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Uncountably Many Quasi-Isometry Classes of Groups of Type FP

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Joint work with

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We build $X = K(G, 1)$ as follows:

- X has a single 0–cell,
- 1–cells of X correspond to generators of G,
- 2–cells of X correspond to relations of G,
- 3–cells of X are added to kill $\pi_2(X)$,
- 4–cells of X are added to kill $\pi_3(X)$,
- etc. . .

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If $X = K(G,1)$, G acts cellularly on $\widetilde{X}$ and we have a long exact sequence

$$\cdots \longrightarrow C_i(\widetilde{X}) \longrightarrow \cdots \longrightarrow C_1(\widetilde{X}) \longrightarrow C_0(\widetilde{X}) \longrightarrow \mathbb{Z} \longrightarrow 0$$

consisting of free $\mathbb{Z}G$–modules. This leads to a definition:

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Bestvina–Brady machine:

Input: A flag simplicial complex L.
Output: A group BB_L with nice properties:

- L is $(n-1)$–connected $\iff$ BB_L is of type F_n,
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L is n–dimensional octahedron (orthoplex) $\implies$ Bieri's example.
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Question 2: How many groups are there of type FP_2?
Answer 1: Up to isomorphism: $2^{\aleph_0}$ (I.Leary'15)
Answer 2: Up to quasiisometry: $2^{\aleph_0}$ (R.Kropholler–I.Leary–S.'17)

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Input: A flag simplicial complex L, a finite collection Γ of directed edge loops in L that normally generates $\pi_1(L)$, a subset $S \subset \mathbb{Z}$.

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Theorem (I.J.Leary)

If L is a flag complex with $\pi_1(L) \neq 1$, then groups $G_L(S)$ form $2^{\aleph_0}$ isomorphism classes. If, in addition, L is aspherical and acyclic, then groups $G_L(S)$ are all of type FP.

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What is a possible example of an aspherical and acyclic flag simplicial complex L?

Take the famous Higman's group:

$$H = \langle a, b, c, d \mid a^b = a^2, b^c = b^2, c^d = c^2, d^a = d^2 \rangle.$$

Let K be its presentation complex. It is aspherical and acyclic. Take L to be the 2nd barycentric subdivision of K. Then L is a flag simplicial complex with 97 vertices, 336 edges and 240 triangles. Thus,

$$G_L(S) = \langle 336 \text{ gen's} \mid 240 \times 2 \text{ triangle relators}, \ 1 \text{ long relator } \forall n \in S \rangle.$$

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Recall that groups G_1, G_2 are **quasiisometric** (qi), if their Cayley graphs are qi as metric spaces, i.e. there exists $f\colon Cay(G_1, d_1) \to Cay(G_2, d_2)$, and $A \geq 1$, $B \geq 0$, $C \geq 0$ such that for all $x, y \in Cay(G_1)$:

$$\frac{1}{A} d_1(x, y) - B \leq d_2(f(x), f(y)) \leq A d_1(x, y) + B,$$

and for all $z \in Cay(G_2)$ there exists $x \in Cay(G_1)$ such that $d_2(z, f(x)) \leq C$.

How to distinguish groups up to qi?

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Let $TL(G)$ denote the spectrum of lengths of taut loops in the Cayley graph of a group G. Bowditch proves that if groups G_1 and G_2 are qi, then there exist constants $A, B, N > 0$ such that for every $l_1 \in TL(G_1)$, $l_1 > N$, there exist an $l_2 \in TL(G_2)$ such that $l_1 \in [Al_2, Bl_2]$ and vice versa.

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In our case, groups $G_L(S)$ do not have the property of small cancellation, so instead we use CAT(0) geometry of branched covers of cubical complexes to get estimates for the taut loops spectra. This information, and the freedom to choose arbitrary subsets $S \subset \mathbb{Z}$ for groups $G_L(S)$ allow us to construct continuously many qi classes of these groups.

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Thank you!