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On the Tightness and Long Directed Limits of Free Topological Algebras

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On the tightness and long directed limits of free topological algebras

Gábor Lukács and Rafael Dahmen

Halifax, NS

June 28, 2017

Gábor Lukács and Rafael Dahmen Tightness and long directed limits of free topological algebras

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- $\mathscr{A} \subseteq \mathscr{T}$, but the inclusion may be strict.
 - For $\lambda = \omega$ and $\mathbf{A} = \mathbf{Grp}(\mathbf{Top})$, one has $\mathscr{A} \neq \mathscr{T}$ in most cases (Yamasaki's Theorem, 1998).

- τ is a cardinal.
- X is a topological space.
- $\operatorname{cl}_{\tau} Y := \bigcup \{ \overline{D} : D \subseteq Y, |D| \le \tau \}$ for $Y \subseteq X$.

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- $f: X \to Y$ is cts whenever $f_{|E}$ is cts for every $E \subseteq X$ with $|E| \le \tau$.

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Tightness of long limits

Observation

If the tightness of $(A, \mathscr{A}) = \operatorname{colim}_{\substack{\alpha < \lambda}}^{\mathbf{A}} A_{\alpha}$ is smaller than the cofinality of λ , then $\mathscr{A} = \mathscr{T}$.

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• Reason: If $E \subseteq A$ is such that $|E| \le \tau$ and $\tau < cf(\lambda)$, then $E \subseteq A_{\alpha}$ for some $\alpha < \lambda$.

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Corollary 2

For $\lambda = \omega_1$, if (A, \mathscr{A}) is countably tight, then $\mathscr{A} = \mathscr{T}$.

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For X ∈ Top, the free topological algebra on X is F(X) ∈ A together with a cts ι_X: X → F(X) such that for every B ∈ A:



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If such an *F*(*X*) exists for every *X* ∈ Top, then one has a free algebra functor *F*: Top → A that is left adjoint to the forgetful functor *U*: A → Top.

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Examples

- Take $X_{\alpha} = \alpha$ for $\alpha < \omega_1$.
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 - $V(\omega_1) = \operatorname{colim}_{\alpha < \omega_1}^{\mathsf{TVS}} V(\alpha).$

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A space X is *sequential* if $Y \subseteq X$ is closed whenever $\lim y_n \in Y$ for every sequence $\{y_n\} \subseteq Y$ such that $\lim y_n$ exists in X.

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Let X be a countably compact space such that X^n is sequential and normal for every n. Then F(X) is sequential.

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 $F(\omega_1)$ and $A(\omega_1)$ are sequential, and thus countably tight.

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Corollary 6

$$\begin{split} F(\omega_1) &= \operatorname{colim}_{\alpha < \omega_1}^{\operatorname{Grp}(\operatorname{Top})} F(\alpha) = \operatorname{colim}_{\alpha < \omega_1}^{\operatorname{Top}} F(\alpha). \\ A(\omega_1) &= \operatorname{colim}_{\alpha < \omega_1}^{\operatorname{Ab}(\operatorname{Top})} A(\alpha) = \operatorname{colim}_{\alpha < \omega_1}^{\operatorname{Top}} A(\alpha). \end{split}$$

Tightness and long directed limits of free topological algebras

Theorem 7

For a Tychonoff space X, the natural cts homomorphism $A([-1,1] \times X) \longrightarrow V(X)$ induced by $(t,x) \mapsto te_x$ is a quotient in Ab(Top), and thus an open surjection.

Gábor Lukács and Rafael Dahmen Tightness and long directed limits of free topological algebras

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Corollary 10

$$V(\omega_1) = \operatorname{colim}_{\alpha < \omega_1}^{\mathsf{TVS}} V(\alpha) = \operatorname{colim}_{\alpha < \omega_1}^{\mathsf{Top}} V(\alpha).$$

Tightness and long directed limits of free topological algebras

If X is an uncountable Tychonoff space, then V(X) has uncountable tightness and is not a *k*-space.

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Counterexample to "Fact" on ArXiv

"Fact" (Gabriyelyan & Morris, 2016)

If X is an uncountable Tychonoff space, then V(X) has uncountable tightness and is not a *k*-space.

• $V(\omega_1)$ is countably tight.

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If X is an uncountable Tychonoff space, then V(X) has uncountable tightness and is not a *k*-space.

- $V(\omega_1)$ is countably tight.
- For every limit ordinal $\alpha < \omega_1$, the product $[-1,1] \times (\alpha+1)$ is compact, and $F([-1,1] \times (\alpha+1))$ is a k_{ω} -space.

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- In particular, V(α+1) is a k-space for every limit ordinal α < ω₁.
- Hence, $V(\omega_1)$ is a *k*-space.

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