6-2017

Normal Images of a Product and Countably Paracompact Condensation

Jila Niknejad
University of Kansas, jilaniknejad@gmail.com

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June 28, 2017

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There have been a few results that are variations of Tamano's theorem, one is a result by Kunen .

Theorem (Kunen, 1984)

In 1984, Kunen proved: For a normal space X, $X \times (|X| + 1)$ is normal iff X is paracompact.

The condensation variation of Kunen's proof was introduced by Buzjakova.

Theorem (Buzjakova, 1997)

Let X be a pseudocompact Tychonoff space and $\kappa = |\beta X|^+$. Then X condenses onto a compact space iff $X \times (\kappa + 1)$ condenses onto a normal space.

The condensation variation of Kunen's proof was introduced by Buzjakova.

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Comparing the results of Buzjakova and Kunen, the question is whether it is possible to prove condensation variation to Kunen's result without the extra assumption that X is pseudocompact.
(If a Tychonoff space is Pseudocompact and Paracompact then the space is compact.)

Definition

A continuous function $f : X \longrightarrow Y$ is a **condensation** iff f is one-to-one and onto Y. Without loss of generality we assume that Y is the same set as X with a coarser topology than the topology of X.

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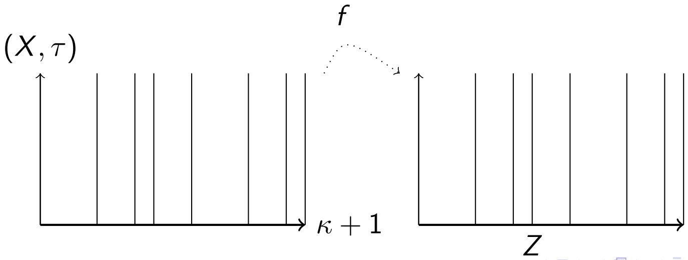
We use the following picture of the condensation $f : X \times (\kappa + 1) \rightarrow Z$ for the rest of this talk.

Condensation



Today we show that for a Tychonoff space X, if $X \times (\kappa + 1)$ condenses onto a normal space, then X condenses onto a countably paracompact space, where $\kappa = (2^{2^{|X|}})^+$.

Buzjakova's argument of a coarser topology depends on the structure of Stone-Čech compactification of the product, which is the product of Stone-Čech compactifications, by assuming that X is **pseudocompact**.
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So we begin with some facts about the structure of the Stone-Čech compactification of the product.

Henceforth, let X be a Tychonoff space and κ be a regular cardinal $> |X|$, and denote $X \times (\kappa + 1)$ by Y.

Definition

A space $A \subseteq X$ is C^*-**embedded** in space X if and only if any bounded real valued continuous function $f : A \to \mathbb{R}$ can be extended to bounded real valued continuous function $h : X \to \mathbb{R}$ such that $h(x) = f(x)$ for all $x \in A$.

Fact

Let X be a Tychonoff space, κ a cardinal, and K a closed subset of $\kappa + 1$. Then $X \times K$ is C^*-embedded in $X \times (\kappa + 1)$.

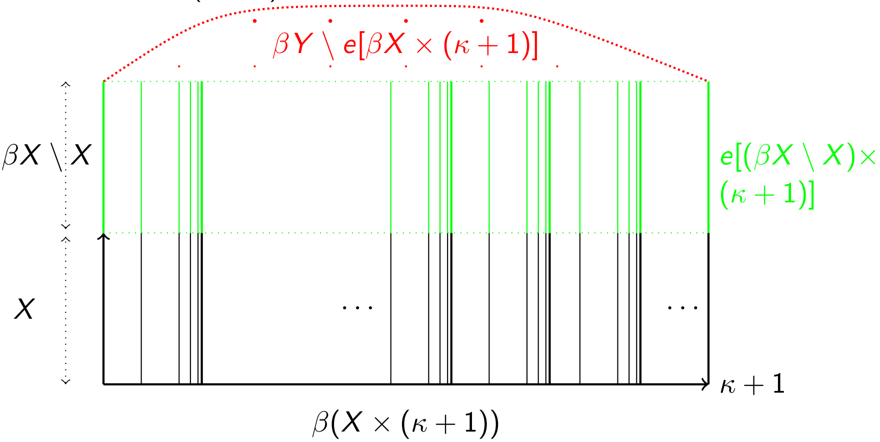
Proof. Let $f \in \mathcal{C}^*(X \times K)$. f is continuously extendable to whole space by

$$\widehat{f}(x, \alpha) = \begin{cases} f(x, \inf(K \setminus \alpha)), & \text{if } \sup K > \alpha; \\ 0, & \text{otherwise.} \end{cases}$$

Notation

For a Tychonoff space X, a cardinal κ, and $\alpha \in \kappa + 1$, by the above fact, $X \times \{\alpha\}$ is C^*-embedded in $X \times (\kappa + 1) =: Y$. As Y is C^*-embedded in βY, it follows that $X \times \{\alpha\}$ is C^*-embedded in βY. So $h_\alpha : \beta X \times \{\alpha\} \approx cl_{\beta Y}(X \times \{\alpha\})$. For $y \in \beta X$, $h_\alpha(y, \alpha) \in \beta Y$. To avoid confusion, we denote $h_\alpha(y, \alpha)$ by $e(y, \alpha)$ and $e[\beta X \times (\kappa + 1)] = \bigcup_{y \in \beta X, \alpha \in \kappa} e(y, \alpha)$.

Using this notation, Stone-Čech compactification of $Y := X \times (\kappa + 1)$ will look like this:



Fact

Let $Y = X \times (\kappa + 1)$ *and* $\alpha \in \kappa + 1$ *be such that* $\mathrm{cf}(\alpha) > |X|$. *Then for* $y \in \beta X \setminus X$ *and* $e(y, \alpha) \in U \in \tau(\beta Y)$, *there exists* $\xi < \alpha$ *and* $V \in \tau(\beta X)$ *such that* $e(y, \alpha) \in e[V \times (\xi, \alpha]] \subseteq U$.

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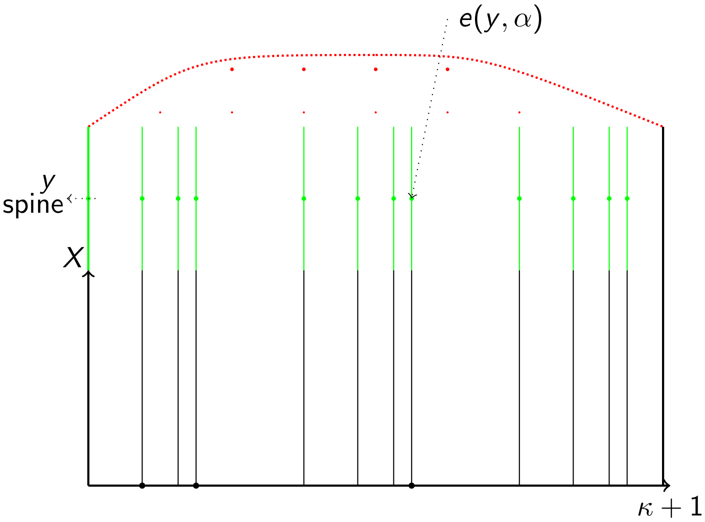
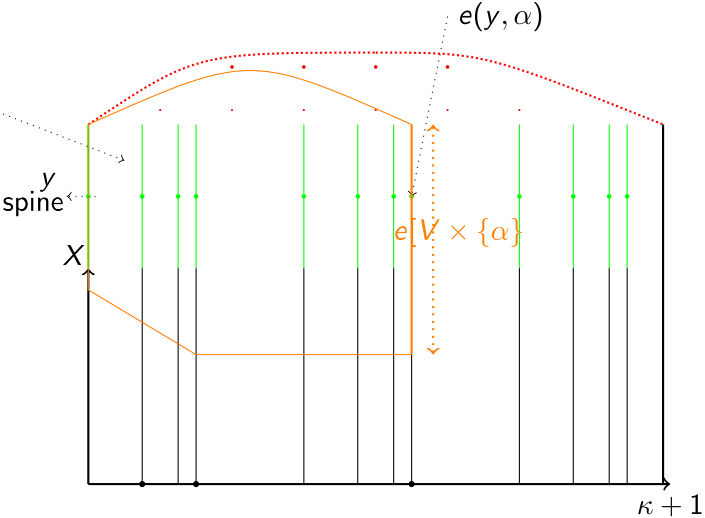
Pictorial Proof:

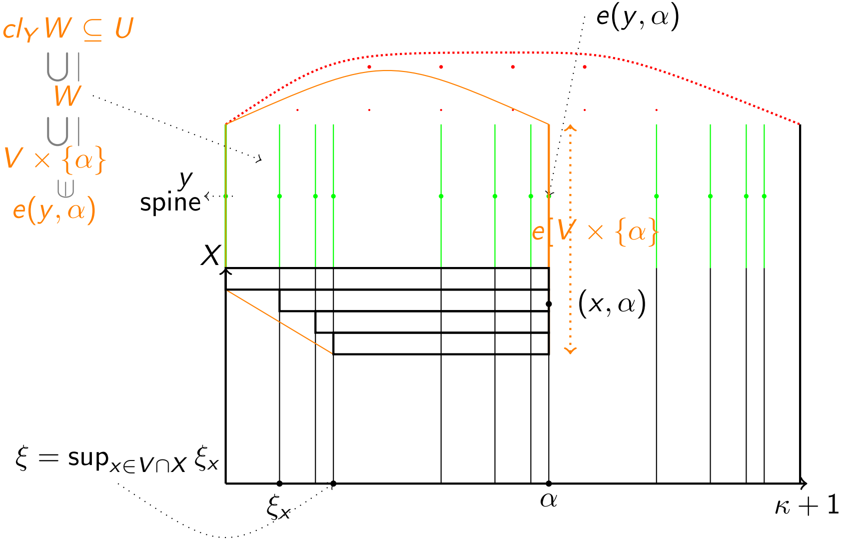

$e(y, \alpha)$
y
spine
X
$\kappa + 1$

□

$$\begin{array}{c} cl_Y W \subseteq U \\ \cup\!| \\ W \\ \cup\!| \\ V \times \{\alpha\} \\ \Cup \\ e(y,\alpha) \end{array}$$



$\square$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$\bigcup|$
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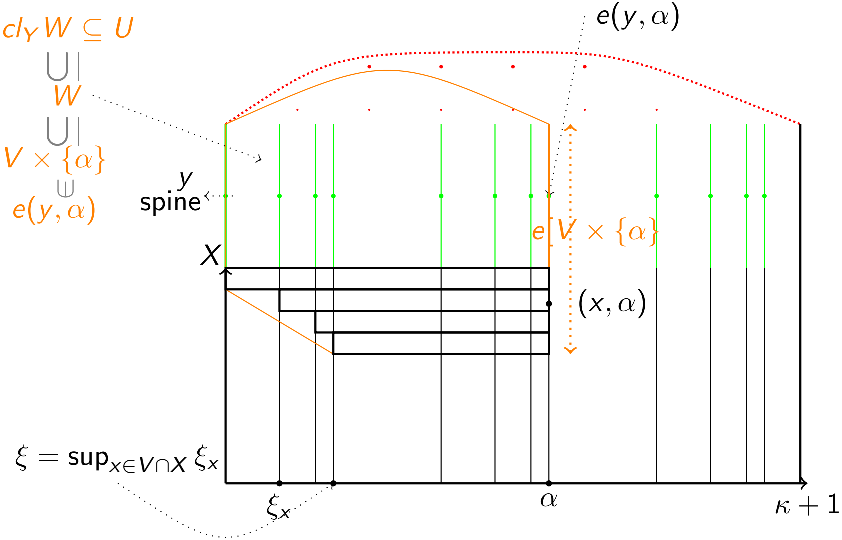
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So some parts of a spine of green points is convergent to specific green points or:

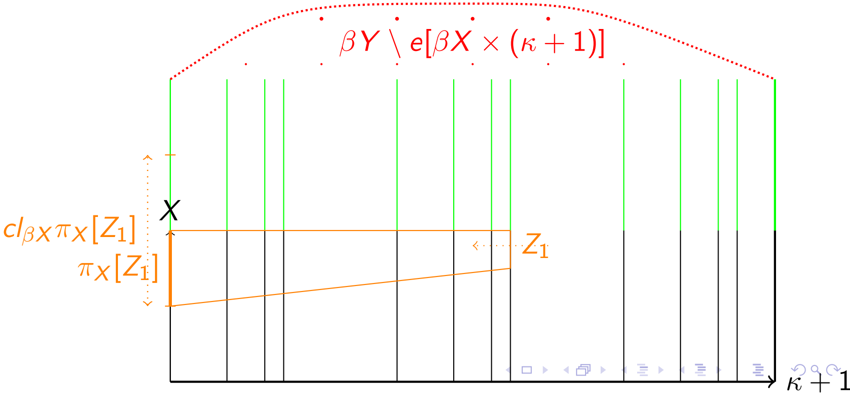
Corollary

Let $Y = X \times (\kappa + 1)$, *and let* $\alpha \in \kappa + 1$ *be such that* $\mathrm{cf}(\alpha) > |X|$. *If* $\{\beta_\gamma : \gamma < \mathrm{cf}(\alpha)\}$ *is a cofinal sequence in* α, *then* $\{e(y, \beta_\gamma) : \gamma < \mathrm{cf}(\alpha)\} \to e(y, \alpha)$. *And for a regular cardinal* α, $e(y, \alpha)$ *is the unique complete accumulation point of* $\{e(y, \beta_\gamma) : \gamma < \alpha\}$.

Fact

For every free $z-$ultrafilter $\mathcal{F}$ on $X \times K$, where K is closed in $\kappa+1$, there exists $y_{\mathcal{F}} \in \beta X$ such that $\bigcap_{F\in\mathcal{F}} cl_{\beta X}\pi_X(F) = \{y_{\mathcal{F}}\}$. Denote $y_{\mathcal{F}}$ as the corresponding βX- element of $\mathcal{F}$.

Let $Z_1, Z_2 \in \mathcal{F}$.



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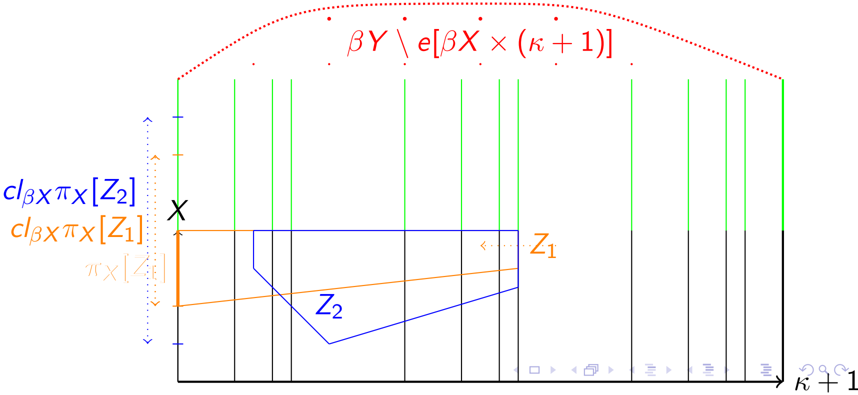

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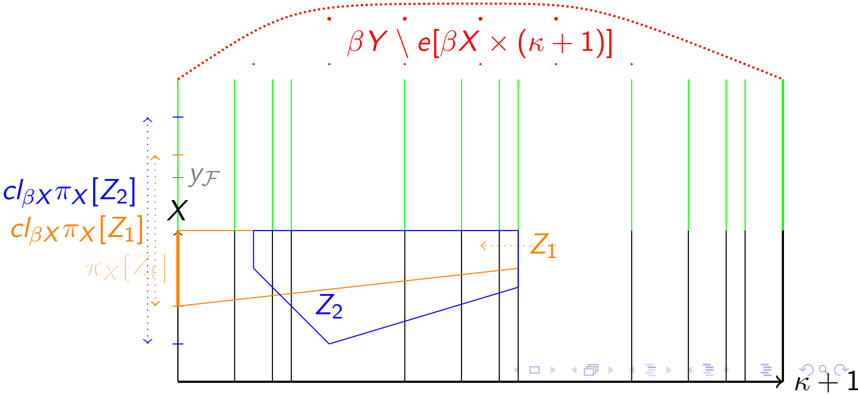

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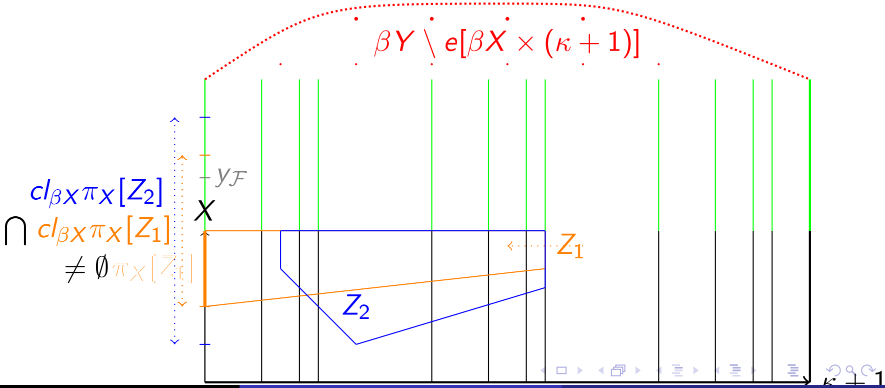
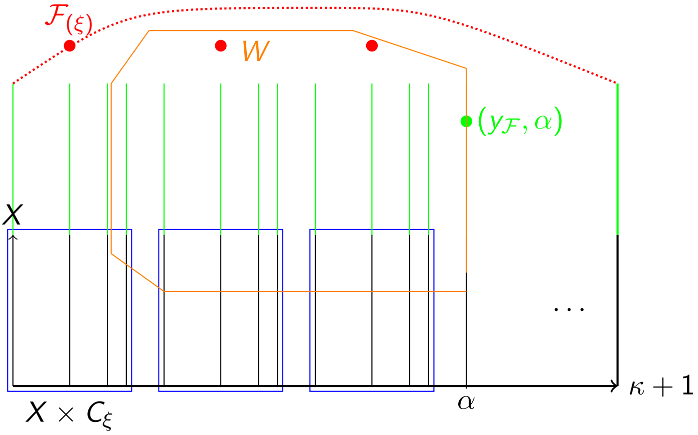


Fact

Let $\eta \in \kappa$ *and* $\{C_\xi : \xi \in \kappa\}$ *be a family of subsets of* $\kappa + 1$ *isomorphic to* $\eta + 1$ *such that* $\min C_\xi > \sup C_\zeta$ *if* $\xi > \zeta$. *Let* $\mathcal{F}$ *be a free* $z-$*ultrafilter on* $X \times (\eta + 1)$ *and* $\{\mathcal{F}_{(\xi)} : \xi \in \kappa\}$ *be a collection of* $z-$*ultrafilter homeomorphic to* $\mathcal{F}$ *on the corresponding* $X \times C_\xi$ *and let* $y_{\mathcal{F}}$ *be as in the previous fact. Then for* ν *such that* $\mathrm{cf}(\nu) > |X|$ *and for* $e(y_{\mathcal{F}}, \alpha) \in W \in \tau(\beta Y)$, *where* $\alpha = \sup_{\xi \in \nu} \cup C_\xi$, *there exists* $\lambda < \alpha$ *such that* $W \cap \mathcal{F}_{(\xi)} \neq \emptyset$ *for all* $\xi > \lambda$. *Moreover, a spine of red points also converges to a green point at level* α.

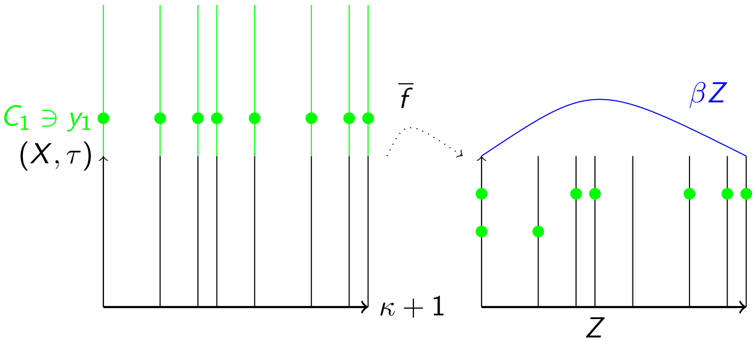

$\mathcal{F}_{(\xi)}$
W
$(y_{\mathcal{F}}, \alpha)$
X
$\dots$
$\kappa + 1$
α
$X \times C_\xi$

Definition

Let X be a Tychonoff space and κ be a cardinal such that $\mathrm{cf}(\kappa) > |\beta X|$ and let $f : X \times (\kappa + 1) \to Z$ be a condensation onto a normal space Z, then let
$C_1 := \{y \in \beta X \setminus X \text{ s.t } |\{\alpha \in \kappa : \overline{f}(e(y, \alpha)) \in Z\}| = \kappa$ and $|\overline{f}[e[\{y\} \times \kappa]] \cap Z| = \kappa\}$.
$C_2 := \{y \in \beta X \setminus X \text{ s.t } |\{\alpha \in \kappa : \overline{f}(e(y, \alpha)) \in Z\}| = \kappa$ and $|\overline{f}[e[\{y\} \times \kappa]] \cap Z| < \kappa\}$.

To prove any type of paracompactness, we need to show that many of the $f[X \times \{\xi\}]$ have the same topology. Setting $\kappa > 2^{2^{|X|}}$ ensures that we have an unbounded number of ξs with the same topology, for there are at most $2^{2^{|X|}}$ many topologies on X.
Also we need another method to show that the closure of a group of η many of these lines will not intersect closure of the limit line of the group. Partitioning $\beta X \setminus X$ into C_1, C_2 and the rest, provides the necessary tools.

Let $y_1 \in C_1$ and $y_2 \in C_2$, $f[e(\{y_1\} \times (\kappa + 1))]$ and $f[e(\{y_2\} \times (\kappa + 1))]$ will look like this:



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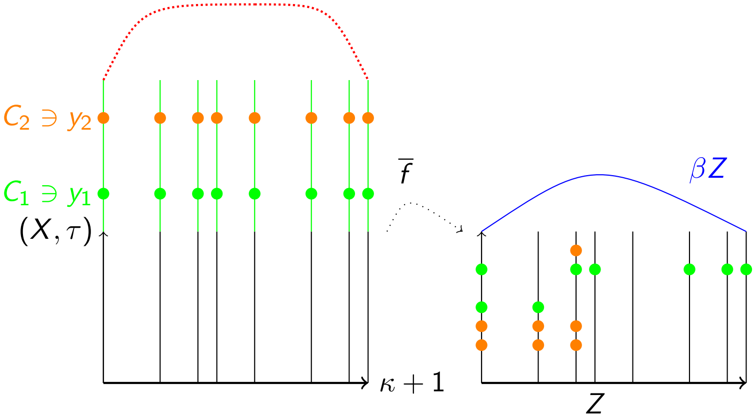
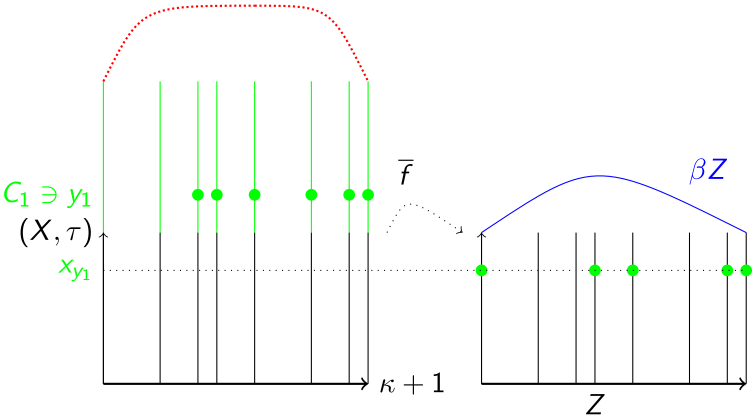


Using the above Fact allows us to draw a few conclusions about spines on points of C_1 or C_2:



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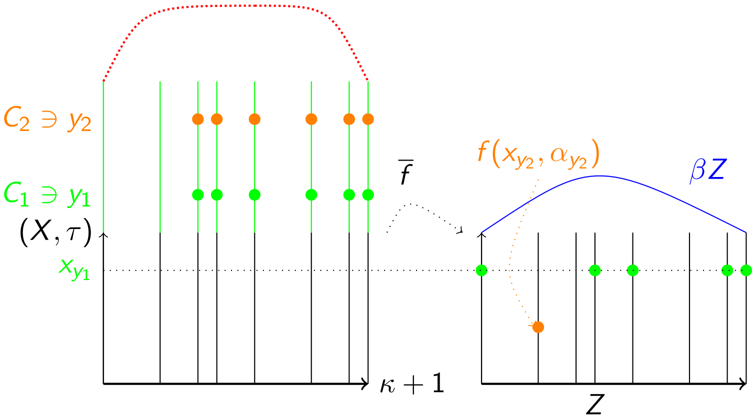


Moreover, by above Fact, we can find unbounded set $B \subseteq \kappa$ such that for $\xi \in B$,

for $y_1 \in C_1$, $\overline{f}(e(y_1, \xi)) = f(x_{y_1}, \xi)$;

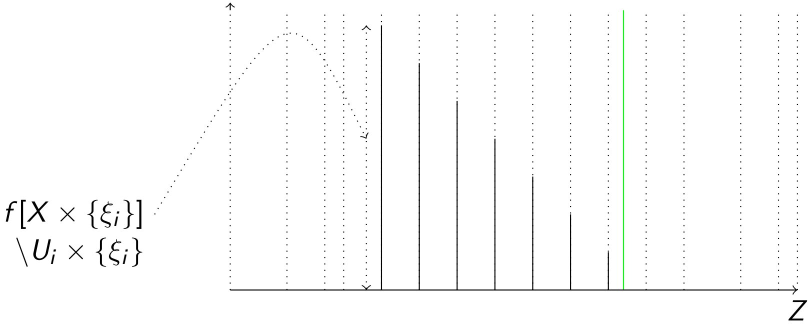
for $y_2 \in C_2$, $\overline{f}(e(y_2, \xi)) = f(x_{y_2}, \alpha_{y_2})$;

for $\alpha \in \lim A$, where $A \subset B$, $cl_Z f[X \times \{\alpha\}] \cap cl_Z f[X \times \{\xi\}] = cl_Z[\overline{f}[e[C_2 \times \{\xi\}]]]$, for $\xi \in B \cap \alpha$.

The topology defined by f on $X \times \{\xi\}$ for $\xi \in B$ is the coarsest topology defined by f on $X \times \{\alpha\}$ for large α.

Now using the above fact, we pick $C \subseteq B$ of size η, where η is the size of the cover, so that image of red points, if it is in Z, will be on the same image of spines as elements of C_1 and C_2.

Now we have enough tools to prove the space $f[X \times \{\xi\}] \cup cl_{\beta Z}[\overline{f}[e[C_2 \times \{\xi\}]]$ is countably paracompact, by choosing C of size ω by the technique mentioned before and using the cover to obtain a triangular set, whose closure does not intersect the closure of the limit line.



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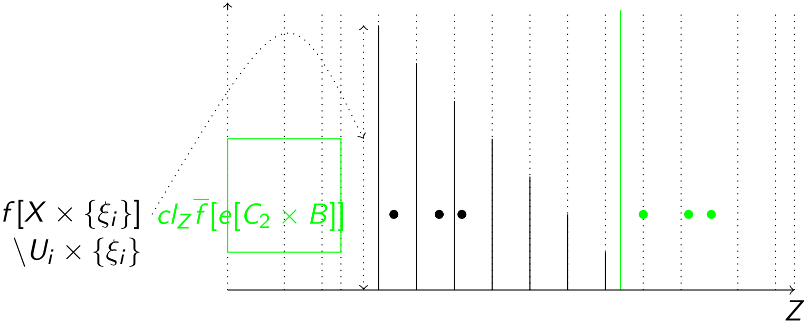

Then we prove $f[X \times \{\xi\}] \cup cl_{\beta Z}[\overline{f}[e[C_2 \times \{\xi\}]]$ maps under a closed continuous function to X and that proves X has a coarser countably paracompact topology.

How to incorporate Buzjakova's insight without pesudocompactness.

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Thank you!