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A Trace Formula for Foliated Flows (working paper)

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A trace formula for foliated flows

work in progress
joint with Yuri Kordyukov & Eric Leichtnam

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Contents

Foliated flows

- M a closed manifold, $\dim M = n$.
- $\mathcal{F}$ a foliation on M, $\operatorname{codim} \mathcal{F} = 1$.
- $\phi = (\phi^t)$ a foliated flow on M, leaves $\to$ leaves.
- M^0 the union of leaves with fixed points.
- $M^1 = M \setminus M^0$.

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Hypotheses

1. The closed orbits are simple: c, any period ℓ, $x \in c$,

$$\det(\mathrm{id} - \phi_*^\ell : T_x\mathcal{F} \to T_x\mathcal{F}) \neq 0 ,$$

   - $\rightsquigarrow \epsilon_\ell(c) = \operatorname{sign}\det$.

2. The fixed points are simple: p,

$$\det(\mathrm{id} - \phi_*^t : T_pM \to T_pM) \neq 0 \quad \forall t \neq 0 ,$$

   - $\rightsquigarrow \epsilon_p = \operatorname{sign}\det$.
   - $\rightsquigarrow \overline{\phi_*^t} = e^{\varkappa_p t}$ on $N_p\mathcal{F} := T_pM/T_p\mathcal{F}$, $\varkappa_p \neq 0$.
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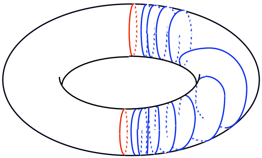

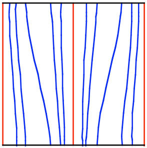

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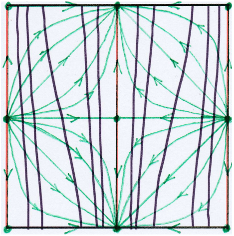

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Guillemin-Sternberg, C. Deninger

- Define:
  - a "leafwise cohomology" H^i, $\rightsquigarrow \phi^* = (\phi^{t*})$ on H^i,
  - a "distributional trace" $\operatorname{Tr}(\phi^*|_{H^i}) \in C^{-\infty}(\mathbb{R})$,
    $\rightsquigarrow$ "Leftschetz distribution"
    $L(\phi) := \operatorname{Tr}^s(\phi^*) := \sum_i (-1)^i \operatorname{Tr}(\phi^*|_{H^i}) \in C^{-\infty}(\mathbb{R})$.
- Prove a trace formula: on $\mathbb{R}^+$,

$$L(\phi) = \sum_c \ell(c) \sum_{k=1}^{\infty} \epsilon_{k\ell(c)}(c)\, \delta_{k\ell(c)} + \sum_p \frac{\epsilon_p}{|1 - e^{\varkappa_p t}|}\,,$$

c runs in the closed orbits and p in the fixed points of ϕ,
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Motivation

- **Guillemin-Sternberg**: Quantization.
- **C. Deninger**: Arithmetic Geometry (Berlin, ICM, 1998).
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Non-singular foliated flows

- ϕ has no fixed point. $\rightsquigarrow$ infinitesimal generator $X \neq 0$.
- $\rightsquigarrow$ a Riemannian metric on M so that $|X| = 1$ and $X \perp \mathcal{F}$.
  $\rightsquigarrow$ $\mathcal{F}$ is defined by local Riemannian submersions:
  a bundle-like metric, a Riemannian foliation.
- Leafwise complex: $C^\infty(M; \Lambda\mathcal{F})$, $\Lambda\mathcal{F} := \bigwedge T^*\mathcal{F}$,
  $d_\mathcal{F}$ defined by the de Rham diff. operator on the leaves.
- $\rightsquigarrow$ Reduced leafwise cohomology: $\overline{H}(\mathcal{F}) = \ker d_\mathcal{F} / \overline{\operatorname{im} d_\mathcal{F}}$.
- $\rightsquigarrow$ $\phi^{t*} : \overline{H}^i(\mathcal{F}) \to \overline{H}^i(\mathcal{F})$. Trace?
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Leafwise Hodge isomorphism

for any Riemannian foliation on a closed manifold, of arbitrary codimension

- $\delta_{\mathcal{F}}$ on $C^\infty(M;\Lambda\mathcal{F})$ defined by the adjoint of $d_{\mathcal{F}}$ on the leaves.
  $\rightsquigarrow$ Leafwise Laplacian $\Delta_{\mathcal{F}} = d_{\mathcal{F}}\delta_{\mathcal{F}} + \delta_{\mathcal{F}}d_{\mathcal{F}}$.
- Bundle-like metric $\Rightarrow$ $\Delta_{\mathcal{F}}$ is symmetric in $L^2(M;\Lambda\mathcal{F})$.
- $\mathcal{H} = \ker \Delta_{\mathcal{F}}$ in $C^\infty(M;\Lambda\mathcal{F})$, $L^2\mathcal{H} = \ker \Delta_{\mathcal{F}}$ in $L^2(M;\Lambda\mathcal{F})$
  $\Pi : L^2(M;\Lambda\mathcal{F}) \to L^2\mathcal{H}$ the orthogonal projection.
- $\exists$ a restriction $\Pi : C^\infty(M;\Lambda\mathcal{F}) \to \mathcal{H}$ inducing $\overline{H}(\mathcal{F}) \cong \mathcal{H}$.
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Leftschetz trace formula
for non-singular foliated flows (**J.A., Y. Kordyukov, 2002**)

- $\forall f \in C_c^\infty(\mathbb{R})$, the operator

$$A_f = \int_{\mathbb{R}} \phi^{t*} \cdot f(t)\, dt \circ \Pi \,.$$

is smoothing ($\rightsquigarrow$ of trace class) ($\phi^{t*} \circ \Pi$ is not.)

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Difficulties

- Recall: $M = M^0 \sqcup M^1$,
  $M^0 =$ (finite) union of (compact) leaves with fixed points.
- $\mathcal{F}$ is not Riemannian,
  $\mathcal{F}^1 := \mathcal{F}|_{M^1}$ is Riemannian,
  $\mathcal{F}$ is a transversely affine foliation almost without holonomy.
- The Schwartz kernel of A_f is not smooth at M^0.
- $\rightsquigarrow (C^\infty(M; \Lambda\mathcal{F}), d_{\mathcal{F}})$ doesn't work,
  $\rightsquigarrow$ another leafwise complex,
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Distributional leafwise forms conormal to M^0

- $\mathfrak{X}(M,\mathcal{F}) = \{$infinitesimal transformations of $(M,\mathcal{F})\}$
$= \{$infinitesimal generators of foliated flows$\}$.
- $\mathfrak{X}(M,\mathcal{F})$ generates the $C^\infty(M)$-module
$\mathfrak{X}(M,M^0) = \{\, Y \in \mathfrak{X}(M) \mid Y$ is tangent to $M^0\,\}$.
- $\mathfrak{X}(M,M^0) \rightsquigarrow \mathrm{Diff}(M,M^0;\Lambda\mathcal{F})$,
$d_{\mathcal{F}} \in \mathrm{Diff}(M,M^0;\Lambda\mathcal{F})$.
- $H^s(M;\Lambda\mathcal{F})$ Sobolev space of order s.

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Distributional leafwise forms conormal to M^0 (contd.)

- Distributional leafwise forms conormal to M^0:

$$I^{[s]}(M, M^0; \Lambda\mathcal{F}) = \{\, \alpha \in H^s(M; \Lambda\mathcal{F}) \mid \mathrm{Diff}(M, M^0; \Lambda\mathcal{F}) \cdot \alpha \subset H^s(M; \Lambda\mathcal{F}) \,\} ,$$

$$I(M, M^0; \Lambda\mathcal{F}) = \bigcup_s I^{[s]}(M, M^0; \Lambda\mathcal{F}) .$$

- $I(M, M^0; \Lambda\mathcal{F})$, $\quad d_{\mathcal{F}}$ $\equiv$ the continuous extension of $d_{\mathcal{F}}$ to $C^{-\infty}(M; \Lambda\mathcal{F})$.
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Canonical short exact sequence

- $\alpha \in I(M, M^0; \Lambda\mathcal{F}) \rightsquigarrow \exists\, \alpha|_{M^1} \in C^\infty(M^1; \Lambda\mathcal{F}^1)$.
- $\rightsquigarrow$ a canonical short exact sequence

$$0 \to \{\, \alpha \in I(M, M^0; \Lambda\mathcal{F}) \mid \operatorname{supp}\alpha \subset M^0 \,\} \to I(M, M^0; \Lambda\mathcal{F}) \to \{\, \alpha|_{M^1} \mid \alpha \in I(M, M^0; \Lambda\mathcal{F}) \,\} \to 0\ .$$

- $\exists$ a non-canonical continuous section of complexes

$$I(M, M^0; \Lambda\mathcal{F}) \leftarrow \{\, \alpha|_{M^1} \mid \alpha \in I(M, M^0; \Lambda\mathcal{F}) \,\}\ .$$

- $\rightsquigarrow$ direct sum decomposition of $\overline{H}(I(M, M^0; \Lambda\mathcal{F}), d_{\mathcal{F}})$.
- $\rightsquigarrow$ define $L(\phi)$ on both terms of the direct sum, and study the corresponding trace formulae.

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Term supported on M^0

- Assume $\mathcal{F}$ transversely oriented $\rightsquigarrow \exists\, \omega, \eta \in C^\infty(M; \Lambda^1 M)$ such that $T\mathcal{F} = \ker \omega$ and $d\omega = \omega \wedge \eta$.
- $\mathcal{F}$ transversely affine $\Leftrightarrow$ we can assume $d\eta = 0$.
- Using δ-sections at M^0 and their transverse derivatives,

$$\{\, \alpha \in I(M, M^0; \Lambda\mathcal{F}) \mid \operatorname{supp} \alpha \subset M^0 \,\} \equiv \bigoplus_{k=0}^{\infty} C^\infty(M^0; \Lambda M^0 \otimes \Omega^{-1} NM^0)\ ,$$

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- $\rightsquigarrow$ Novikov complexes on the compact manifold M^0 ...
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Term supported on M^1

- $\mathcal{F}$ almost without holonomy: only the compact leaves in M^0 have holonomy.
- $\rightsquigarrow$ "cutting" M through M^0, we get a finite number of compact foliated manifolds with boundary $(M_l, \mathcal{F}_l)$, $M^1 \equiv \bigsqcup_l \mathring{M}_l$, $\mathcal{F}^1 \equiv \bigsqcup_l \mathring{\mathcal{F}}_l$. (**Hector**)
- $\exists\, g^1$ appropriate bundle-like metric for $(M^1, \mathcal{F}^1)$ of bounded geometry.
- $\rightsquigarrow$ the Hodge isomorphism

$$\begin{aligned}\overline{H}(H^\infty(M^1; \Lambda\mathcal{F}^1), d_{\mathcal{F}^1}) &\equiv \bigoplus_l \overline{H}(H^\infty(\mathring{M}_l; \Lambda\mathring{\mathcal{F}}_l), d_{\mathring{\mathcal{F}}_l}) \\ &\cong \bigoplus_l \ker \Delta_{\mathring{\mathcal{F}}_l} \quad (\text{in } H^\infty(\mathring{M}_l; \Lambda\mathring{\mathcal{F}}_l)) .\end{aligned}$$

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Term supported on M^1 (contd.)

- $\rightsquigarrow A_f$ is defined as above in every $L^2(\mathring{M}_l;\Lambda\mathring{\mathcal{F}}_l)$.
- Difficulty: M^1 is not compact,
  $\rightsquigarrow$ smoothing operators are not of trace class.
- $\rightsquigarrow g^1 \equiv$ a b-metric of the manifolds with boundary M_l (b-calculus, **Melrose, 1993**)
- $\rightsquigarrow A_f \in \Psi_b^{-\infty}(M_l;\Lambda\mathcal{F}_l)$.
- A_f has a b-trace $\rightsquigarrow {}^{\mathrm{b}}\mathrm{Tr}^{\mathrm{s}}(A_f) \rightsquigarrow$ a part of $L(\phi)$.
- Description of this part:
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- $\exists\ \rho \in C^\infty(M^l)$ such that $\partial M_l = \{\rho = 0\}$ and $d\rho \neq 0$ on ∂M_l, a defining function of ∂M_l.
  We can also assume $d\rho = \rho\eta$.
- Then

$$\{\, \alpha|_{M^1} \mid \alpha \in I(M, M^0; \Lambda\mathcal{F})\,\} = \bigoplus_l \bigcup_{m=0}^{\infty} \rho^{-m} H^\infty(\mathring{M}_l; \mathring{\mathcal{F}}_l)\,.$$

- Multiplication by ρ^m defines an isomorphism

$$(\rho^{-m} H^\infty(\mathring{M}_l; \mathring{\mathcal{F}}_l), d_{\mathring{\mathcal{F}}_l}) \cong (H^\infty(\mathring{M}_l; \mathring{\mathcal{F}}_l), d_{\mathring{\mathcal{F}}_l} + m\,\eta\wedge)\,.$$

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- We solved the case where $m = 0$ with the above argument using A_f.
- $\rightsquigarrow$ Novikov's complex versions of the above argument ...

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Thank you very much!