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Spaces with No S or L Subspaces

Joan Hart
University of Wisconsin-Oshkosh, hartj@uwosh.edu

Kenneth Kunen

Spaces with no S or L Subspaces

We consider spaces that contain neither an S-space nor an L-space.

Outline of talk

1. History
2. ESLC spaces
3. Spoiling an ESLC product
4. Questions

Reference

J. Hart & K. Kunen, Spaces with no S or L Subspaces, preprint...

See my home page
`http://www.uwosh.edu/faculty_staff/hartj/`

All spaces are T_3 (Hausdorff and regular).

History

A space X is
hereditarily separable (*HS*) iff all subspaces of X are separable, and
hereditarily Lindelöf (*HL*) iff all subspaces of X are Lindelöf.
Also, X is *strongly* HS/HL iff X^n is HS/HL for all $n \in \omega$.
Then, X is an *S-space* iff X is HS but not HL,
and X is a *strong S-space* iff in addition X is strongly HS.
Similarly, X is an *L-space* iff X is HL but not HS,
and X is a *strong L-space* iff in addition X is strongly HL.

In ZFC, a strong S-space exists iff a strong L-space exists [Zenor 1980].
Strong S-spaces are refuted by MA($\aleph_1$) [Kunen 1976],
but exist under CH [Kunen 1975].
L-spaces exist in ZFC [J. T. Moore 2006].
S-spaces are consistent with MA($\aleph_1$) [Szentmiklóssy 1983]
but are refuted by PFA [Todorčević 1981].

First S- or L-spaces were from a Suslin line:
L-space: Kurepa (1935) S-space: M.E. Rudin (1972)
Con($\exists$ a Suslin line): Tennenbaum, Jech, Jensen (1967,'68)
S- or L-spaces using Cohen forcing:
S-space: Hajnal and Juhász (1971-72); strong L-space: Roitman (1979)

For more background:
Roitman, Basic S and L, *Handbook of set-theoretic topology* (1984)
OR
Juhász, A survey of S- and L-spaces, *Topology, Vol. II*
(Proc. Fourth Colloq., Budapest, 1978)
OR
Juhász, *Topology and its Applications*, 158 (2011) 2460–2462

No S or L subspaces:

Def. The space X is *ESLC* iff every subspace of X is either both HS and HL or neither HS nor HL.

$$\text{HS} \rightarrow \text{HC} \leftarrow \text{HL}$$

The space X is ***HC*** (hereditarily ccc) iff every subspace of X has the ccc (countable chain condition) iff X has countable spread iff X has no uncountable discrete subspaces.

HC $\not\to$ HS and HC $\not\to$ HL: S-space $\oplus$ L-space is HC

X is HC iff X has no discrete ω_1-sequences.
X is HS iff X has no left separated ω_1-sequences.
X is HL iff X has no right separated ω_1-sequences.
A sequence $\langle x_\alpha : \alpha < \omega_1 \rangle$ is
discrete provided that each $x_\alpha \notin \mathrm{cl}(\{x_\xi : \xi \neq \alpha\})$, and
left separated provided that each $x_\alpha \notin \mathrm{cl}(\{x_\xi : \xi < \alpha\})$, and
right separated provided that each $x_\alpha \notin \mathrm{cl}(\{x_\xi : \xi > \alpha\})$.

X is ESLC iff HS $\leftrightarrow$ HL $\leftrightarrow$ HC holds for all subspaces of X.
Pf: $\Longleftarrow$: by def of ESLC. $\Longrightarrow$: Suppose X is ESLC and $Y \subseteq X$.
If Y is HS, then Y is also HC and by ESLC is HL.
If Y is not HS, apply ESLC to get a discrete ω_1-sequence in Y:
By not HS, Y has a left separated $\langle x_\alpha : \alpha < \omega_1 \rangle$.
By ESLC, this left separated $\{x_\alpha : \alpha < \omega_1\}$ is also not HL, and hence has a right separated $\langle x_{\alpha_\beta} : \beta < \omega_1 \rangle$.

ESLC examples:

0. *HS+HL spaces*:
Sorgenfrey line is HS + HL

$\mathsf{PFA}(S)[S] \to$ each compact ccc T_5 space is HS + HL [Todorčević]
T_5 = hereditarily normal
page 39: `http://www.math.toronto.edu/~stevo/todorcevic_chain_cond.pdf`

$\mathsf{MA} + \neg\mathsf{CH} \to$ each separable hereditarily supercompact space is HS + HL [Banakh,Kosztolowicz,Turek, 2014]
A space X is hereditarily supercompact if every *closed* subspace of X is supercompact; it is supercompact if it has a subbase $\mathcal{S}$ so that each cover of X by elements of $\mathcal{S}$ has a 2-element subcover.

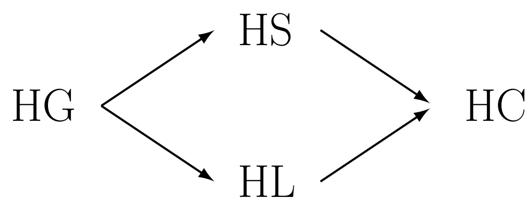
1. *metrizable spaces*: Every metric space is either second countable or has an uncountable discrete subspace.

2. *separately continuously semi-metrizable spaces*:
(semi-metrizable = symmetrizable and first countable)
Every HC separately continuously semi-metrizable space is also HG (more later ...).

3. *countable products of monarch butterfly spaces*
Special case: $\prod_{n\in\omega} X_n$ with each X_n a subspace of the Sorgenfrey line.
In ZFC, such products cannot contain an S- or L-space. It's consistent that these products can be HS + HL, or neither HS nor HL.

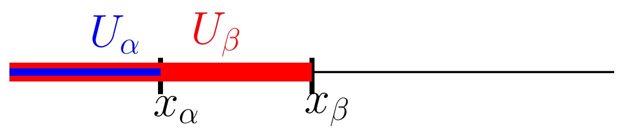
In contrast to #3:
Theorem (CH *or* V[one Cohen real]) There are X, Y that are stHG (so stHS and stHL) such that $X \times Y$ contains strong S- and L-spaces.

The fourth property:



The space X is ***HG*** iff there are no bad sequences in X.
A sequence $\langle x_\alpha : \alpha < \omega_1 \rangle$ in X is *bad* iff there are open $U_\alpha \ni x_\alpha$ for $\alpha < \omega_1$ such that for all $\{\alpha, \beta\} \in [\omega_1]^2$, $x_\alpha \notin U_\beta$ or $x_\beta \notin U_\alpha$.

HS + HL $\not\to$ HG : the Sorgenfrey line is not HG.



To see Sorgenfrey line E is HL: Suppose $Y \subseteq E$.
Recall that if $\mathcal{U}$ is a cover of Y by basic clopen $U_x = (x - \varepsilon_x, x] \cap Y$,
then $|Y \setminus \bigcup\{U_x \setminus \{x\} : x \in Y\}| \leq \aleph_0$ (because $\mathbb{R}$ is ccc).
Also, letting $V_x = U_x \setminus \{x\}$,
$\exists \mathcal{V} \in [\{V_x : x \in Y\}]^{\leq \aleph_0}$ with $\bigcup \mathcal{V} = \bigcup\{V_x : x \in Y\}$ ($\mathbb{R}$ is HL).

Recap: All four properties start

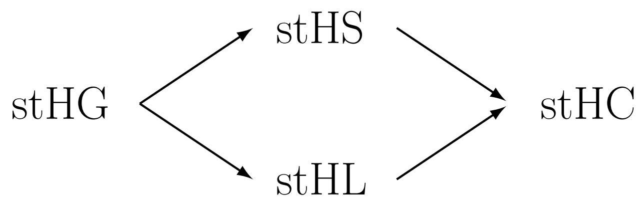
$$\text{“if } \{x_\alpha : \alpha < \omega_1\} \subseteq X \text{ and } \forall \alpha < \omega_1 \;\; x_\alpha \in U_\alpha \overset{open}{\subseteq} X\text{”}$$

and then conclude that X is:

HS iff	$\exists \alpha < \beta \, [x_\alpha \in U_\beta]$	no left separated sequence
HL iff	$\exists \alpha < \beta \, [x_\beta \in U_\alpha]$	no right separated sequence
HC iff	$\exists \alpha \neq \beta \, [x_\beta \in U_\alpha]$	no discrete sequence
HG iff	$\exists \alpha \neq \beta \, [x_\beta \in U_\alpha \;\&\; x_\alpha \in U_\beta]$	no bad sequence

The four strong properties:

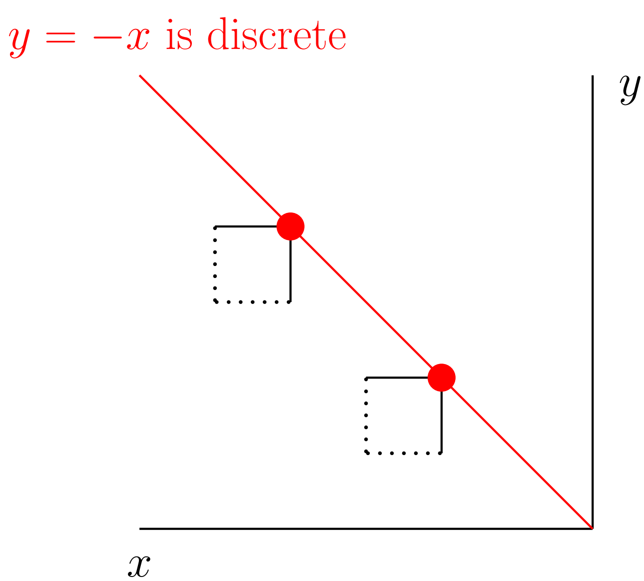
Def. X is *strongly* $\mathcal{P}$ iff X^n is $\mathcal{P}$ for all $n \in \omega$.
Ex: X is *strongly* HG/HC iff X^n is HG/HC for all $n \in \omega$.



CH: None of the four $\to$ reverses:

stHC $\not\to$ stHS:	strong L-space
stHC $\not\to$ stHL:	strong S-space
stHS + stHL $\not\to$ stHG:	Sorgenfrey $X \in [\mathbb{R}]^{\aleph_1}$
	that is n-entangled $\forall n \in \omega$

Without n-entangled:
(Sorgenfrey line) $\times$ (Sorgenfrey line) is not HC.



Separately continuous semi-metrizable spaces

Def. For any space X, let $d : X \times X \to \mathbb{R}$, and $B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$. The function d is a *scsymmetric* for X iff d satisfies the following four conditions:

1. $d(x, y) \geq 0$, and $d(x, y) = 0 \leftrightarrow x = y$.
2. $d(x, y) = d(y, x)$.
3. For each $x \in X$, $\{B(x, \varepsilon) : \varepsilon > 0\}$ is a local base at x.
4. For each x, the map $y \mapsto d(x, y)$ is continuous.

Then X is *scsymmetrizable* iff there exists a scsymmetric for X.

Note: each $B(x, \varepsilon)$ is open by #4.

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symmetric = #1 & #2; semi-metric = #1 & #2 & #3

Davis, Gruenhage, Nyikos (1978): $\exists$ a symmetrizable space in which some closed set is not a G_δ.

Our scsymmetrizable space: Every closed set is a G_δ.

Lemma. For X scsymmetrizable, the four properties (HG, HS, HL, HC) are equivalent. So every scsymmetrizable X is ESLC.

Proof: HC $\to$ HG:

Suppose $\langle x_\alpha : \alpha < \omega_1 \rangle$ is a bad sequence with open $U_\alpha \ni x_\alpha$ for $\alpha < \omega_1$ such that $\forall \{\alpha, \beta\} \in [\omega_1]^2$ $x_\alpha \notin U_\beta$ or $x_\beta \notin U_\alpha$.

By #3, each $U_\alpha \supseteq B(x_\alpha, 2^{-n_\alpha})$ for some $n_\alpha \in \omega$.

Passing to a subsequence, $n_\alpha = n \in \omega$ for all α.

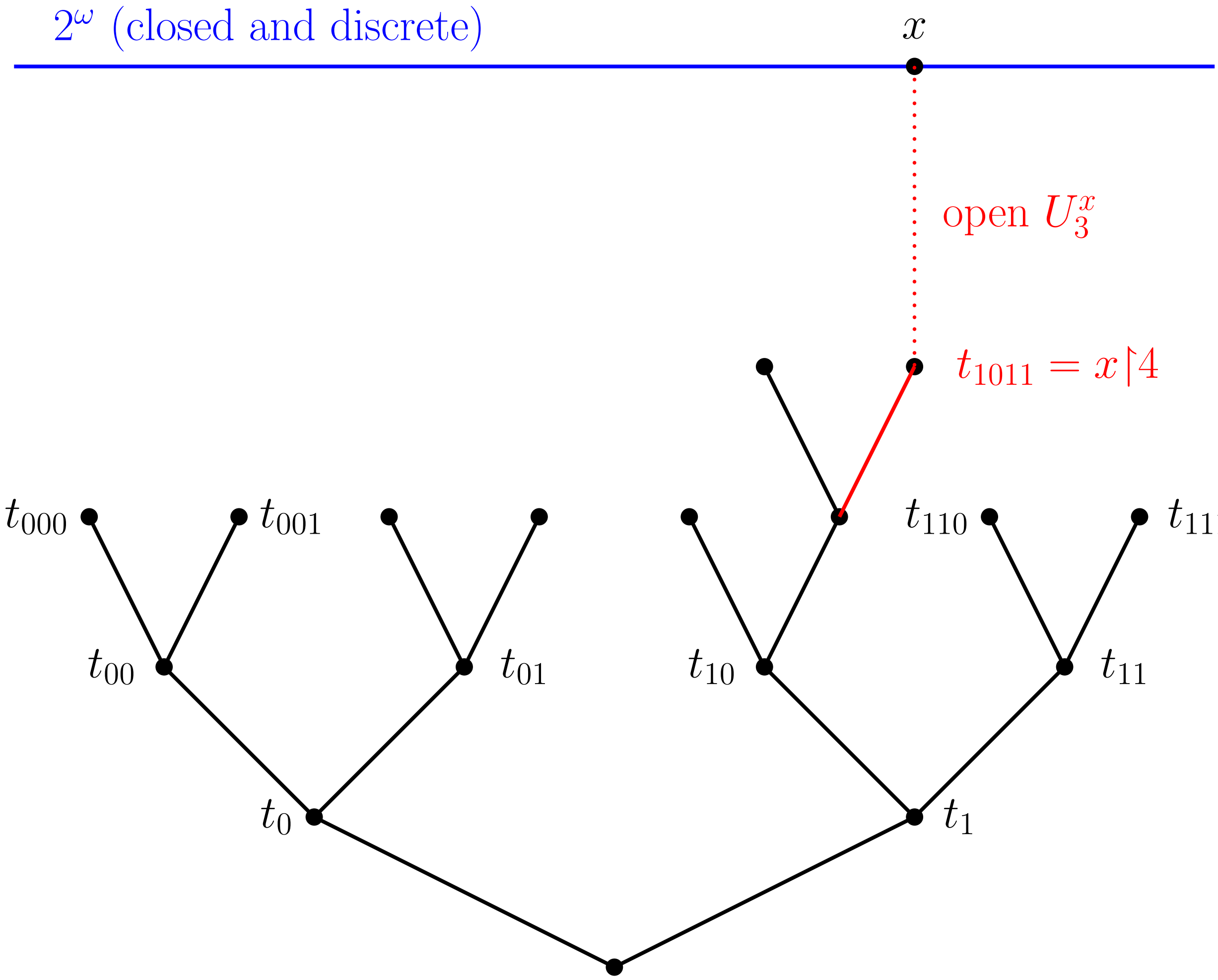
Now $d(x_\alpha, x_\beta) \geq 2^{-n}$, so $x_\alpha \notin B(x_\beta, 2^{-n})$, for all $\{\alpha, \beta\} \in [\omega_1]^2$.

By #4, the sequence $\langle x_\alpha : \alpha < \omega_1 \rangle$ is discrete.

Sorgenfrey line is ESLC, but is *not* scsymmetrizable (fails #3) ...

Simple scsymmetric example:

The *Cantor Tree Space* $X = 2^\omega \cup 2^{<\omega}$



For $x, y \in X$ with $x \not\subseteq y$: $\quad d(x,y) = d(y,x) = 2$

For $s, t \in 2^{<\omega}$ and $s \subseteq t$: $\quad d(s,t) = d(t,s) = |2^{-\mathrm{lh}(s)} - 2^{-\mathrm{lh}(t)}|$

For $x \in 2^\omega$: $\quad d(x, x\restriction n) = d(x\restriction n, x) = 2^{-n}$

d is a *scsymmetric* with basic nbds:

$2^{<\omega}$: $U_s = \{s\}$ $\qquad 2^\omega$: $U_n^x := \{x\restriction\nu : n \leq \nu \leq \omega\} = B(x, 2^{-n+1})$

d is not a metric: X separable, but 2^ω discrete $\rightarrow$ X not HC

For $Y \subseteq X$:

the *five* properties HC, HS, HL, HG, and *countable* are equivalent.

Preserving products

Theorem Every countable product of monarch spaces is ESLC.

A *butterfly* or *bow-tie*: Alexandroff and Niemytzki [1938] (Engelking 3.1.I).
Burke and van Douwen [1980]

Another version:

Def. A *monarch* space is a set X with a butterfly refinement of some separable metric on X. A *butterfly refinement* of a separable metric space $(X, \mathcal{T})$ is a topology $\widehat{\mathcal{T}}$ on X with base $\{U_x^n : x \in X \ \& \ n \in \omega\}$ satisfying:

1. $x \in U_x^n$,
2. $U_x^n \setminus \{x\}$ is $\mathcal{T}$ open,
3. $\operatorname{diam}(U_x^n) \searrow_n 0$, and
4. $\operatorname{cl}(U_x^{n+1}, \mathcal{T}) \subseteq U_x^n$.

Example: $X \subseteq \mathbb{R}$ with the Sorgenfrey topology is a monarch space:

$$U_x^n = X \cap (x - 2^{-n}, x]$$

Trivial example: $\widehat{\mathcal{T}} = \mathcal{T}$

Another trivial example: $\widehat{\mathcal{T}}$ is discrete

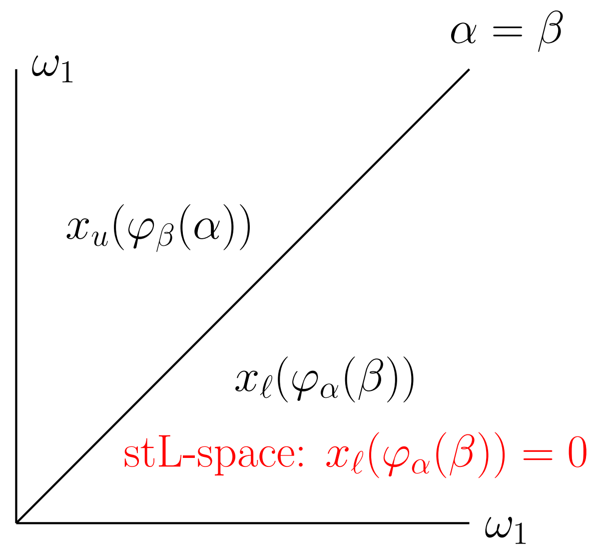
Spoiling ESLC products

Theorem (CH or V[one Cohen real]) There are X, Y that are strongly HG (hence ESLC) such that $X \times Y$ contains strong L- and S-spaces.

V[one Cohen real] Pf: Adapt strong L-space to get stHG spaces:
Our strongly HG spaces will be $\mathcal{F}^\Psi = \{f_\beta^\Psi : \beta \in \omega_1\} \subseteq \omega^{\omega_1}$,
where each $f_\beta^\Psi : \omega_1 \to 2$ by $f_\beta^\Psi(\alpha) = \Psi(\alpha, \beta)$,
and $\Psi : \omega_1 \times \omega_1 \to 2$ is defined from Cohen reals.
Instead of forcing with $\mathrm{Fn}(\omega, 2)$ to add a Cohen real $x : \omega \to 2$,
use $\mathrm{Fn}(\omega, 2) \times \mathrm{Fn}(\omega, 2)$ to add Cohen reals $x_u : \omega \to 2$, $x_\ell : \omega \to 2$.
In V, choose $\varphi_\beta : \beta \xrightarrow{1-1} \omega$ for $\beta < \omega_1$
with $\alpha \neq \beta \to |\mathrm{ran}(\varphi_\alpha) \cap \mathrm{ran}(\varphi_\beta)| < \aleph_0$.

Define $\Psi : \omega_1 \times \omega_1 \to 2$ by

$$\Psi(\alpha, \beta) = \begin{cases} 0 & \text{if } \alpha = \beta, \\ x_u(\varphi_\beta(\alpha)) & \text{if } \alpha < \beta, \\ x_\ell(\varphi_\alpha(\beta)) & \text{if } \alpha > \beta. \end{cases}$$



Now get $X \neq Y$ from x_u^i, x_ℓ^i for $i = 0, 1$, as above, with $x_\ell^0 = x_\ell^1$.
So $X = \mathcal{F}^{\Psi_0}$ and $Y = \mathcal{F}^{\Psi_1}$, where $\Psi_i : \omega_1 \times \omega_1 \to 2$. Let $f_\beta^i = f_\beta^{\Psi_i}$.
Then $\Delta = \{(f_\beta^0, f_\beta^1) : \beta < \omega_1\} \subseteq X \times Y$. Note Δ is constructed from $\widehat{\Psi} : \omega_1 \times \omega_1 \to 2 \times 2$ by $\widehat{\Psi}(\alpha, \beta) = (\Psi_0(\alpha, \beta), \Psi_1(\alpha, \beta))$.
Δ is a strong L-space because $\widehat{\Psi}(\text{lower right}) \subsetneq \widehat{\Psi}(\text{upper left})$:

$$x_\ell^0 = x_\ell^1 \text{ generic } \to \widehat{\Psi}(\text{lower right}) = \{(0,0), (1,1)\} \cong 2$$
$$\text{and } x_u^i \text{ generic } \to \widehat{\Psi}(\text{upper left}) = \{(0,0), (1,1), (1,0), (0,1)\} \cong 4.$$

Questions

1. Is HG equivalent to "hereditarily $\mathcal{P}$" for some known property?

2. Is HG $\rightarrow$ strongly HG consistent? is it provable from $\mathsf{MA}(\aleph_1)$?

3. ?? $\times$ ESLC $\rightarrow$ ESLC
   Example: Does ?? = scsymmetric work?

Theorem metric $\times$ ESLC $\rightarrow$ ESLC

Pf: Use the fact that in a metric space, every uncountable set has an uncountable subset that is either discrete or second countable.