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On Product Stability of Asymptotic Property C

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Plan of the Talk

Large-scale dimension

- Motivation comes from Gromov's approach to groups as metric spaces.
- Discrete metric spaces have uninteresting topology.
- However, Gromov considered the topological properties of such discrete spaces "when viewed from infinitely far away."

Example

$\mathbb{R}$ and $\mathbb{Z}$ look the same when viewed from infinitely far away.

- This gives rise to large-scale topological properties, like large-scale connectedness, or large-scale dimension.

Product Stability

- Given a large-scale topological property, it is natural to want to know whether this property is preserved by products.
- Thus, if G and H have property $\mathscr{P}$ ...
  - Does $G \times H$ have $\mathscr{P}$?
  - Does $G * H$ have $\mathscr{P}$?

Example

Given two spaces with finite large-scale dimension, their product will have finite large-scale dimension.

- We make these notions more precise presently.

Uniformly bounded and R-disjoint

Definition

A family $\mathcal{U}$ of subsets of a metric space X is said to be uniformly bounded if $\sup\{\operatorname{diam}(U) \mid U \in \mathcal{U}\} < \infty$.

Definition

Let $R > 0$ be given. A family of subsets $\mathcal{U}$ of X is said to be R-disjoint if $d(U, V) > R$ whenever $U \neq V$ are elements of $\mathcal{U}$. Here, $d(U, V) = \inf\{d(u, v) \mid u \in U, v \in V\}$.

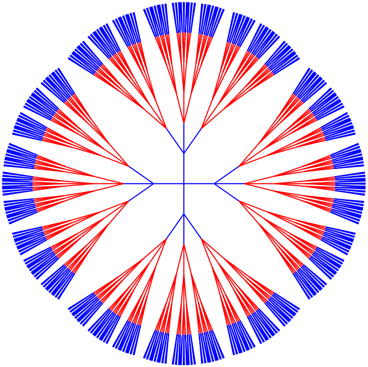
Asymptotic Dimension

Definition

A metric space X is said to have asymptotic dimension no more than n, expressed $\operatorname{asdim} X \leq n$ if $\forall R > 0 \ \exists \ n+1$-many families of subsets of X $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_n$ such that

1. $\operatorname{diam}(U)$ is uniformly bounded for $U \in \mathcal{U}_i$, ($i = 0, 1, \ldots, n$);
2. each family $\mathcal{U}_i$ is R-disjoint; and
3. $\mathcal{U}_0 \cup \cdots \cup \mathcal{U}_n$ covers X.

$\operatorname{asdim} T = 1$



$\operatorname{asdim} F_2 = 1.$

- Cover the Cayley graph of F_2 by concentric $3R$-thick annuli alternating red and blue.
- Then, take R-connected components.
- We have 2 uniformly bounded R-disjoint families of subsets that cover.

Groups with (In)Finite Asymptotic Dimension

Examples.

The following is a (non-exhaustive) list of classes of groups with finite asdim.

- Hyperbolic Groups
- Braid Groups
- Arithmetic Groups
- Relatively hyperbolic...
- Artin Groups
- Coxeter Groups
- Mapping Class Groups
- One-relator Groups

Examples.

- Thompson's group F has infinite asymptotic dimension.
- The first Grigorchuk group has infinite asymptotic dimension.

Asymptotic Property C

Definition (Dranishnikov)

A metric space X has asymptotic property C if for any number sequence $R_1, R_2, \dots$ there is an integer n and families of subsets $\mathcal{U}_0, \mathcal{U}_1, \dots, \mathcal{U}_n$ of X such that

1. $\mathrm{diam}(U)$ is uniformly bounded for $U \in \mathcal{U}_i$, $(i = 0, 1, \dots, n)$; and
2. each $\mathcal{U}_i$ is R_i-disjoint; and
3. $\mathcal{U}_0 \cup \cdots \cup \mathcal{U}_n$ covers X.

Theorem

Spaces with finite asymptotic dimension have asymptotic property C.

Examples

- $\bigoplus nZ$ has APC (Yamauchi).
- $\bigsqcup nZ^n$ can be metrized to have APC.
- Are there finitely generated groups with infinite asymptotic dimension and APC?

Asymptotic property C is preserved by direct products

Theorem (B.-Nagórko, Davilla)

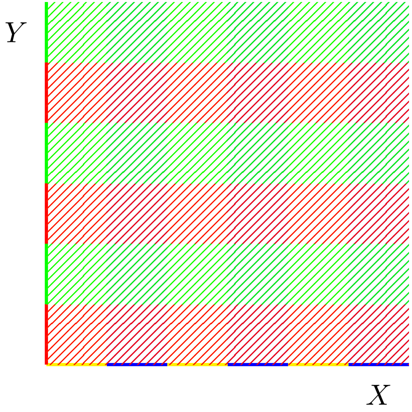
Let X and Y be metric spaces with asymptotic property C. Then, $X \times Y$ has asymptotic property C.

Compare to Topological Property C

- (E. Pol & R. Pol, 2009) There exists a separable complete metric C-space X such that X^2 fails to have C.
- (R. Pol, 1986) There is a separable metric C-space whose product with the irrationals fails to have C.
- (Rohm, 1990) If X and Y are compact C-spaces, then $X \times Y$ has C.

Asymptotic Dimension of Products

The simple way to prove that the product of spaces with finite asymptotic dimension has finite asymptotic dimension is suggested by the following picture.



Given a cover uniformly bounded R-disjoint cover of X and a uniformly bounded R-disjoint cover for Y we can construct a uniformly bounded cover for $X \times Y$ with cardinality $(\operatorname{asdim} X+1)(\operatorname{asdim} Y+1)$.

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Asymptotic Property C

Unfortunately, such a picture won't work to show APC:

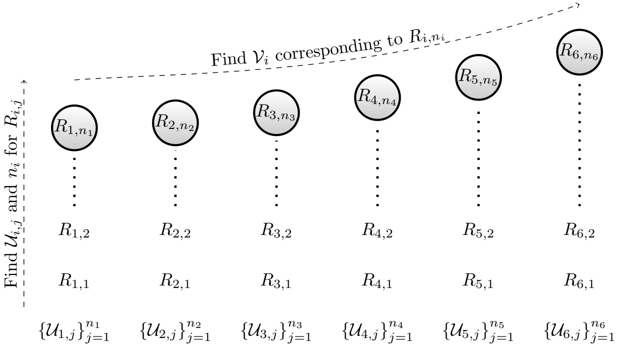
- the number of sets n for X and the number of sets m for Y may change with the sequence $R_1, R_2, \ldots$ and
- the disjointness of these rectangles is governed by the minimum of the disjointness in the X-factor and the Y-factor.

Sketch of the Proof

Let $R_1 \leq R_2 \leq \cdots$ be a given sequence of positive numbers. Rearrange the sequence into subsequences $R_{i,1} \leq R_{i,2} \leq \cdots$ like this:

R_{21}	R_{27}	R_{34}	R_{42}	R_{51}	R_{61}	$R_{1,6}$	$R_{2,6}$	$R_{3,6}$	$R_{4,6}$	$R_{5,6}$	$R_{6,6}$
R_{15}	R_{20}	R_{26}	R_{33}	R_{41}	R_{50}	$R_{1,5}$	$R_{2,5}$	$R_{3,5}$	$R_{4,5}$	$R_{5,5}$	$R_{6,5}$
R_{10}	R_{14}	R_{19}	R_{25}	R_{32}	R_{40}	$R_{1,4}$	$R_{2,4}$	$R_{3,4}$	$R_{4,4}$	$R_{5,4}$	$R_{6,4}$
R_6	R_9	R_{13}	R_{18}	R_{24}	R_{31}	$R_{1,3}$	$R_{2,3}$	$R_{3,3}$	$R_{4,3}$	$R_{5,3}$	$R_{6,3}$
R_3	R_5	R_8	R_{12}	R_{17}	R_{23}	$R_{1,2}$	$R_{2,2}$	$R_{3,2}$	$R_{4,2}$	$R_{5,2}$	$R_{6,2}$
R_1	R_2	R_4	R_7	R_{11}	R_{16}	$R_{1,1}$	$R_{2,1}$	$R_{3,1}$	$R_{4,1}$	$R_{5,1}$	$R_{6,1}$

Sketch of Proof



We find covers $\{\mathcal{U}_{i,j}\}_{j=1}^{n_i}$ of X for each column $R_{i,1}, R_{i,2}, \ldots$, and then construct a cover $\{\mathcal{V}_i\}_{i=1}^{m}$ of Y corresponding to R_{i,n_i}

Sketch of Proof

- Let $\mathcal{W}_{i,j} = \{U \times V : U \in \mathcal{U}_{i,j}, V \in \mathcal{V}_i\}$.
- Check that the collection $\bigcup_{i,j} \mathcal{W}_{i,j}$ covers $X \times Y$.
- $\operatorname{diam}(W)^2 \leq \operatorname{mesh}(\mathcal{U}_{i,j})^2 + \operatorname{mesh}(\mathcal{V}_i)^2$.
- Each family $\mathcal{W}_{i,j}$ is $R_{i,j}$-disjoint.
- Finally, we rearrange the $\mathcal{W}_{i,j}$ back into a single sequence as above.

Fibering Theorem

Our fibering theorem was inspired by old results of Hurewicz on maps that lower dimension.

Theorem (Hurewicz, 1927)

If $f : X \to Y$ is a closed map of separable metric spaces and if there is some k so that $\dim f^{-1}(y)) \leq k$ for all $y \in Y$, then

$$\dim X \leq \dim Y + k.$$

Fibering Theorem

Fibering results

In coarse geometry there are several results of the type:

- Let $f : X \to Y$ be a coarse map.
- Let Y have some coarse property $\mathscr{P}$.
- Assume that for all $R > 0$, $f^{-1}(B_R(y))$ has property $\mathscr{P}$ uniformly in y.

Then X has property P.

Theorem (B.-Nagórko)

If $f : X \to Y$ is uniformly expansive, if Y has APC, and if coarse fibers of f have uniform APC, then X has APC.

Free products

As a corollary of the fibering theorem we obtain:

Theorem (B.-Nagórko)

Let G and H be groups (in proper left-invariant metrics) with APC. Then $G * H$ has APC.

Using entirely different techniques, we can show this theorem for "free products" of general spaces:

Theorem (B.-Nagórko)

Let X and Y be metric spaces with APC. Then $X \hat{*} Y$ has APC.