0. Therefore X is contractible with respect to Y. The converse is trivial. □

Corollary

Let X be an ANR space. Then X is pseudo-contractible if and only if X is contractible.

Proof.

Suppose that X is pseudo-contractible with respect to Y. Let $f : X \to Y$ be a mapping, then by Definition, f is pseudo-homotopic to a constant y_0. Since Y is an ANR space, f is homotopic to a constant y_0. Therefore X is contractible with respect to Y. The converse is trivial. □

Corollary

Let X be an ANR space. Then X is pseudo-contractible if and only if X is contractible.

Corollary

Let X be a compact metric space. Then X has trivial shape if and only if X is pseudo-contractible with respect to each ANR space.

Theorem

If X is a pseudo-contractible continuum then it has trivial shape.

Proof.

Since X is a pseudo-contractible continuum, then the space X is pseudo-contractible respect each compact metric space. In particular X is a pseudo-contractible with respect to each ANR space. Therefore X has trivial shape. □

Trivial Shape and Pseudo-contractibility

It is well known that S^1 does not have trivial shape. So, S^1 is not pseudo-contractible. Notice that if $X \approx^E S^1$, then by Corollary 4, X is not pseudo-contractible. The following continua are some examples of non pseudo-contractible continua because all of them are the same type of homotopy that S^1.

1. The annulus $A = \{(x, y) : 1 \leq x^2 + y^2 \leq 2)\}$.
2. Solid Torus $S^1 \times D^2$.
3. Möbius strip.

In general, if X is a continuum such that it has no trivial shape and $Y \approx_P^E X$, then Y is not pseudo-contractible.

Theorem

Let X be a continuum. If X is a proper circle-like continuum, then X is not pseudo-contractible.

Proof.

If X is a proper circle-like continuum, there exists an essential mapping from X onto S^1 (J. Krasinkiewicz, 1974). Therefore X is not pseudo-contractible. □

Notice that the pseudo-circle is not pseudo-contractible because it is a proper circle-like continuum.

Property $b)$ and pseudo-contractibility

Definition

Let X be a topological space. We say that X has the property b) provided that for each mapping $f : X \to S^1$, there exists a mapping $g : X \to \mathbb{R}$ such that $f = \exp \circ g$, where $\exp : \mathbb{R} \to S^1$ is defined by $\exp(t) = (\cos(2\pi t), \sin(2\pi t))$ and $\mathbb{R}$ denote the real line. The mapping g is called a lift of f.

The following result is known.

Theorem

Let X be a compact metric space. The space X is contractible with respect to S^1 if and only if X has the property b)

Corollary

Let X be a compact metric space. The following conditions are equivalents:

1. *X is pseudo-contractible with respect to S^1;*
2. *X is contractible with respect to S^1;*
3. *X has the property b).*
4. *$C(X, S^1)$ is arcwise connected*

Theorem

Let X be a compact metric space. If X is pseudo-contractible then X has property b).

Proof.

If X is pseudo-contractible, X is pseudo-contractible with respect to S^1. Hence X has property b).

□

So, if X does not have Property b) then every space Y such that $Y \approx^E X$ is not pseudo-contractible.

Theorem

Every connected space X having the property b) is unicoherent.

Corollary

Let X be a continuum. If X is pseudo-contractible then it is unicoherent.

In this way, if X is not unicoherent then every space Y such that $Y \approx^E X$ is not pseudo-contractible.

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In this way, if X is not unicoherent then every space Y such that $Y \approx^E X$ is not pseudo-contractible.

Definition

A continuum X is acyclic if $\check{H}^1(X, \mathbb{Z}) = 0$; i.e., the first Cěch cohomology group with integer coefficients is trivial.

If a continuum X has property b) then it is acyclic (C. H. Dowker, 1947).

As a consequence we have the following result.

Corollary

Let X be a continuum. If X is pseudo-contractible, then X is acyclic.

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Corollary

Let X be a continuum. If X is pseudo-contractible, then X is acyclic.

So, if X is not acyclic, then every space Y such that $Y \approx^E X$, is not pseudo-contractible.

Finally, we are going to consider a continuum X, when X is a curve.

Theorem

If X is a pseudo-contractible curve then it is hereditarily unicoherent.

Proof.

Let X be a curve. If X is pseudo-contractible, then X is a curve with trivial shape. Thus X is tree like (J. Krasinkiewicz, 1975). Therefore X is hereditarily unicoherent (J. H. Case and R. E. Chamberlin, 1960). □

The following continua are not pseudo-contractible.

1. Menger sponge.
2. Sierpinski carpet.
3. Compactification of an arc with remainder a circle.

In general if X is a non hereditarily unicoherent curve. Then every space Y such that $Y \approx^E X$ is not pseudo-contractible.
Since Solenoids are hereditarily unicoherent, circle-like, and non acyclic curves, they are not pseudo-contractibles.

It is known that every hereditarily decomposable continuum is a curve, then we have the following result.

Corollary

Let X be a hereditarily decomposable continuum. If X is pseudo-contractible then X is a λ-dendroid.

The converse is not true, $sin(1/x)$ curve .

Definition

A metric space X is homogeneous provided that for each pair of points $x,\ y \in X$, there exists a homeomorphism $h : X \to X$ such that $h(x) = y$.

Theorem

Let X be a continuum. If X is a hereditarily decomposable pseudo-contractible continuum. Then X is not homogeneous.

Proof.

If X is homogeneous and hereditarily decomposable, there exists an essential mapping from X onto S^1 (S. Macias and S. B. Nadler Jr., 2009), a contradiction. □

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Note that the circle of pseudo-arcs is not pseudo-contractible because it is a decomposable homogeneous curve.

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Note that the circle of pseudo-arcs is not pseudo-contractible because it is a decomposable homogeneous curve.

Corollary

Let X be a curve. The following propositions are true:

1. *If X is pseudo-contractible with factor arcwise-connected space, then X is a uniformly arcwise-connected dendroid. Moreover. The curve X is contractible and it is a uniformly arcwise connected dendroid.*
2. *If X is pseudo-contractible and arcwise-connected then X is a dendroid.*
3. *The space X is locally connected and pseudo-contractible if and only if X is a dendrite.*

Questions

Question

(J. J. Charatonik, Selected problems in continuum theory. Topology Proc. 27 (1) (2003) 51-78.) Question 4.10.) Is every pseudo-contractible dendroid also contractible?

In general:

Question

W. Lewis, Continuum theory problems, Proceedings of the 1983 topology conference (Houston, Tex., 1983), Topology Proc. 8 (2) (1983), 361-394. Problem 118.
Does there exist a curve which is pseudo-contractible but not contractible?

Question

Does there exist an arcwise-connected continuum which is pseudo-contractible but not contractible? Or equivalently, every pseudo-contractible arcwise connected continuum is contractible?

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(J. J. Charatonik, Selected problems in continuum theory. Topology Proc. 27 (1) (2003) 51-78.) Question 4.10.) Is every pseudo-contractible dendroid also contractible?

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(E. M. Pearl (Ed.), 2011, Open problems in topology II. Elsevier. Question 19). Does there exist a nondegenerate (hereditarily) indecomposable continuum which is pseudo-contractible?