6-2017

Entropy in Topological Groups, Part 2

Dikran Dikranjan
University of Udine, dikran.dikranjan@uniud.it

Entropy in Topological Groups

Dikran Dikranjan

W o r k s h o p a t
Summer Topology Conference
Dayton, Ohio (USA),
June 28, 2017

Weiss [1976] *Let $\phi : K \to K$ a continuous endomorphism of a totally disconnected compact abelian group K. If $\widehat{\phi} : \widehat{K} \to \widehat{K}$ is the Pontryagin dual of ϕ. Then*

$$h_{top}(\phi) = h_{alg}(\widehat{\phi}). \qquad (\dagger)$$

Peters [1979] proved ($\dagger$) when G is compact metrizable and ϕ is a continuous automorphism (Peters [Pac.J.Math. 1980] LCA groups).

Theorem (Giordano Bruno - DD)

Let $\phi : G \to G$ be a continuous endomorphism of a LCA group G. Then ($\dagger$) holds if one of the following condition is fulfilled:

(a) *G is totally disconnected* (generalizes Weiss);

(b) *G is compact* (generalizes Peters).

Question

Does ($\dagger$) hold true for every LCA group G?

Yes, for automorphisms (for actions of amenable groups), Virili '13

Weiss [1976] *Let $\phi : K \to K$ a continuous endomorphism of a totally disconnected compact abelian group K. If $\widehat{\phi} : \widehat{K} \to \widehat{K}$ is the Pontryagin dual of ϕ. Then*

$$h_{top}(\phi) = h_{alg}(\widehat{\phi}). \qquad (\dagger)$$

Peters [1979] proved $(\dagger)$ when G is compact metrizable and ϕ is a continuous automorphism (Peters [Pac.J.Math. 1980] LCA groups).

Theorem (Giordano Bruno - DD)

Let $\phi : G \to G$ be a continuous endomorphism of a LCA group G. Then $(\dagger)$ holds if one of the following condition is fulfilled:

(a) *G is totally disconnected* (generalizes Weiss);

(b) *G is compact* (generalizes Peters).

Question

Does $(\dagger)$ hold true for every LCA group G?

Yes, for automorphisms (for actions of amenable groups), Virili '13

Weiss [1976] *Let $\phi : K \to K$ a continuous endomorphism of a totally disconnected compact abelian group K. If $\widehat{\phi} : \widehat{K} \to \widehat{K}$ is the Pontryagin dual of ϕ. Then*

$$h_{top}(\phi) = h_{alg}(\widehat{\phi}). \qquad (\dagger)$$

Peters [1979] proved ($\dagger$) when G is compact metrizable and ϕ is a continuous automorphism (Peters [Pac.J.Math. 1980] LCA groups).

Theorem (Giordano Bruno - DD)

Let $\phi : G \to G$ be a continuous endomorphism of a LCA group G. Then ($\dagger$) holds if one of the following condition is fulfilled:

(a) *G is totally disconnected* (generalizes Weiss);

(b) *G is compact* (generalizes Peters).

Question

Does ($\dagger$) hold true for every LCA group G?

Yes, for automorphisms (for actions of amenable groups), Virili '13

Weiss [1976] *Let $\phi : K \to K$ a continuous endomorphism of a totally disconnected compact abelian group K. If $\widehat{\phi} : \widehat{K} \to \widehat{K}$ is the Pontryagin dual of ϕ. Then*

$$h_{top}(\phi) = h_{alg}(\widehat{\phi}). \qquad (\dagger)$$

Peters [1979] proved ($\dagger$) when G is compact metrizable and ϕ is a continuous automorphism (Peters [Pac.J.Math. 1980] LCA groups).

Theorem (Giordano Bruno - DD)

Let $\phi : G \to G$ be a continuous endomorphism of a LCA group G. Then ($\dagger$) holds if one of the following condition is fulfilled:

(a) *G is totally disconnected* (generalizes Weiss);

(b) *G is compact* (generalizes Peters).

Question

Does ($\dagger$) hold true for every LCA group G?

Yes, for automorphisms (for actions of amenable groups), Virili '13

Weiss [1976] *Let $\phi : K \to K$ a continuous endomorphism of a totally disconnected compact abelian group K. If $\widehat{\phi} : \widehat{K} \to \widehat{K}$ is the Pontryagin dual of ϕ. Then*

$$h_{top}(\phi) = h_{alg}(\widehat{\phi}). \qquad (\dagger)$$

Peters [1979] proved ($\dagger$) when G is compact metrizable and ϕ is a continuous automorphism (Peters [Pac.J.Math. 1980] LCA groups).

Theorem (Giordano Bruno - DD)

Let $\phi : G \to G$ be a continuous endomorphism of a LCA group G. Then ($\dagger$) holds if one of the following condition is fulfilled:

(a) *G is totally disconnected* (generalizes Weiss);

(b) *G is compact* (generalizes Peters).

Question

Does ($\dagger$) hold true for every LCA group G?

Yes, for automorphisms (for actions of amenable groups), Virili '13

Weiss [1976] *Let $\phi : K \to K$ a continuous endomorphism of a totally disconnected compact abelian group K. If $\widehat{\phi} : \widehat{K} \to \widehat{K}$ is the Pontryagin dual of ϕ. Then*

$$h_{top}(\phi) = h_{alg}(\widehat{\phi}). \qquad (\dagger)$$

Peters [1979] proved ($\dagger$) when G is compact metrizable and ϕ is a continuous automorphism (Peters [Pac.J.Math. 1980] LCA groups).

Theorem (Giordano Bruno - DD)

Let $\phi : G \to G$ be a continuous endomorphism of a LCA group G. Then ($\dagger$) holds if one of the following condition is fulfilled:

(a) *G is totally disconnected* (generalizes Weiss);

(b) *G is compact* (generalizes Peters).

Question

Does ($\dagger$) hold true for every LCA group G?

Yes, for automorphisms (for actions of amenable groups), Virili '13

Let X be a set and $\lambda : X \to X$ a selfmap. For a finite subset D of X and $n \in \mathbb{N}_+$ the n-th λ-trajectory of D is

$$\mathfrak{T}_n(\lambda, D) = D \cup \lambda(D) \cup \cdots \cup \lambda^{n-1}(D),$$

while the λ-trajectory ([positive] orbit) of D under λ is

$$\mathfrak{T}(\lambda, D) = \bigcup_{n\in\mathbb{N}} \lambda^n(D) = \bigcup_{n\in\mathbb{N}_+} \mathfrak{T}_n(\lambda, D).$$

This is the smallest λ-invariant subset of X containig D.
One can define similarly the inverse n-th λ-trajectory of D by

$$\mathfrak{T}^*_n(\lambda, D) = D \cup \lambda^{-1}(D) \cup \cdots \cup \lambda^{-n+1}(D)$$

Let X be a set and $\lambda : X \to X$ a selfmap. For a finite subset D of X and $n \in \mathbb{N}_+$ the n-th λ-trajectory of D is

$$\mathfrak{T}_n(\lambda, D) = D \cup \lambda(D) \cup \cdots \cup \lambda^{n-1}(D),$$

while the λ-trajectory ([positive] orbit) of D under λ is

$$\mathfrak{T}(\lambda, D) = \bigcup_{n\in\mathbb{N}} \lambda^n(D) = \bigcup_{n\in\mathbb{N}_+} \mathfrak{T}_n(\lambda, D).$$

This is the smallest λ-invariant subset of X containig D.
One can define similarly the inverse n-th λ-trajectory of D by

$$\mathfrak{T}^*_n(\lambda, D) = D \cup \lambda^{-1}(D) \cup \cdots \cup \lambda^{-n+1}(D)$$

Let X be a set and $\lambda : X \to X$ a selfmap. For a finite subset D of X and $n \in \mathbb{N}_+$ the n-th λ-trajectory of D is

$$\mathfrak{T}_n(\lambda, D) = D \cup \lambda(D) \cup \cdots \cup \lambda^{n-1}(D),$$

while the λ-trajectory ([positive] orbit) of D under λ is

$$\mathfrak{T}(\lambda, D) = \bigcup_{n\in\mathbb{N}} \lambda^n(D) = \bigcup_{n\in\mathbb{N}_+} \mathfrak{T}_n(\lambda, D).$$

This is the smallest λ-invariant subset of X containig D.
One can define similarly the inverse n-th λ-trajectory of D by

$$\mathfrak{T}^*_n(\lambda, D) = D \cup \lambda^{-1}(D) \cup \cdots \cup \lambda^{-n+1}(D)$$

Let X be a set and $\lambda : X \to X$ a selfmap. For a finite subset D of X and $n \in \mathbb{N}_+$ the n-th λ-trajectory of D is

$$\mathfrak{T}_n(\lambda, D) = D \cup \lambda(D) \cup \cdots \cup \lambda^{n-1}(D),$$

while the λ-trajectory ([positive] orbit) of D under λ is

$$\mathfrak{T}(\lambda, D) = \bigcup_{n \in \mathbb{N}} \lambda^n(D) = \bigcup_{n \in \mathbb{N}_+} \mathfrak{T}_n(\lambda, D).$$

This is the smallest λ-invariant subset of X containig D.
One can define similarly the inverse n-th λ-trajectory of D by

$$\mathfrak{T}_n^*(\lambda, D) = D \cup \lambda^{-1}(D) \cup \cdots \cup \lambda^{-n+1}(D)$$

Let X be a set and $\lambda : X \to X$ a selfmap.

(a) For a finite subset D of X the (covariant) combinatorial entropy of λ with respect to D is
$$h_c(\lambda, D) = \lim_{n\to\infty} \frac{|\mathfrak{T}_n(\lambda, D)|}{n} \leq |D|.$$

(b) The number $h_c(\lambda) = \sup\{h_c(\lambda, D) : D \in [X]^{<\omega}\}$ is the (covariant) combinatorial entropy of λ.

If $\lambda : X \to X$ is finitely many-to-one, the (contravariant) combinatorial entropy $h_c^*(\lambda)$ of λ can be defined similarly, by making use of $\mathfrak{T}_n^*(\lambda, D)$ in place of $\mathfrak{T}_n(\lambda, D)$.

Example (Generalized shifts)

Let K be a finite group (set) and $\lambda : X \to X$ be a selfmap, $X \neq \emptyset$. Define the generalized shift $\sigma_\lambda : K^X \to K^X$ by $\sigma_\lambda(g) = g \circ \lambda$ for $g : X \to K$.

(a) $h_{top}(\sigma_\lambda) = h_c(\lambda) \log |K|$ (this remains true also for compositions $\psi \circ \sigma_\lambda$ or $\sigma_\lambda \circ \psi$, where $\psi = (\psi_i) \in Sym(K)^I$).

(b) if $\lambda : X \to X$ is finitely many-to-one, then the direct sum $\bigoplus_X K$ is σ_λ-invariant in K^X and $h_{alg}(\sigma_\lambda \restriction_{\bigoplus_X K}) = h_c^*(\lambda) \log |K|$.

Let X be a set and $\lambda : X \to X$ a selfmap.

(a) For a finite subset D of X the (covariant) combinatorial entropy of λ with respect to D is

$$h_c(\lambda, D) = \lim_{n\to\infty} \frac{|\mathfrak{T}_n(\lambda,D)|}{n} \leq |D|.$$

(b) The number $h_c(\lambda) = \sup\{h_c(\lambda, D) : D \in [X]^{<\omega}\}$ is the (covariant) combinatorial entropy of λ.

If $\lambda : X \to X$ is finitely many-to-one, the (contravariant) combinatorial entropy $h_c^*(\lambda)$ of λ can be defined similarly, by making use of $\mathfrak{T}_n^*(\lambda, D)$ in place of $\mathfrak{T}_n(\lambda, D)$.

Example (Generalized shifts)

Let K be a finite group (set) and $\lambda : X \to X$ be a selfmap, $X \neq \emptyset$. Define the generalized shift $\sigma_\lambda : K^X \to K^X$ by $\sigma_\lambda(g) = g \circ \lambda$ for $g : X \to K$.

(a) $h_{top}(\sigma_\lambda) = h_c(\lambda) \log |K|$ (this remains true also for compositions $\psi \circ \sigma_\lambda$ or $\sigma_\lambda \circ \psi$, where $\psi = (\psi_i) \in Sym(K)^I$).

(b) if $\lambda : X \to X$ is finitely many-to-one, then the direct sum $\bigoplus_X K$ is σ_λ-invariant in K^X and $h_{alg}(\sigma_\lambda \restriction_{\bigoplus_X K}) = h_c^*(\lambda) \log |K|$.

Let X be a set and $\lambda : X \to X$ a selfmap.

(a) For a finite subset D of X the (covariant) combinatorial entropy of λ with respect to D is
$$h_c(\lambda, D) = \lim_{n\to\infty} \frac{|\mathfrak{T}_n(\lambda,D)|}{n} \leq |D|.$$

(b) The number $h_c(\lambda) = \sup\{h_c(\lambda, D) : D \in [X]^{<\omega}\}$ is the (covariant) combinatorial entropy of λ.

If $\lambda : X \to X$ is finitely many-to-one, the (contravariant) combinatorial entropy $h_c^*(\lambda)$ of λ can be defined similarly, by making use of $\mathfrak{T}_n^*(\lambda, D)$ in place of $\mathfrak{T}_n(\lambda, D)$.

Example (Generalized shifts)

Let K be a finite group (set) and $\lambda : X \to X$ be a selfmap, $X \neq \emptyset$. Define the generalized shift $\sigma_\lambda : K^X \to K^X$ by $\sigma_\lambda(g) = g \circ \lambda$ for $g : X \to K$.

(a) $h_{top}(\sigma_\lambda) = h_c(\lambda) \log |K|$ (this remains true also for compositions $\psi \circ \sigma_\lambda$ or $\sigma_\lambda \circ \psi$, where $\psi = (\psi_i) \in Sym(K)^I$).

(b) if $\lambda : X \to X$ is finitely many-to-one, then the direct sum $\bigoplus_X K$ is σ_λ-invariant in K^X and $h_{alg}(\sigma_\lambda \restriction_{\bigoplus_X K}) = h_c^*(\lambda) \log |K|$.

Let X be a set and $\lambda : X \to X$ a selfmap.

(a) For a finite subset D of X the (covariant) combinatorial entropy of λ with respect to D is
$$h_c(\lambda, D) = \lim_{n\to\infty} \frac{|\mathfrak{T}_n(\lambda,D)|}{n} \leq |D|.$$

(b) The number $h_c(\lambda) = \sup\{h_c(\lambda, D) : D \in [X]^{<\omega}\}$ is the (covariant) combinatorial entropy of λ.

If $\lambda : X \to X$ is finitely many-to-one, the (contravariant) combinatorial entropy $h_c^*(\lambda)$ of λ can be defined similarly, by making use of $\mathfrak{T}_n^*(\lambda, D)$ in place of $\mathfrak{T}_n(\lambda, D)$.

Example (Generalized shifts)

Let K be a finite group (set) and $\lambda : X \to X$ be a selfmap, $X \neq \emptyset$. Define the generalized shift $\sigma_\lambda : K^X \to K^X$ by $\sigma_\lambda(g) = g \circ \lambda$ for $g : X \to K$.

(a) $h_{top}(\sigma_\lambda) = h_c(\lambda) \log |K|$ (this remains true also for compositions $\psi \circ \sigma_\lambda$ or $\sigma_\lambda \circ \psi$, where $\psi = (\psi_i) \in Sym(K)^I$).

(b) if $\lambda : X \to X$ is finitely many-to-one, then the direct sum $\bigoplus_X K$ is σ_λ-invariant in K^X and $h_{alg}(\sigma_\lambda \restriction_{\bigoplus_X K}) = h_c^*(\lambda) \log |K|$.

Let X be a set and $\lambda : X \to X$ a selfmap.

(a) For a finite subset D of X the (covariant) combinatorial entropy of λ with respect to D is
$$h_c(\lambda, D) = \lim_{n\to\infty} \frac{|\mathfrak{T}_n(\lambda, D)|}{n} \leq |D|.$$

(b) The number $h_c(\lambda) = \sup\{h_c(\lambda, D) : D \in [X]^{<\omega}\}$ is the (covariant) combinatorial entropy of λ.

If $\lambda : X \to X$ is finitely many-to-one, the (contravariant) combinatorial entropy $h_c^*(\lambda)$ of λ can be defined similarly, by making use of $\mathfrak{T}_n^*(\lambda, D)$ in place of $\mathfrak{T}_n(\lambda, D)$.

Example (Generalized shifts)

Let K be a finite group (set) and $\lambda : X \to X$ be a selfmap, $X \neq \emptyset$. Define the generalized shift $\sigma_\lambda : K^X \to K^X$ by $\sigma_\lambda(g) = g \circ \lambda$ for $g : X \to K$.

(a) $h_{top}(\sigma_\lambda) = h_c(\lambda) \log |K|$ (this remains true also for compositions $\psi \circ \sigma_\lambda$ or $\sigma_\lambda \circ \psi$, where $\psi = (\psi_i) \in Sym(K)^I$).

(b) if $\lambda : X \to X$ is finitely many-to-one, then the direct sum $\bigoplus_X K$ is σ_λ-invariant in K^X and $h_{alg}(\sigma_\lambda \restriction_{\bigoplus_X K}) = h_c^*(\lambda) \log |K|$.

Let X be a set and $\lambda : X \to X$ a selfmap.

(a) For a finite subset D of X the (covariant) combinatorial entropy of λ with respect to D is
$$h_c(\lambda, D) = \lim_{n\to\infty} \frac{|\mathfrak{T}_n(\lambda,D)|}{n} \leq |D|.$$

(b) The number $h_c(\lambda) = \sup\{h_c(\lambda, D) : D \in [X]^{<\omega}\}$ is the (covariant) combinatorial entropy of λ.

If $\lambda : X \to X$ is finitely many-to-one, the (contravariant) combinatorial entropy $h_c^*(\lambda)$ of λ can be defined similarly, by making use of $\mathfrak{T}_n^*(\lambda, D)$ in place of $\mathfrak{T}_n(\lambda, D)$.

Example (Generalized shifts)

Let K be a finite group (set) and $\lambda : X \to X$ be a selfmap, $X \neq \emptyset$. Define the generalized shift $\sigma_\lambda : K^X \to K^X$ by $\sigma_\lambda(g) = g \circ \lambda$ for $g : X \to K$.

(a) $h_{top}(\sigma_\lambda) = h_c(\lambda)\log|K|$ (this remains true also for compositions $\psi \circ \sigma_\lambda$ or $\sigma_\lambda \circ \psi$, where $\psi = (\psi_i) \in Sym(K)^I$).

(b) if $\lambda : X \to X$ is finitely many-to-one, then the direct sum $\bigoplus_X K$ is σ_λ-invariant in K^X and $h_{alg}(\sigma_\lambda \restriction_{\bigoplus_X K}) = h_c^*(\lambda)\log|K|$.

Let X be a set and $\lambda : X \to X$ a selfmap.

(a) For a finite subset D of X the (covariant) combinatorial entropy of λ with respect to D is

$$h_c(\lambda, D) = \lim_{n\to\infty} \frac{|\mathfrak{T}_n(\lambda,D)|}{n} \leq |D|.$$

(b) The number $h_c(\lambda) = \sup\{h_c(\lambda, D) : D \in [X]^{<\omega}\}$ is the (covariant) combinatorial entropy of λ.

If $\lambda : X \to X$ is finitely many-to-one, the (contravariant) combinatorial entropy $h_c^*(\lambda)$ of λ can be defined similarly, by making use of $\mathfrak{T}_n^*(\lambda, D)$ in place of $\mathfrak{T}_n(\lambda, D)$.

Example (Generalized shifts)

Let K be a finite group (set) and $\lambda : X \to X$ be a selfmap, $X \neq \emptyset$. Define the generalized shift $\sigma_\lambda : K^X \to K^X$ by $\sigma_\lambda(g) = g \circ \lambda$ for $g : X \to K$.

(a) $h_{top}(\sigma_\lambda) = h_c(\lambda) \log |K|$ (this remains true also for compositions $\psi \circ \sigma_\lambda$ or $\sigma_\lambda \circ \psi$, where $\psi = (\psi_i) \in Sym(K)^I$).

(b) if $\lambda : X \to X$ is finitely many-to-one, then the direct sum $\bigoplus_X K$ is σ_λ-invariant in K^X and $h_{alg}(\sigma_\lambda \restriction_{\bigoplus_X K}) = h_c^*(\lambda) \log |K|$.

Let X be a set and $\lambda : X \to X$ a selfmap.

(a) For a finite subset D of X the (covariant) combinatorial entropy of λ with respect to D is
$$h_c(\lambda, D) = \lim_{n\to\infty} \frac{|\mathfrak{T}_n(\lambda,D)|}{n} \leq |D|.$$

(b) The number $h_c(\lambda) = \sup\{h_c(\lambda, D) : D \in [X]^{<\omega}\}$ is the (covariant) combinatorial entropy of λ.

If $\lambda : X \to X$ is finitely many-to-one, the (contravariant) combinatorial entropy $h_c^*(\lambda)$ of λ can be defined similarly, by making use of $\mathfrak{T}_n^*(\lambda, D)$ in place of $\mathfrak{T}_n(\lambda, D)$.

Example (Generalized shifts)

Let K be a finite group (set) and $\lambda : X \to X$ be a selfmap, $X \neq \emptyset$. Define the generalized shift $\sigma_\lambda : K^X \to K^X$ by $\sigma_\lambda(g) = g \circ \lambda$ for $g : X \to K$.

(a) $h_{top}(\sigma_\lambda) = h_c(\lambda) \log |K|$ (this remains true also for compositions $\psi \circ \sigma_\lambda$ or $\sigma_\lambda \circ \psi$, where $\psi = (\psi_i) \in Sym(K)^I$).

(b) if $\lambda : X \to X$ is finitely many-to-one, then the direct sum $\bigoplus_X K$ is σ_λ-invariant in K^X and $h_{alg}(\sigma_\lambda \restriction_{\bigoplus_X K}) = h_c^*(\lambda) \log |K|$.

Call a compact group *strictly reductive* if it is isomorphic to a cartesian product of simple compact groups.

Theorem (Countable Layer Theorem, Hofmann-Morris)

Any compact profinite group G has a canonical countable descending sequence

$$G = \Omega_0(G) \supseteq \ldots \supseteq \Omega_n(G) \supseteq \ldots$$

of closed characteristic subgroups of G such that:
(1) $\bigcap_{n=1} \Omega_n(G) = \{e\}$,
(2) each layer $L_n = \Omega_{n-1}(G)/\Omega_n(G)$ is a strictly reductive group.

The computation of the topological entropy of an automorphism $f : G \to G$ of a compact profinite group G can be reduced to the case of a strictly reductive compact group L. Indeed, f induces an automorphism $f_n : L_n \to L_n$ of the strictly reductive group L_n and $h_{top}(f) = \sum_{n=1}^{\infty} h_{top}(f_n)$, as $G = \varprojlim G/\Omega_n(G)$ and the induced automorphism $\overline{f}$ of $G/\Omega_n(G)$ has $h_{top}(\overline{f}) = \sum_{k=1}^{n} h_{top}(f_k)$.

Call a compact group *strictly reductive* if it is isomorphic to a cartesian product of simple compact groups.

Theorem (Countable Layer Theorem, Hofmann-Morris)

Any compact profinite group G has a canonical countable descending sequence

$$G = \Omega_0(G) \supseteq \ldots \supseteq \Omega_n(G) \supseteq \ldots$$

of closed characteristic subgroups of G such that:
(1) $\bigcap_{n=1} \Omega_n(G) = \{e\}$,
(2) each layer $L_n = \Omega_{n-1}(G)/\Omega_n(G)$ *is a strictly reductive group.*

The computation of the topological entropy of an automorphism $f : G \to G$ of a compact profinite group G can be reduced to the case of a strictly reductive compact group L. Indeed, f induces an automorphism $f_n : L_n \to L_n$ of the strictly reductive group L_n and $h_{top}(f) = \sum_{n=1}^{\infty} h_{top}(f_n)$, as $G = \varprojlim G/\Omega_n(G)$ and the induced automorphism $\overline{f}$ of $G/\Omega_n(G)$ has $h_{top}(\overline{f}) = \sum_{k=1}^{n} h_{top}(f_k)$.

Call a compact group *strictly reductive* if it is isomorphic to a cartesian product of simple compact groups.

Theorem (Countable Layer Theorem, Hofmann-Morris)

Any compact profinite group G has a canonical countable descending sequence

$$G = \Omega_0(G) \supseteq \ldots \supseteq \Omega_n(G) \supseteq \ldots$$

of closed characteristic subgroups of G such that:
(1) $\bigcap_{n=1} \Omega_n(G) = \{e\}$,
(2) each layer $L_n = \Omega_{n-1}(G)/\Omega_n(G)$ *is a strictly reductive group.*

The computation of the topological entropy of an automorphism $f : G \to G$ of a compact profinite group G can be reduced to the case of a strictly reductive compact group L. Indeed, f induces an automorphism $f_n : L_n \to L_n$ of the strictly reductive group L_n and $h_{top}(f) = \sum_{n=1}^{\infty} h_{top}(f_n)$, as $G = \varprojlim G/\Omega_n(G)$ and the induced automorphism $\overline{f}$ of $G/\Omega_n(G)$ has $h_{top}(\overline{f}) = \sum_{k=1}^{n} h_{top}(f_k)$.

An automorphism f of a compact group L induces automorphisms of L' and L/L', so by using AT (when $L' = \overline{L}'$), one can assume wlog that either $L = L'$ or L is abelian when computing $h_{top}(f)$.
A strictly reductive compact group with $L = L'$ has the form $\prod_{j\in J} K_j$, where $K_j = F_j^{I_j}$, for some simple finite non-abelian group F_j and $I_j \neq \emptyset \neq J$. Then f induces automorphisms f_j of K_j so that $h_{top}(f) = \sum_{j\in J}^{\infty} h_{top}(f_j)$. Each f_j induces a bijection λ_j of I_j, so that $\psi_j := \sigma_{\lambda_j}^{-1} \circ f_j$ acts coordinatewise on $F_j^{I_j}$. Thus,

$$h_{top}(f_j) = h_{top}(\sigma_{\lambda_j} \circ \psi_j) = h_{top}(\sigma_{\lambda_j}) = h_c(\lambda_j) \log |F_j|.$$

In case L is abelian, it has the form $L = \prod_{p\in\pi} K_p$, where $K_p = \mathbb{Z}_p^{\kappa_p}$ for some set π of primes. Now each $f_p : K_p \to K_p$ is conjugated to a direct product of generalized shifts of $\mathbb{Z}_p^{\kappa_p}$.
Note that in both cases these generalized shifts are just products of periodic automorphsims and Bernoulli automorphsims.

An automorphism f of a compact group L induces automorphisms of L' and L/L', so by using AT (when $L' = \overline{L'}$), one can assume wlog that either $L = L'$ or L is abelian when computing $h_{top}(f)$. A strictly reductive compact group with $L = L'$ has the form $\prod_{j\in J} K_j$, where $K_j = F_j^{I_j}$, for some simple finite non-abelian group F_j and $I_j \neq \emptyset \neq J$. Then f induces automorphisms f_j of K_j so that $h_{top}(f) = \sum_{j\in J}^{\infty} h_{top}(f_j)$. Each f_j induces a bijection λ_j of I_j, so that $\psi_j := \sigma_{\lambda_j}^{-1} \circ f_j$ acts coordinatewise on $F_j^{I_j}$. Thus,

$$h_{top}(f_j) = h_{top}(\sigma_{\lambda_j} \circ \psi_j) = h_{top}(\sigma_{\lambda_j}) = h_c(\lambda_j) \log |F_j|.$$

In case L is abelian, it has the form $L = \prod_{p\in\pi} K_p$, where $K_p = \mathbb{Z}_p^{\kappa_p}$ for some set π of primes. Now each $f_p : K_p \to K_p$ is conjugated to a direct product of generalized shifts of $\mathbb{Z}_p^{\kappa_p}$.

Note that in both cases these generalized shifts are just products of periodic automorphsims and Bernoulli automorphsims.

An automorphism f of a compact group L induces automorphisms of L' and L/L', so by using AT (when $L' = \overline{L'}$), one can assume wlog that either $L = L'$ or L is abelian when computing $h_{top}(f)$.
A strictly reductive compact group with $L = L'$ has the form $\prod_{j \in J} K_j$, where $K_j = F_j^{I_j}$, for some simple finite non-abelian group F_j and $I_j \neq \emptyset \neq J$. Then f induces automorphisms f_j of K_j so that $h_{top}(f) = \sum_{j \in J}^{\infty} h_{top}(f_j)$. Each f_j induces a bijection λ_j of I_j, so that $\psi_j := \sigma_{\lambda_j}^{-1} \circ f_j$ acts coordinatewise on $F_j^{I_j}$. Thus,

$$h_{top}(f_j) = h_{top}(\sigma_{\lambda_j} \circ \psi_j) = h_{top}(\sigma_{\lambda_j}) = h_c(\lambda_j) \log |F_j|.$$

In case L is abelian, it has the form $L = \prod_{p \in \pi} K_p$, where $K_p = \mathbb{Z}_p^{\kappa_p}$ for some set π of primes. Now each $f_p : K_p \to K_p$ is conjugated to a direct product of generalized shifts of $\mathbb{Z}_p^{\kappa_p}$.
Note that in both cases these generalized shifts are just products of periodic automorphsims and Bernoulli automorphsims.

An automorphism f of a compact group L induces automorphisms of L' and L/L', so by using AT (when $L' = \overline{L}'$), one can assume wlog that either $L = L'$ or L is abelian when computing $h_{top}(f)$. A strictly reductive compact group with $L = L'$ has the form $\prod_{j\in J} K_j$, where $K_j = F_j^{I_j}$, for some simple finite non-abelian group F_j and $I_j \neq \emptyset \neq J$. Then f induces automorphisms f_j of K_j so that $h_{top}(f) = \sum_{j\in J}^{\infty} h_{top}(f_j)$. Each f_j induces a bijection λ_j of I_j, so that $\psi_j := \sigma_{\lambda_j}^{-1} \circ f_j$ acts coordinatewise on $F_j^{I_j}$. Thus,

$$h_{top}(f_j) = h_{top}(\sigma_{\lambda_j} \circ \psi_j) = h_{top}(\sigma_{\lambda_j}) = h_c(\lambda_j) \log |F_j|.$$

In case L is abelian, it has the form $L = \prod_{p\in\pi} K_p$, where $K_p = \mathbb{Z}_p^{\kappa_p}$ for some set π of primes. Now each $f_p : K_p \to K_p$ is conjugated to a direct product of generalized shifts of $\mathbb{Z}_p^{\kappa_p}$.

Note that in both cases these generalized shifts are just products of periodic automorphsims and Bernoulli automorphsims.

An automorphism f of a compact group L induces automorphisms of L' and L/L', so by using AT (when $L' = \overline{L'}$), one can assume wlog that either $L = L'$ or L is abelian when computing $h_{top}(f)$.
A strictly reductive compact group with $L = L'$ has the form $\prod_{j\in J} K_j$, where $K_j = F_j^{I_j}$, for some simple finite non-abelian group F_j and $I_j \neq \emptyset \neq J$. Then f induces automorphisms f_j of K_j so that $h_{top}(f) = \sum_{j\in J}^{\infty} h_{top}(f_j)$. Each f_j induces a bijection λ_j of I_j, so that $\psi_j := \sigma_{\lambda_j}^{-1} \circ f_j$ acts coordinatewise on $F_j^{I_j}$. Thus,

$$h_{top}(f_j) = h_{top}(\sigma_{\lambda_j} \circ \psi_j) = h_{top}(\sigma_{\lambda_j}) = h_c(\lambda_j) \log |F_j|.$$

In case L is abelian, it has the form $L = \prod_{p\in\pi} K_p$, where $K_p = \mathbb{Z}_p^{\kappa_p}$ for some set π of primes. Now each $f_p : K_p \to K_p$ is conjugated to a direct product of generalized shifts of $\mathbb{Z}_p^{\kappa_p}$.
Note that in both cases these generalized shifts are just products of periodic automorphsims and Bernoulli automorphsims.

Similarly, one can compute $h_{top}(f)$ when G is a compact connected group. As mentioned above, we can reduce to the cases when G is abelian or $G' = G$ (note that G' is closed and connected). The abelian case can be reduced, via the Bridge theorem, to the computation of $h_{alg}(\widehat{f})$.

Since $Z(G)$ is characteristic, the computation of $h_{top}(f)$ can be reduced, due to AT, to the case when G is center-free, as $Z(G/Z(G)) = \{e\}$. In such a case the group G is, again, strictly reductive, i.e., $G = \prod_{i \in I} F_i^{I_i}$, where F_i are pairwise non-isomorphic compact connected simple Lie groups with trivial center.

As above, f_j induces a bijection λ_j of I_j, so that $\psi_j := g_{\lambda_j}^{-1} \circ f_j$ acts coordinatewise on $F_j^{I_j}$. Now $h_{top}(f)$ is computed as above, but here one has a dichotomy:

- either $h_{top}(f) = 0$ (if all $h_{\lambda_j}(f) = 0$), or
- $h_{top}(f) = \infty$ otherwise (i.e., some λ_j has infinite orbits).

Similarly, one can compute $h_{top}(f)$ when G is a compact connected group. As mentioned above, we can reduce to the cases when G is abelian or $G' = G$ (note that G' is closed and connected). The abelian case can be reduced, via the Bridge theorem, to the computation of $h_{alg}(\widehat{f})$.
Since $Z(G)$ is characteristic, the computation of $h_{top}(f)$ can be reduced, due to AT, to the case when G is center-free, as $Z(G/Z(G)) = \{e\}$. In such a case the group G is, again, strictly reductive, i.e., $G = \prod_{i \in I} F_i^{I_i}$, where F_i are pairwise non-isomorphic compact connected simple Lie groups with trivial center.
As above, f_j induces a bijection λ_j of I_j, so that $\psi_j := g_{\lambda_j}^{-1} \circ f_j$ acts coordinatewise on $F_j^{I_j}$. Now $h_{top}(f)$ is computed as above, but here one has a dichotomy:

- either $h_{top}(f) = 0$ (if all $h_{\lambda_j}(f) = 0$), or
- $h_{top}(f) = \infty$ otherwise (i.e., some λ_j has infinite orbits).

Similarly, one can compute $h_{top}(f)$ when G is a compact connected group. As mentioned above, we can reduce to the cases when G is abelian or $G' = G$ (note that G' is closed and connected). The abelian case can be reduced, via the Bridge theorem, to the computation of $h_{alg}(\widehat{f})$.

Since $Z(G)$ is characteristic, the computation of $h_{top}(f)$ can be reduced, due to AT, to the case when G is center-free, as $Z(G/Z(G)) = \{e\}$. In such a case the group G is, again, strictly reductive, i.e., $G = \prod_{i \in I} F_i^{I_i}$, where F_i are pairwise non-isomorphic compact connected simple Lie groups with trivial center.

As above, f_j induces a bijection λ_j of I_j, so that $\psi_j := g_{\lambda_j}^{-1} \circ f_j$ acts coordinatewise on $F_j^{I_j}$. Now $h_{top}(f)$ is computed as above, but here one has a dichotomy:

- either $h_{top}(f) = 0$ (if all $h_{\lambda_j}(f) = 0$), or
- $h_{top}(f) = \infty$ otherwise (i.e., some λ_j has infinite orbits).

Similarly, one can compute $h_{top}(f)$ when G is a compact connected group. As mentioned above, we can reduce to the cases when G is abelian or $G' = G$ (note that G' is closed and connected). The abelian case can be reduced, via the Bridge theorem, to the computation of $h_{alg}(\widehat{f})$.
Since $Z(G)$ is characteristic, the computation of $h_{top}(f)$ can be reduced, due to AT, to the case when G is center-free, as $Z(G/Z(G)) = \{e\}$. In such a case the group G is, again, strictly reductive, i.e., $G = \prod_{i \in I} F_i^{I_j}$, where F_i are pairwise non-isomorphic compact connected simple Lie groups with trivial center.
As above, f_j induces a bijection λ_j of I_j, so that $\psi_j := g_{\lambda_j}^{-1} \circ f_j$ acts coordinatewise on $F_j^{I_j}$. Now $h_{top}(f)$ is computed as above, but here one has a dichotomy:

- either $h_{top}(f) = 0$ (if all $h_{\lambda_j}(f) = 0$), or
- $h_{top}(f) = \infty$ otherwise (i.e., some λ_j has infinite orbits).

Definition

A *normed semigroup* is a commutative semigroup $(S,+)$ provided with a map (*norm*) $v: S \to \mathbb{R}_{\geq 0} = \{r \in \mathbb{R} : r \geq 0\}$ satisfying

$$v(x+y) \leq v(x) + v(y)$$

for all $x, y \in S$.

The category $\mathfrak{S}$ of normed semigroups has as morphisms all contractive semigroup homomorphism $f : (S, v) \to (S_1, v_1)$ (i.e., $\phi(x+y) = \phi(x) + \phi(y)$ and $v_1(\phi(x)) \leq v(x)$ hold for every $x, y \in S$).

For $(S, v) \in \mathfrak{S}$ we say that the norm is *s-monotone*, if

$$\max\{v(x), v(y)\} \leq v(x+y) \leq v(x) + v(y) \qquad \text{for all } x, y \in S.$$

Definition

A *normed semigroup* is a commutative semigroup $(S, +)$ provided with a map (*norm*) $v : S \to \mathbb{R}_{\geq 0} = \{r \in \mathbb{R} : r \geq 0\}$ satisfying

$$v(x + y) \leq v(x) + v(y)$$

for all $x, y \in S$.

The category $\mathfrak{S}$ of normed semigroups has as morphisms all contractive semigroup homomorphism $f : (S, v) \to (S_1, v_1)$ (i.e., $\phi(x + y) = \phi(x) + \phi(y)$ and $v_1(\phi(x)) \leq v(x)$ hold for every $x, y \in S$).
For $(S, v) \in \mathfrak{S}$ we say that the norm is *s-monotone*, if

$$\max\{v(x), v(y)\} \leq v(x + y) \leq v(x) + v(y) \quad \text{for all } x, y \in S.$$

Definition

A *normed semigroup* is a commutative semigroup $(S,+)$ provided with a map (*norm*) $v : S \to \mathbb{R}_{\geq 0} = \{r \in \mathbb{R} : r \geq 0\}$ satisfying

$$v(x+y) \leq v(x) + v(y)$$

for all $x, y \in S$.

The category $\mathfrak{S}$ of normed semigroups has as morphisms all contractive semigroup homomorphism $f : (S,v) \to (S_1, v_1)$ (i.e., $\phi(x+y) = \phi(x) + \phi(y)$ and $v_1(\phi(x)) \leq v(x)$ hold for every $x, y \in S$).

For $(S,v) \in \mathfrak{S}$ we say that the norm is *s-monotone*, if

$$\max\{v(x), v(y)\} \leq v(x+y) \leq v(x) + v(y) \quad \text{for all } x, y \in S.$$

Definition

A *normed semigroup* is a commutative semigroup $(S, +)$ provided with a map (*norm*) $v : S \to \mathbb{R}_{\geq 0} = \{r \in \mathbb{R} : r \geq 0\}$ satisfying

$$v(x + y) \leq v(x) + v(y)$$

for all $x, y \in S$.

The category $\mathfrak{S}$ of normed semigroups has as morphisms all contractive semigroup homomorphism $f : (S, v) \to (S_1, v_1)$ (i.e., $\phi(x + y) = \phi(x) + \phi(y)$ and $v_1(\phi(x)) \leq v(x)$ hold for every $x, y \in S$).
For $(S, v) \in \mathfrak{S}$ we say that the norm is *s-monotone*, if

$$\max\{v(x), v(y)\} \leq v(x + y) \leq v(x) + v(y) \quad \text{for all } x, y \in S.$$

For $(S, v) \in \mathfrak{S}$, $x \in S$ and $n \in \mathbb{N}_+$ consider the *n-th trajectory of x under* ϕ

$$T_n(\phi, x) = x + \phi(x) + \ldots + \phi^{n-1}(x) \text{ and } c_n(\phi, x) = v(T_n(\phi, x)).$$

Then $(c_n(\phi, x))$ is subadditive and $c_n \leq n \cdot v(x)$, so the growth of the function $n \mapsto c_n(\phi, x)$ is at most linear.

Theorem

Let $\phi : S \to S$ be an endomorphism in $\mathfrak{S}$. Then for every $x \in S$ the limit $h_{\mathfrak{S}}(\phi, x) := \lim_n \frac{c_n(\phi, x)}{n}$ exists and satisfies $h_{\mathfrak{S}}(\phi, x) \leq v(x)$.

The existence of the limit is ensured by Fekete Lemma.

Definition

Let $\phi : S \to S$ be an endomorphism in $\mathfrak{S}$. The semigroup entropy of ϕ is

$$h_{\mathfrak{S}}(\phi) = \sup_{x \in S} h_{\mathfrak{S}}(\phi, x).$$

For $(S, v) \in \mathfrak{S}$, $x \in S$ and $n \in \mathbb{N}_+$ consider the *n-th trajectory of x under ϕ*

$$T_n(\phi, x) = x + \phi(x) + \ldots + \phi^{n-1}(x) \text{ and } c_n(\phi, x) = v(T_n(\phi, x)).$$

Then $(c_n(\phi, x))$ is subadditive and $c_n \leq n \cdot v(x)$, so the growth of the function $n \mapsto c_n(\phi, x)$ is at most linear.

Theorem

Let $\phi : S \to S$ be an endomorphism in $\mathfrak{S}$. Then for every $x \in S$ the limit $h_{\mathfrak{S}}(\phi, x) := \lim_n \frac{c_n(\phi, x)}{n}$ exists and satisfies $h_{\mathfrak{S}}(\phi, x) \leq v(x)$.

The existence of the limit is ensured by Fekete Lemma.

Definition

Let $\phi : S \to S$ be an endomorphism in $\mathfrak{S}$. The semigroup entropy of ϕ is

$$h_{\mathfrak{S}}(\phi) = \sup_{x \in S} h_{\mathfrak{S}}(\phi, x).$$

For $(S, v) \in \mathfrak{S}$, $x \in S$ and $n \in \mathbb{N}_+$ consider the *n-th trajectory of x under* ϕ

$$T_n(\phi, x) = x + \phi(x) + \ldots + \phi^{n-1}(x) \text{ and } c_n(\phi, x) = v(T_n(\phi, x)).$$

Then $(c_n(\phi, x))$ is subadditive and $c_n \leq n \cdot v(x)$, so the growth of the function $n \mapsto c_n(\phi, x)$ is at most linear.

Theorem

Let $\phi : S \to S$ be an endomorphism in $\mathfrak{S}$. Then for every $x \in S$ the limit $h_{\mathfrak{S}}(\phi, x) := \lim_n \frac{c_n(\phi,x)}{n}$ exists and satisfies $h_{\mathfrak{S}}(\phi, x) \leq v(x)$.

The existence of the limit is ensured by Fekete Lemma.

Definition

Let $\phi : S \to S$ be an endomorphism in $\mathfrak{S}$. The semigroup entropy of ϕ is

$$h_{\mathfrak{S}}(\phi) = \sup_{x \in S} h_{\mathfrak{S}}(\phi, x).$$

For $(S, v) \in \mathfrak{S}$, $x \in S$ and $n \in \mathbb{N}_+$ consider the *n-th trajectory of x under* ϕ

$$T_n(\phi, x) = x + \phi(x) + \ldots + \phi^{n-1}(x) \text{ and } c_n(\phi, x) = v(T_n(\phi, x)).$$

Then $(c_n(\phi, x))$ is subadditive and $c_n \leq n \cdot v(x)$, so the growth of the function $n \mapsto c_n(\phi, x)$ is at most linear.

Theorem

Let $\phi : S \to S$ be an endomorphism in $\mathfrak{S}$. Then for every $x \in S$ the limit $h_{\mathfrak{S}}(\phi, x) := \lim_n \frac{c_n(\phi,x)}{n}$ exists and satisfies $h_{\mathfrak{S}}(\phi, x) \leq v(x)$.

The existence of the limit is ensured by Fekete Lemma.

Definition

Let $\phi : S \to S$ be an endomorphism in $\mathfrak{S}$. The semigroup entropy of ϕ is

$$h_{\mathfrak{S}}(\phi) = \sup_{x \in S} h_{\mathfrak{S}}(\phi, x).$$

For $(S, v) \in \mathfrak{S}$, $x \in S$ and $n \in \mathbb{N}_+$ consider the *n-th trajectory of x under* ϕ

$$T_n(\phi, x) = x + \phi(x) + \ldots + \phi^{n-1}(x) \text{ and } c_n(\phi, x) = v(T_n(\phi, x)).$$

Then $(c_n(\phi, x))$ is subadditive and $c_n \leq n \cdot v(x)$, so the growth of the function $n \mapsto c_n(\phi, x)$ is at most linear.

Theorem

Let $\phi : S \to S$ be an endomorphism in $\mathfrak{S}$. Then for every $x \in S$ the limit $h_{\mathfrak{S}}(\phi, x) := \lim_n \frac{c_n(\phi,x)}{n}$ exists and satisfies $h_{\mathfrak{S}}(\phi, x) \leq v(x)$.

The existence of the limit is ensured by Fekete Lemma.

Definition

Let $\phi : S \to S$ be an endomorphism in $\mathfrak{S}$. The semigroup entropy of ϕ is

$$h_{\mathfrak{S}}(\phi) = \sup_{x \in S} h_{\mathfrak{S}}(\phi, x).$$

For $(S, v) \in \mathfrak{S}$, $x \in S$ and $n \in \mathbb{N}_+$ consider the *n-th trajectory of x under* ϕ

$$T_n(\phi, x) = x + \phi(x) + \ldots + \phi^{n-1}(x) \text{ and } c_n(\phi, x) = v(T_n(\phi, x)).$$

Then $(c_n(\phi, x))$ is subadditive and $c_n \leq n \cdot v(x)$, so the growth of the function $n \mapsto c_n(\phi, x)$ is at most linear.

Theorem

Let $\phi : S \to S$ be an endomorphism in $\mathfrak{S}$. Then for every $x \in S$ the limit $h_{\mathfrak{S}}(\phi, x) := \lim_n \frac{c_n(\phi,x)}{n}$ exists and satisfies $h_{\mathfrak{S}}(\phi, x) \leq v(x)$.

The existence of the limit is ensured by Fekete Lemma.

Definition

Let $\phi : S \to S$ be an endomorphism in $\mathfrak{S}$. The semigroup entropy of ϕ is

$$h_{\mathfrak{S}}(\phi) = \sup_{x \in S} h_{\mathfrak{S}}(\phi, x).$$

For $(S, v) \in \mathfrak{S}$, $x \in S$ and $n \in \mathbb{N}_+$ consider the *n-th trajectory of x under* ϕ

$$T_n(\phi, x) = x + \phi(x) + \ldots + \phi^{n-1}(x) \text{ and } c_n(\phi, x) = v(T_n(\phi, x)).$$

Then $(c_n(\phi, x))$ is subadditive and $c_n \leq n \cdot v(x)$, so the growth of the function $n \mapsto c_n(\phi, x)$ is at most linear.

Theorem

Let $\phi : S \to S$ be an endomorphism in $\mathfrak{S}$. Then for every $x \in S$ the limit $h_{\mathfrak{S}}(\phi, x) := \lim_n \frac{c_n(\phi,x)}{n}$ exists and satisfies $h_{\mathfrak{S}}(\phi, x) \leq v(x)$.

The existence of the limit is ensured by Fekete Lemma.

Definition

Let $\phi : S \to S$ be an endomorphism in $\mathfrak{S}$. The semigroup entropy of ϕ is

$$h_{\mathfrak{S}}(\phi) = \sup_{x \in S} h_{\mathfrak{S}}(\phi, x).$$

Lemma ($h_{\mathfrak{S}}$ is monotone under taking quotients)

If $\phi : S \to S$ and $\psi : T \to T$ are endomorphisms in $\mathfrak{S}$ and $\alpha : T \to S$ is a surjective homomorphism between normed semigroups such that $\alpha \circ \psi = \phi \circ \alpha$, then $h_{\mathfrak{S}}(\phi) \leq h_{\mathfrak{S}}(\psi)$.

Corollary ($h_{\mathfrak{S}}$ is invariant under conjugation)

If $\phi : S \to S$ is an endomorphism in $\mathfrak{S}$ and $\alpha : T \to S$ is an isomorphism in $\mathfrak{S}$, then $h_{\mathfrak{S}}(\phi) = h_{\mathfrak{S}}(\alpha \circ \phi \circ \alpha^{-1})$.

Lemma ($h_{\mathfrak{S}}$ is invariant under inversion)

If $\phi : S \to S$ is an isomorphism in $\mathfrak{S}$, then $h_{\mathfrak{S}}(\phi^{-1}) = h_{\mathfrak{S}}(\phi)$.

Lemma (Logarithmic Law)

Let (S, v) be a normed semigroup and $\phi : S \to S$ an endomorphism. Then $h_{\mathfrak{S}}(\phi^k) \leq k \cdot h_{\mathfrak{S}}(\phi)$ for every $k \in \mathbb{N}$. Furthermore equality holds if v is s-monotone.

Lemma ($h_{\mathfrak{S}}$ is monotone under taking quotients)

If $\phi : S \to S$ and $\psi : T \to T$ are endomorphisms in $\mathfrak{S}$ and $\alpha : T \to S$ is a surjective homomorphism between normed semigroups such that $\alpha \circ \psi = \phi \circ \alpha$, then $h_{\mathfrak{S}}(\phi) \leq h_{\mathfrak{S}}(\psi)$.

Corollary ($h_{\mathfrak{S}}$ is invariant under conjugation)

If $\phi : S \to S$ is an endomorphism in $\mathfrak{S}$ and $\alpha : T \to S$ is an isomorphism in $\mathfrak{S}$, then $h_{\mathfrak{S}}(\phi) = h_{\mathfrak{S}}(\alpha \circ \phi \circ \alpha^{-1})$.

Lemma ($h_{\mathfrak{S}}$ is invariant under inversion)

If $\phi : S \to S$ is an isomorphism in $\mathfrak{S}$, then $h_{\mathfrak{S}}(\phi^{-1}) = h_{\mathfrak{S}}(\phi)$.

Lemma (Logarithmic Law)

Let (S, v) be a normed semigroup and $\phi : S \to S$ an endomorphism. Then $h_{\mathfrak{S}}(\phi^k) \leq k \cdot h_{\mathfrak{S}}(\phi)$ for every $k \in \mathbb{N}$. Furthermore equality holds if v is s-monotone.

Lemma ($h_{\mathfrak{S}}$ is monotone under taking quotients)

If $\phi : S \to S$ and $\psi : T \to T$ are endomorphisms in $\mathfrak{S}$ and $\alpha : T \to S$ is a surjective homomorphism between normed semigroups such that $\alpha \circ \psi = \phi \circ \alpha$, then $h_{\mathfrak{S}}(\phi) \leq h_{\mathfrak{S}}(\psi)$.

Corollary ($h_{\mathfrak{S}}$ is invariant under conjugation)

If $\phi : S \to S$ is an endomorphism in $\mathfrak{S}$ and $\alpha : T \to S$ is an isomorphism in $\mathfrak{S}$, then $h_{\mathfrak{S}}(\phi) = h_{\mathfrak{S}}(\alpha \circ \phi \circ \alpha^{-1})$.

Lemma ($h_{\mathfrak{S}}$ is invariant under inversion)

If $\phi : S \to S$ is an isomorphism in $\mathfrak{S}$, then $h_{\mathfrak{S}}(\phi^{-1}) = h_{\mathfrak{S}}(\phi)$.

Lemma (Logarithmic Law)

Let (S, v) be a normed semigroup and $\phi : S \to S$ an endomorphism. Then $h_{\mathfrak{S}}(\phi^k) \leq k \cdot h_{\mathfrak{S}}(\phi)$ for every $k \in \mathbb{N}$. Furthermore equality holds if v is s-monotone.

Lemma ($h_{\mathfrak{S}}$ is monotone under taking quotients)

If $\phi : S \to S$ and $\psi : T \to T$ are endomorphisms in $\mathfrak{S}$ and $\alpha : T \to S$ is a surjective homomorphism between normed semigroups such that $\alpha \circ \psi = \phi \circ \alpha$, then $h_{\mathfrak{S}}(\phi) \leq h_{\mathfrak{S}}(\psi)$.

Corollary ($h_{\mathfrak{S}}$ is invariant under conjugation)

If $\phi : S \to S$ is an endomorphism in $\mathfrak{S}$ and $\alpha : T \to S$ is an isomorphism in $\mathfrak{S}$, then $h_{\mathfrak{S}}(\phi) = h_{\mathfrak{S}}(\alpha \circ \phi \circ \alpha^{-1})$.

Lemma ($h_{\mathfrak{S}}$ is invariant under inversion)

If $\phi : S \to S$ is an isomorphism in $\mathfrak{S}$, then $h_{\mathfrak{S}}(\phi^{-1}) = h_{\mathfrak{S}}(\phi)$.

Lemma (Logarithmic Law)

Let (S, v) be a normed semigroup and $\phi : S \to S$ an endomorphism. Then $h_{\mathfrak{S}}(\phi^k) \leq k \cdot h_{\mathfrak{S}}(\phi)$ for every $k \in \mathbb{N}$. Furthermore equality holds if v is s-monotone.

Lemma ($h_{\mathfrak{S}}$ is monotone under taking quotients)

If $\phi : S \to S$ and $\psi : T \to T$ are endomorphisms in $\mathfrak{S}$ and $\alpha : T \to S$ is a surjective homomorphism between normed semigroups such that $\alpha \circ \psi = \phi \circ \alpha$, then $h_{\mathfrak{S}}(\phi) \leq h_{\mathfrak{S}}(\psi)$.

Corollary ($h_{\mathfrak{S}}$ is invariant under conjugation)

If $\phi : S \to S$ is an endomorphism in $\mathfrak{S}$ and $\alpha : T \to S$ is an isomorphism in $\mathfrak{S}$, then $h_{\mathfrak{S}}(\phi) = h_{\mathfrak{S}}(\alpha \circ \phi \circ \alpha^{-1})$.

Lemma ($h_{\mathfrak{S}}$ is invariant under inversion)

If $\phi : S \to S$ is an isomorphism in $\mathfrak{S}$, then $h_{\mathfrak{S}}(\phi^{-1}) = h_{\mathfrak{S}}(\phi)$.

Lemma (Logarithmic Law)

Let (S, v) be a normed semigroup and $\phi : S \to S$ an endomorphism. Then $h_{\mathfrak{S}}(\phi^k) \leq k \cdot h_{\mathfrak{S}}(\phi)$ for every $k \in \mathbb{N}$. Furthermore equality holds if v is s-monotone.

Lemma ($h_{\mathfrak{S}}$ is monotone under taking quotients)

If $\phi : S \to S$ and $\psi : T \to T$ are endomorphisms in $\mathfrak{S}$ and $\alpha : T \to S$ is a surjective homomorphism between normed semigroups such that $\alpha \circ \psi = \phi \circ \alpha$, then $h_{\mathfrak{S}}(\phi) \leq h_{\mathfrak{S}}(\psi)$.

Corollary ($h_{\mathfrak{S}}$ is invariant under conjugation)

If $\phi : S \to S$ is an endomorphism in $\mathfrak{S}$ and $\alpha : T \to S$ is an isomorphism in $\mathfrak{S}$, then $h_{\mathfrak{S}}(\phi) = h_{\mathfrak{S}}(\alpha \circ \phi \circ \alpha^{-1})$.

Lemma ($h_{\mathfrak{S}}$ is invariant under inversion)

If $\phi : S \to S$ is an isomorphism in $\mathfrak{S}$, then $h_{\mathfrak{S}}(\phi^{-1}) = h_{\mathfrak{S}}(\phi)$.

Lemma (Logarithmic Law)

Let (S, v) be a normed semigroup and $\phi : S \to S$ an endomorphism. Then $h_{\mathfrak{S}}(\phi^k) \leq k \cdot h_{\mathfrak{S}}(\phi)$ for every $k \in \mathbb{N}$. Furthermore equality holds if v is s-monotone.

Lemma ($h_{\mathfrak{S}}$ is monotone under taking quotients)

If $\phi : S \to S$ and $\psi : T \to T$ are endomorphisms in $\mathfrak{S}$ and $\alpha : T \to S$ is a surjective homomorphism between normed semigroups such that $\alpha \circ \psi = \phi \circ \alpha$, then $h_{\mathfrak{S}}(\phi) \leq h_{\mathfrak{S}}(\psi)$.

Corollary ($h_{\mathfrak{S}}$ is invariant under conjugation)

If $\phi : S \to S$ is an endomorphism in $\mathfrak{S}$ and $\alpha : T \to S$ is an isomorphism in $\mathfrak{S}$, then $h_{\mathfrak{S}}(\phi) = h_{\mathfrak{S}}(\alpha \circ \phi \circ \alpha^{-1})$.

Lemma ($h_{\mathfrak{S}}$ is invariant under inversion)

If $\phi : S \to S$ is an isomorphism in $\mathfrak{S}$, then $h_{\mathfrak{S}}(\phi^{-1}) = h_{\mathfrak{S}}(\phi)$.

Lemma (Logarithmic Law)

Let (S, v) be a normed semigroup and $\phi : S \to S$ an endomorphism. Then $h_{\mathfrak{S}}(\phi^k) \leq k \cdot h_{\mathfrak{S}}(\phi)$ for every $k \in \mathbb{N}$. Furthermore equality holds if v is s-monotone.

Lemma ($h_{\mathfrak{S}}$ is monotone under taking quotients)

If $\phi : S \to S$ and $\psi : T \to T$ are endomorphisms in $\mathfrak{S}$ and $\alpha : T \to S$ is a surjective homomorphism between normed semigroups such that $\alpha \circ \psi = \phi \circ \alpha$, then $h_{\mathfrak{S}}(\phi) \leq h_{\mathfrak{S}}(\psi)$.

Corollary ($h_{\mathfrak{S}}$ is invariant under conjugation)

If $\phi : S \to S$ is an endomorphism in $\mathfrak{S}$ and $\alpha : T \to S$ is an isomorphism in $\mathfrak{S}$, then $h_{\mathfrak{S}}(\phi) = h_{\mathfrak{S}}(\alpha \circ \phi \circ \alpha^{-1})$.

Lemma ($h_{\mathfrak{S}}$ is invariant under inversion)

If $\phi : S \to S$ is an isomorphism in $\mathfrak{S}$, then $h_{\mathfrak{S}}(\phi^{-1}) = h_{\mathfrak{S}}(\phi)$.

Lemma (Logarithmic Law)

Let (S, v) be a normed semigroup and $\phi : S \to S$ an endomorphism. Then $h_{\mathfrak{S}}(\phi^k) \leq k \cdot h_{\mathfrak{S}}(\phi)$ for every $k \in \mathbb{N}$. Furthermore equality holds if v is s-monotone.

Lemma ($h_{\mathfrak{S}}$ is monotone under taking quotients)

If $\phi : S \to S$ and $\psi : T \to T$ are endomorphisms in $\mathfrak{S}$ and $\alpha : T \to S$ is a surjective homomorphism between normed semigroups such that $\alpha \circ \psi = \phi \circ \alpha$, then $h_{\mathfrak{S}}(\phi) \leq h_{\mathfrak{S}}(\psi)$.

Corollary ($h_{\mathfrak{S}}$ is invariant under conjugation)

If $\phi : S \to S$ is an endomorphism in $\mathfrak{S}$ and $\alpha : T \to S$ is an isomorphism in $\mathfrak{S}$, then $h_{\mathfrak{S}}(\phi) = h_{\mathfrak{S}}(\alpha \circ \phi \circ \alpha^{-1})$.

Lemma ($h_{\mathfrak{S}}$ is invariant under inversion)

If $\phi : S \to S$ is an isomorphism in $\mathfrak{S}$, then $h_{\mathfrak{S}}(\phi^{-1}) = h_{\mathfrak{S}}(\phi)$.

Lemma (Logarithmic Law)

Let (S, v) be a normed semigroup and $\phi : S \to S$ an endomorphism. Then $h_{\mathfrak{S}}(\phi^k) \leq k \cdot h_{\mathfrak{S}}(\phi)$ for every $k \in \mathbb{N}$. Furthermore equality holds if v is s-monotone.

For a topological space X the family $\mathrm{cov}(X)$ of all open covers of X is a commutative monoid $(\mathrm{cov}(X), \vee, \mathcal{E})$, where $\vee$ is defined as before and $\mathcal{E} = \{X\}$ is the trivial cover.
One has a natural a preorder $\mathcal{U} \prec \mathcal{V}$ on $\mathrm{cov}(C)$ ($\mathcal{V}$ *refines* $\mathcal{U}$, i.e, if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subseteq U$), that is not an order. It has bottom element $\mathcal{E}$. In general, $\mathcal{U} \vee \mathcal{U} \neq \mathcal{U}$ (so $\mathrm{cov}(C)$ is not a semilattice), yet $\mathcal{U} \vee \mathcal{U} \sim \mathcal{U}$ (where $\mathcal{U} \sim \mathcal{V}$ means $\mathcal{U} \prec \mathcal{V}$ abd $\mathcal{V} \prec \mathcal{U}$)
For a continuous map $\phi : X \to Y$ and $\mathcal{U} \in \mathrm{cov}(Y)$ let

$$\phi^{-1}(\mathcal{U}) = \{\phi^{-1}(U) : U \in \mathcal{U}\}.$$

The assignment $\mathcal{U} \mapsto \phi^{-1}(\mathcal{U})$ gives a semigroup homomorphism $\mathrm{cov}(\phi) : \mathrm{cov}(Y) \to \mathrm{cov}(X)$ (as $\phi^{-1}(\mathcal{U} \vee \mathcal{V}) = \phi^{-1}(\mathcal{U}) \vee \phi^{-1}(\mathcal{V})$).
This defines a contravariant functor cov from the category of all topological spaces to the category of commutative semigroups.

For a topological space X the family $\mathrm{cov}(X)$ of all open covers of X is a commutative monoid $(\mathrm{cov}(X), \vee, \mathcal{E})$, where $\vee$ is defined as before and $\mathcal{E} = \{X\}$ is the trivial cover.
One has a natural a preorder $\mathcal{U} \prec \mathcal{V}$ on $\mathrm{cov}(C)$ ($\mathcal{V}$ *refines* $\mathcal{U}$, i.e, if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subseteq U$), that is not an order. It has bottom element $\mathcal{E}$. In general, $\mathcal{U} \vee \mathcal{U} \neq \mathcal{U}$ (so $\mathrm{cov}(C)$ is not a semilattice), yet $\mathcal{U} \vee \mathcal{U} \sim \mathcal{U}$ (where $\mathcal{U} \sim \mathcal{V}$ means $\mathcal{U} \prec \mathcal{V}$ abd $\mathcal{V} \prec \mathcal{U}$)
For a continuous map $\phi : X \to Y$ and $\mathcal{U} \in \mathrm{cov}(Y)$ let

$$\phi^{-1}(\mathcal{U}) = \{\phi^{-1}(U) : U \in \mathcal{U}\}.$$

The assignment $\mathcal{U} \mapsto \phi^{-1}(\mathcal{U})$ gives a semigroup homomorphism $\mathrm{cov}(\phi) : \mathrm{cov}(Y) \to \mathrm{cov}(X)$ (as $\phi^{-1}(\mathcal{U} \vee \mathcal{V}) = \phi^{-1}(\mathcal{U}) \vee \phi^{-1}(\mathcal{V})$).
This defines a contravariant functor cov from the category of all topological spaces to the category of commutative semigroups.

For a topological space X the family $\mathrm{cov}(X)$ of all open covers of X is a commutative monoid $(\mathrm{cov}(X), \vee, \mathcal{E})$, where $\vee$ is defined as before and $\mathcal{E} = \{X\}$ is the trivial cover.

One has a natural a preorder $\mathcal{U} \prec \mathcal{V}$ on $\mathrm{cov}(C)$ ($\mathcal{V}$ *refines* $\mathcal{U}$, i.e, if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subseteq U$), that is not an order. It has bottom element $\mathcal{E}$. In general, $\mathcal{U} \vee \mathcal{U} \neq \mathcal{U}$ (so $\mathrm{cov}(C)$ is not a semilattice), yet $\mathcal{U} \vee \mathcal{U} \sim \mathcal{U}$ (where $\mathcal{U} \sim \mathcal{V}$ means $\mathcal{U} \prec \mathcal{V}$ abd $\mathcal{V} \prec \mathcal{U}$)

For a continuous map $\phi : X \to Y$ and $\mathcal{U} \in \mathrm{cov}(Y)$ let

$$\phi^{-1}(\mathcal{U}) = \{\phi^{-1}(U) : U \in \mathcal{U}\}.$$

The assignment $\mathcal{U} \mapsto \phi^{-1}(\mathcal{U})$ gives a semigroup homomorphism $\mathrm{cov}(\phi) : \mathrm{cov}(Y) \to \mathrm{cov}(X)$ (as $\phi^{-1}(\mathcal{U} \vee \mathcal{V}) = \phi^{-1}(\mathcal{U}) \vee \phi^{-1}(\mathcal{V})$). This defines a contravariant functor cov from the category of all topological spaces to the category of commutative semigroups.

For a topological space X the family $\mathrm{cov}(X)$ of all open covers of X is a commutative monoid $(\mathrm{cov}(X), \vee, \mathcal{E})$, where $\vee$ is defined as before and $\mathcal{E} = \{X\}$ is the trivial cover.
One has a natural a preorder $\mathcal{U} \prec \mathcal{V}$ on $\mathrm{cov}(C)$ ($\mathcal{V}$ *refines* $\mathcal{U}$, i.e, if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subseteq U$), that is not an order. It has bottom element $\mathcal{E}$. In general, $\mathcal{U} \vee \mathcal{U} \neq \mathcal{U}$ (so $\mathrm{cov}(C)$ is not a semilattice), yet $\mathcal{U} \vee \mathcal{U} \sim \mathcal{U}$ (where $\mathcal{U} \sim \mathcal{V}$ means $\mathcal{U} \prec \mathcal{V}$ abd $\mathcal{V} \prec \mathcal{U}$)
For a continuous map $\phi : X \to Y$ and $\mathcal{U} \in \mathrm{cov}(Y)$ let

$$\phi^{-1}(\mathcal{U}) = \{\phi^{-1}(U) : U \in \mathcal{U}\}.$$

The assignment $\mathcal{U} \mapsto \phi^{-1}(\mathcal{U})$ gives a semigroup homomorphism $\mathrm{cov}(\phi) : \mathrm{cov}(Y) \to \mathrm{cov}(X)$ (as $\phi^{-1}(\mathcal{U} \vee \mathcal{V}) = \phi^{-1}(\mathcal{U}) \vee \phi^{-1}(\mathcal{V})$). This defines a contravariant functor cov from the category of all topological spaces to the category of commutative semigroups.

For a topological space X the family $\mathrm{cov}(X)$ of all open covers of X is a commutative monoid $(\mathrm{cov}(X), \vee, \mathcal{E})$, where $\vee$ is defined as before and $\mathcal{E} = \{X\}$ is the trivial cover.
One has a natural a preorder $\mathcal{U} \prec \mathcal{V}$ on $\mathrm{cov}(C)$ ($\mathcal{V}$ *refines* $\mathcal{U}$, i.e, if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subseteq U$), that is not an order. It has bottom element $\mathcal{E}$. In general, $\mathcal{U} \vee \mathcal{U} \neq \mathcal{U}$ (so $\mathrm{cov}(C)$ is not a semilattice), yet $\mathcal{U} \vee \mathcal{U} \sim \mathcal{U}$ (where $\mathcal{U} \sim \mathcal{V}$ means $\mathcal{U} \prec \mathcal{V}$ abd $\mathcal{V} \prec \mathcal{U}$)
For a continuous map $\phi : X \to Y$ and $\mathcal{U} \in \mathrm{cov}(Y)$ let

$$\phi^{-1}(\mathcal{U}) = \{\phi^{-1}(U) : U \in \mathcal{U}\}.$$

The assignment $\mathcal{U} \mapsto \phi^{-1}(\mathcal{U})$ gives a semigroup homomorphism $\mathrm{cov}(\phi) : \mathrm{cov}(Y) \to \mathrm{cov}(X)$ (as $\phi^{-1}(\mathcal{U} \vee \mathcal{V}) = \phi^{-1}(\mathcal{U}) \vee \phi^{-1}(\mathcal{V})$). This defines a contravariant functor cov from the category of all topological spaces to the category of commutative semigroups.

For a topological space X the family $\mathrm{cov}(X)$ of all open covers of X is a commutative monoid $(\mathrm{cov}(X), \vee, \mathcal{E})$, where $\vee$ is defined as before and $\mathcal{E} = \{X\}$ is the trivial cover.
One has a natural a preorder $\mathcal{U} \prec \mathcal{V}$ on $\mathrm{cov}(C)$ ($\mathcal{V}$ *refines* $\mathcal{U}$, i.e, if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subseteq U$), that is not an order. It has bottom element $\mathcal{E}$. In general, $\mathcal{U} \vee \mathcal{U} \neq \mathcal{U}$ (so $\mathrm{cov}(C)$ is not a semilattice), yet $\mathcal{U} \vee \mathcal{U} \sim \mathcal{U}$ (where $\mathcal{U} \sim \mathcal{V}$ means $\mathcal{U} \prec \mathcal{V}$ abd $\mathcal{V} \prec \mathcal{U}$)
For a continuous map $\phi : X \to Y$ and $\mathcal{U} \in \mathrm{cov}(Y)$ let

$$\phi^{-1}(\mathcal{U}) = \{\phi^{-1}(U) : U \in \mathcal{U}\}.$$

The assignment $\mathcal{U} \mapsto \phi^{-1}(\mathcal{U})$ gives a semigroup homomorphism $\mathrm{cov}(\phi) : \mathrm{cov}(Y) \to \mathrm{cov}(X)$ (as $\phi^{-1}(\mathcal{U} \vee \mathcal{V}) = \phi^{-1}(\mathcal{U}) \vee \phi^{-1}(\mathcal{V})$).
This defines a contravariant functor cov from the category of all topological spaces to the category of commutative semigroups.

For a topological space X the family $\mathrm{cov}(X)$ of all open covers of X is a commutative monoid $(\mathrm{cov}(X), \vee, \mathcal{E})$, where $\vee$ is defined as before and $\mathcal{E} = \{X\}$ is the trivial cover.
One has a natural a preorder $\mathcal{U} \prec \mathcal{V}$ on $\mathrm{cov}(C)$ ($\mathcal{V}$ *refines* $\mathcal{U}$, i.e, if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subseteq U$), that is not an order. It has bottom element $\mathcal{E}$. In general, $\mathcal{U} \vee \mathcal{U} \neq \mathcal{U}$ (so $\mathrm{cov}(C)$ is not a semilattice), yet $\mathcal{U} \vee \mathcal{U} \sim \mathcal{U}$ (where $\mathcal{U} \sim \mathcal{V}$ means $\mathcal{U} \prec \mathcal{V}$ abd $\mathcal{V} \prec \mathcal{U}$)
For a continuous map $\phi : X \to Y$ and $\mathcal{U} \in \mathrm{cov}(Y)$ let

$$\phi^{-1}(\mathcal{U}) = \{\phi^{-1}(U) : U \in \mathcal{U}\}.$$

The assignment $\mathcal{U} \mapsto \phi^{-1}(\mathcal{U})$ gives a semigroup homomorphism $\mathrm{cov}(\phi) : \mathrm{cov}(Y) \to \mathrm{cov}(X)$ (as $\phi^{-1}(\mathcal{U} \vee \mathcal{V}) = \phi^{-1}(\mathcal{U}) \vee \phi^{-1}(\mathcal{V})$).
This defines a contravariant functor cov from the category of all topological spaces to the category of commutative semigroups.

To get a norm on the semigroup $\mathrm{cov}(X)$ we restrict this functor to the subcategory **CTop** of *compact spaces*. For $X \in$ **CTop**, $\mathcal{U} \in \mathrm{cov}(X)$ let $v(\mathcal{U}) = N(\mathcal{U})$.

Lemma

For a compact space X, $(\mathrm{cov}(X), \vee, v)$ is an normed semigroup. For every continuous map $\phi : X \to Y$ of compact spaces the inequality $v(\phi^{-1}(\mathcal{W})) \leq v(\mathcal{W})$ holds for every $\mathcal{W} \in \mathrm{cov}(Y)$.

By the lemma $\mathrm{cov}(\phi) : \mathrm{cov}(Y) \to \mathrm{cov}(X)$ is a morphism in $\mathfrak{S}$, so that the assignement $X \mapsto \mathrm{cov}(X)$ defines a contravariant functor

$$cov : \textbf{CTop} \to \mathfrak{S},$$

that sends embeddings in **CTop** to surjective morphisms in $\mathfrak{S}$ and sends surjective maps in **CTop** to embeddings in $\mathfrak{S}$.

To get a norm on the semigroup $\mathrm{cov}(X)$ we restrict this functor to the subcategory **CTop** of *compact spaces*. For $X \in$ **CTop**, $\mathcal{U} \in \mathrm{cov}(X)$ let $v(\mathcal{U}) = N(\mathcal{U})$.

Lemma

For a compact space X, $(\mathrm{cov}(X), \vee, v)$ is an normed semigroup. For every continuous map $\phi : X \to Y$ of compact spaces the inequality $v(\phi^{-1}(\mathcal{W})) \leq v(\mathcal{W})$ holds for every $\mathcal{W} \in \mathrm{cov}(Y)$.

By the lemma $\mathrm{cov}(\phi) : \mathrm{cov}(Y) \to \mathrm{cov}(X)$ is a morphism in $\mathfrak{S}$, so that the assignement $X \mapsto \mathrm{cov}(X)$ defines a contravariant functor

$$cov : \mathbf{CTop} \to \mathfrak{S},$$

that sends embeddings in **CTop** to surjective morphisms in $\mathfrak{S}$ and sends surjective maps in **CTop** to embeddings in $\mathfrak{S}$.

To get a norm on the semigroup $\mathrm{cov}(X)$ we restrict this functor to the subcategory **CTop** of *compact spaces*. For $X \in$ **CTop**, $\mathcal{U} \in \mathrm{cov}(X)$ let $v(\mathcal{U}) = N(\mathcal{U})$.

Lemma

For a compact space X, $(\mathrm{cov}(X), \vee, v)$ is an normed semigroup. For every continuous map $\phi : X \to Y$ of compact spaces the inequality $v(\phi^{-1}(\mathcal{W})) \leq v(\mathcal{W})$ holds for every $\mathcal{W} \in \mathrm{cov}(Y)$.

By the lemma $\mathrm{cov}(\phi) : \mathrm{cov}(Y) \to \mathrm{cov}(X)$ is a morphism in $\mathfrak{S}$, so that the assignement $X \mapsto \mathrm{cov}(X)$ defines a contravariant functor

$$cov : \mathbf{CTop} \to \mathfrak{S},$$

that sends embeddings in **CTop** to surjective morphisms in $\mathfrak{S}$ and sends surjective maps in **CTop** to embeddings in $\mathfrak{S}$.

To get a norm on the semigroup $\mathrm{cov}(X)$ we restrict this functor to the subcategory **CTop** of *compact spaces*. For $X \in$ **CTop**, $\mathcal{U} \in \mathrm{cov}(X)$ let $v(\mathcal{U}) = N(\mathcal{U})$.

Lemma

For a compact space X, $(\mathrm{cov}(X), \vee, v)$ is an normed semigroup.
For every continuous map $\phi : X \to Y$ of compact spaces the inequality $v(\phi^{-1}(\mathcal{W})) \leq v(\mathcal{W})$ holds for every $\mathcal{W} \in \mathrm{cov}(Y)$.

By the lemma $\mathrm{cov}(\phi) : \mathrm{cov}(Y) \to \mathrm{cov}(X)$ is a morphism in $\mathfrak{S}$, so that the assignement $X \mapsto \mathrm{cov}(X)$ defines a contravariant functor

$$cov : \mathbf{CTop} \to \mathfrak{S},$$

that sends embeddings in **CTop** to surjective morphisms in $\mathfrak{S}$ and sends surjective maps in **CTop** to embeddings in $\mathfrak{S}$.

To get a norm on the semigroup $\mathrm{cov}(X)$ we restrict this functor to the subcategory **CTop** of *compact spaces*. For $X \in$ **CTop**, $\mathcal{U} \in \mathrm{cov}(X)$ let $v(\mathcal{U}) = N(\mathcal{U})$.

Lemma

For a compact space X, $(\mathrm{cov}(X), \vee, v)$ is an normed semigroup. For every continuous map $\phi : X \to Y$ of compact spaces the inequality $v(\phi^{-1}(\mathcal{W})) \leq v(\mathcal{W})$ holds for every $\mathcal{W} \in \mathrm{cov}(Y)$.

By the lemma $\mathrm{cov}(\phi) : \mathrm{cov}(Y) \to \mathrm{cov}(X)$ is a morphism in $\mathfrak{S}$, so that the assignement $X \mapsto \mathrm{cov}(X)$ defines a contravariant functor

$$cov : \mathbf{CTop} \to \mathfrak{S},$$

that sends embeddings in **CTop** to surjective morphisms in $\mathfrak{S}$ and sends surjective maps in **CTop** to embeddings in $\mathfrak{S}$.

To get a norm on the semigroup $\mathrm{cov}(X)$ we restrict this functor to the subcategory **CTop** of *compact spaces*. For $X \in$ **CTop**, $\mathcal{U} \in \mathrm{cov}(X)$ let $v(\mathcal{U}) = N(\mathcal{U})$.

Lemma

For a compact space X, $(\mathrm{cov}(X), \vee, v)$ is an normed semigroup. For every continuous map $\phi : X \to Y$ of compact spaces the inequality $v(\phi^{-1}(\mathcal{W})) \leq v(\mathcal{W})$ holds for every $\mathcal{W} \in \mathrm{cov}(Y)$.

By the lemma $\mathrm{cov}(\phi) : \mathrm{cov}(Y) \to \mathrm{cov}(X)$ is a morphism in $\mathfrak{S}$, so that the assignement $X \mapsto \mathrm{cov}(X)$ defines a contravariant functor

$$\mathfrak{cov} : \mathbf{CTop} \to \mathfrak{S},$$

that sends embeddings in **CTop** to surjective morphisms in $\mathfrak{S}$ and sends surjective maps in **CTop** to embeddings in $\mathfrak{S}$.

To get a norm on the semigroup $\mathrm{cov}(X)$ we restrict this functor to the subcategory **CTop** of *compact spaces*. For $X \in$ **CTop**, $\mathcal{U} \in \mathrm{cov}(X)$ let $v(\mathcal{U}) = N(\mathcal{U})$.

Lemma

For a compact space X, $(\mathrm{cov}(X), \vee, v)$ is an normed semigroup. For every continuous map $\phi : X \to Y$ of compact spaces the inequality $v(\phi^{-1}(\mathcal{W})) \leq v(\mathcal{W})$ holds for every $\mathcal{W} \in \mathrm{cov}(Y)$.

By the lemma $\mathrm{cov}(\phi) : \mathrm{cov}(Y) \to \mathrm{cov}(X)$ is a morphism in $\mathfrak{S}$, so that the assignement $X \mapsto \mathrm{cov}(X)$ defines a contravariant functor

$$\mathfrak{cov} : \mathbf{CTop} \to \mathfrak{S},$$

that sends embeddings in **CTop** to surjective morphisms in $\mathfrak{S}$ and sends surjective maps in **CTop** to embeddings in $\mathfrak{S}$.

Let $F : \mathcal{X} \to \mathfrak{S}$ a functor. Define the entropy function $\mathfrak{h}_F$ in the category $\mathcal{X}$ by

$$\mathfrak{h}_F(\phi) = h_{\mathfrak{S}}(F\phi),$$

for an endomorphism $\phi : X \to X$ in $\mathcal{X}$. The functor F preserves commutative squares and isomorphisms. So, with $X, Y \in \mathcal{X}$ and $\phi \in \mathrm{End}_{\mathcal{X}}(X)$, the entropy $\mathfrak{h}_F$ will satisfy:

[Invariance under conjugation] If $\alpha : Y \to X$ is an isomorphism, then $\mathfrak{h}_F(\phi) = \mathfrak{h}_F(\alpha^{-1} \circ \phi \circ \alpha)$.

[Invariance under inversion] $\mathfrak{h}_F(\phi^{-1}) = \mathfrak{h}_F(\phi)$, if ϕ is an isomorphism.

[Logaritmic law] If the norm of the semigroup $F(X)$ is s-monotone, then $\mathfrak{h}_F(\phi^k) = k \cdot \mathfrak{h}_F(\phi)$. for all $k \in \mathbb{N}$.

Let $F : \mathcal{X} \to \mathfrak{S}$ a functor. Define the entropy function $\mathfrak{h}_F$ in the category $\mathcal{X}$ by

$$\mathfrak{h}_F(\phi) = h_{\mathfrak{S}}(F\phi),$$

for an endomorphism $\phi : X \to X$ in $\mathcal{X}$. The functor F preserves commutative squares and isomorphisms. So, with $X, Y \in \mathcal{X}$ and $\phi \in \mathrm{End}_{\mathcal{X}}(X)$, the entropy $\mathfrak{h}_F$ will satisfy:

[Invariance under conjugation] If $\alpha : Y \to X$ is an isomorphism, then $\mathfrak{h}_F(\phi) = \mathfrak{h}_F(\alpha^{-1} \circ \phi \circ \alpha)$.

[Invariance under inversion] $\mathfrak{h}_F(\phi^{-1}) = \mathfrak{h}_F(\phi)$, if ϕ is an isomorphism.

[Logaritmic law] If the norm of the semigroup $F(X)$ is s-monotone, then $\mathfrak{h}_F(\phi^k) = k \cdot \mathfrak{h}_F(\phi)$. for all $k \in \mathbb{N}$.

Let $F : \mathcal{X} \to \mathfrak{S}$ a functor. Define the entropy function $\mathfrak{h}_F$ in the category $\mathcal{X}$ by

$$\mathfrak{h}_F(\phi) = h_{\mathfrak{S}}(F\phi),$$

for an endomorphism $\phi : X \to X$ in $\mathcal{X}$. The functor F preserves commutative squares and isomorphisms. So, with $X, Y \in \mathcal{X}$ and $\phi \in \mathrm{End}_{\mathcal{X}}(X)$, the entropy $\mathfrak{h}_F$ will satisfy:

[Invariance under conjugation] If $\alpha : Y \to X$ is an isomorphism, then $\mathfrak{h}_F(\phi) = \mathfrak{h}_F(\alpha^{-1} \circ \phi \circ \alpha)$.

[Invariance under inversion] $\mathfrak{h}_F(\phi^{-1}) = \mathfrak{h}_F(\phi)$, if ϕ is an isomorphism.

[Logaritmic law] If the norm of the semigroup $F(X)$ is s-monotone, then $\mathfrak{h}_F(\phi^k) = k \cdot \mathfrak{h}_F(\phi)$. for all $k \in \mathbb{N}$.

Let $F : \mathcal{X} \to \mathfrak{S}$ a functor. Define the entropy function $\mathfrak{h}_F$ in the category $\mathcal{X}$ by

$$\mathfrak{h}_F(\phi) = h_{\mathfrak{S}}(F\phi),$$

for an endomorphism $\phi : X \to X$ in $\mathcal{X}$. The functor F preserves commutative squares and isomorphisms. So, with $X, Y \in \mathcal{X}$ and $\phi \in \mathrm{End}_{\mathcal{X}}(X)$, the entropy $\mathfrak{h}_F$ will satisfy:

[Invariance under conjugation] If $\alpha : Y \to X$ is an isomorphism, then $\mathfrak{h}_F(\phi) = \mathfrak{h}_F(\alpha^{-1} \circ \phi \circ \alpha)$.

[Invariance under inversion] $\mathfrak{h}_F(\phi^{-1}) = \mathfrak{h}_F(\phi)$, if ϕ is an isomorphism.

[Logaritmic law] If the norm of the semigroup $F(X)$ is s-monotone, then $\mathfrak{h}_F(\phi^k) = k \cdot \mathfrak{h}_F(\phi)$. for all $k \in \mathbb{N}$.

Let $F : \mathcal{X} \to \mathfrak{S}$ a functor. Define the entropy function $\mathfrak{h}_F$ in the category $\mathcal{X}$ by

$$\mathfrak{h}_F(\phi) = h_{\mathfrak{S}}(F\phi),$$

for an endomorphism $\phi : X \to X$ in $\mathcal{X}$. The functor F preserves commutative squares and isomorphisms. So, with $X, Y \in \mathcal{X}$ and $\phi \in \mathrm{End}_{\mathcal{X}}(X)$, the entropy $\mathfrak{h}_F$ will satisfy:

[Invariance under conjugation] If $\alpha : Y \to X$ is an isomorphism, then $\mathfrak{h}_F(\phi) = \mathfrak{h}_F(\alpha^{-1} \circ \phi \circ \alpha)$.

[Invariance under inversion] $\mathfrak{h}_F(\phi^{-1}) = \mathfrak{h}_F(\phi)$, if ϕ is an isomorphism.

[Logaritmic law] If the norm of the semigroup $F(X)$ is s-monotone, then $\mathfrak{h}_F(\phi^k) = k \cdot \mathfrak{h}_F(\phi)$. for all $k \in \mathbb{N}$.

Let $F : \mathcal{X} \to \mathfrak{S}$ a functor. Define the entropy function $\mathfrak{h}_F$ in the category $\mathcal{X}$ by

$$\mathfrak{h}_F(\phi) = h_{\mathfrak{S}}(F\phi),$$

for an endomorphism $\phi : X \to X$ in $\mathcal{X}$. The functor F preserves commutative squares and isomorphisms. So, with $X, Y \in \mathcal{X}$ and $\phi \in \mathrm{End}_{\mathcal{X}}(X)$, the entropy $\mathfrak{h}_F$ will satisfy:

[Invariance under conjugation] If $\alpha : Y \to X$ is an isomorphism, then $\mathfrak{h}_F(\phi) = \mathfrak{h}_F(\alpha^{-1} \circ \phi \circ \alpha)$.

[Invariance under inversion] $\mathfrak{h}_F(\phi^{-1}) = \mathfrak{h}_F(\phi)$, if ϕ is an isomorphism.

[Logaritmic law] If the norm of the semigroup $F(X)$ is s-monotone, then $\mathfrak{h}_F(\phi^k) = k \cdot \mathfrak{h}_F(\phi)$. for all $k \in \mathbb{N}$.

Let $F : \mathcal{X} \to \mathfrak{S}$ a functor. Define the entropy function $\mathfrak{h}_F$ in the category $\mathcal{X}$ by

$$\mathfrak{h}_F(\phi) = h_{\mathfrak{S}}(F\phi),$$

for an endomorphism $\phi : X \to X$ in $\mathcal{X}$. The functor F preserves commutative squares and isomorphisms. So, with $X, Y \in \mathcal{X}$ and $\phi \in \mathrm{End}_{\mathcal{X}}(X)$, the entropy $\mathfrak{h}_F$ will satisfy:

[Invariance under conjugation] If $\alpha : Y \to X$ is an isomorphism, then $\mathfrak{h}_F(\phi) = \mathfrak{h}_F(\alpha^{-1} \circ \phi \circ \alpha)$.

[Invariance under inversion] $\mathfrak{h}_F(\phi^{-1}) = \mathfrak{h}_F(\phi)$, if ϕ is an isomorphism.

[Logaritmic law] If the norm of the semigroup $F(X)$ is s-monotone, then $\mathfrak{h}_F(\phi^k) = k \cdot \mathfrak{h}_F(\phi)$. for all $k \in \mathbb{N}$.

Let $F : \mathcal{X} \to \mathfrak{S}$ a functor. Define the entropy function $\mathfrak{h}_F$ in the category $\mathcal{X}$ by

$$\mathfrak{h}_F(\phi) = h_{\mathfrak{S}}(F\phi),$$

for an endomorphism $\phi : X \to X$ in $\mathcal{X}$. The functor F preserves commutative squares and isomorphisms. So, with $X, Y \in \mathcal{X}$ and $\phi \in \mathrm{End}_{\mathcal{X}}(X)$, the entropy $\mathfrak{h}_F$ will satisfy:

[Invariance under conjugation] If $\alpha : Y \to X$ is an isomorphism, then $\mathfrak{h}_F(\phi) = \mathfrak{h}_F(\alpha^{-1} \circ \phi \circ \alpha)$.

[Invariance under inversion] $\mathfrak{h}_F(\phi^{-1}) = \mathfrak{h}_F(\phi)$, if ϕ is an isomorphism.

[Logaritmic law] If the norm of the semigroup $F(X)$ is s-monotone, then $\mathfrak{h}_F(\phi^k) = k \cdot \mathfrak{h}_F(\phi)$. for all $k \in \mathbb{N}$.

Let $F : \mathcal{X} \to \mathfrak{S}$ a functor. Define the entropy function $\mathfrak{h}_F$ in the category $\mathcal{X}$ by

$$\mathfrak{h}_F(\phi) = h_{\mathfrak{S}}(F\phi),$$

for an endomorphism $\phi : X \to X$ in $\mathcal{X}$. The functor F preserves commutative squares and isomorphisms. So, with $X, Y \in \mathcal{X}$ and $\phi \in \mathrm{End}_{\mathcal{X}}(X)$, the entropy $\mathfrak{h}_F$ will satisfy:

[Invariance under conjugation] If $\alpha : Y \to X$ is an isomorphism, then $\mathfrak{h}_F(\phi) = \mathfrak{h}_F(\alpha^{-1} \circ \phi \circ \alpha)$.

[Invariance under inversion] $\mathfrak{h}_F(\phi^{-1}) = \mathfrak{h}_F(\phi)$, if ϕ is an isomorphism.

[Logaritmic law] If the norm of the semigroup $F(X)$ is s-monotone, then $\mathfrak{h}_F(\phi^k) = k \cdot \mathfrak{h}_F(\phi)$. for all $k \in \mathbb{N}$.

Further properties of $\mathfrak{h}_F$ depend on properties of the functor F. We start by monotonicity under taking invariant subobjects or factor flows.

[Monotonicity under taking invariant subobjects] If F sends subobject embeddings in $\mathcal{X}$ to embeddings in $\mathfrak{S}$ or to surjective maps in $\mathfrak{S}$, then $\mathfrak{h}_F$ is monotone under taking invariant subobjects (i.e., if Y is a ϕ-invariant subobject of X, then $\mathfrak{h}_F(\phi\restriction_Y) \leq \mathfrak{h}_F(\phi)$).

[Monotonicity under taking quotients] If F sends quotients in $\mathcal{X}$ to surjective maps in $\mathfrak{S}$ or to embeddings in $\mathfrak{S}$, then $\mathfrak{h}_F$ is monotone under taking quotients.

["Continuity" under direct limits] If F is covariant and sends direct limits to direct limits, then $\mathfrak{h}_F(\phi) = \sup \mathfrak{h}_F(\phi|_{X_i})$ whenever $X = \varinjlim X_i$ and each X_i is a ϕ-invariant subobject of X

Further properties of $\mathfrak{h}_F$ depend on properties of the functor F.

We start by monotonicity under taking invariant subobjects or factor flows.

[Monotonicity under taking invariant subobjects] If F sends subobject embeddings in $\mathcal{X}$ to embeddings in $\mathfrak{S}$ or to surjective maps in $\mathfrak{S}$, then $\mathfrak{h}_F$ is monotone under taking invariant subobjects (i.e., if Y is a ϕ-invariant subobject of X, then $\mathfrak{h}_F(\phi \restriction_Y) \leq \mathfrak{h}_F(\phi)$).

[Monotonicity under taking quotients] If F sends quotients in $\mathcal{X}$ to surjective maps in $\mathfrak{S}$ or to embeddings in $\mathfrak{S}$, then $\mathfrak{h}_F$ is monotone under taking quotients.

[“Continuity” under direct limits] If F is covariant and sends direct limits to direct limits, then $\mathfrak{h}_F(\phi) = \sup \mathfrak{h}_F(\phi|_{X_i})$ whenever $X = \varinjlim X_i$ and each X_i is a ϕ-invariant subobject of X

Further properties of $\mathfrak{h}_F$ depend on properties of the functor F. We start by monotonicity under taking invariant subobjects or factor flows.

[Monotonicity under taking invariant subobjects] If F sends subobject embeddings in $\mathfrak{X}$ to embeddings in $\mathfrak{S}$ or to surjective maps in $\mathfrak{S}$, then $\mathfrak{h}_F$ is monotone under taking invariant subobjects (i.e., if Y is a ϕ-invariant subobject of X, then $\mathfrak{h}_F(\phi\restriction_Y) \leq \mathfrak{h}_F(\phi)$).

[Monotonicity under taking quotients] If F sends quotients in $\mathfrak{X}$ to surjective maps in $\mathfrak{S}$ or to embeddings in $\mathfrak{S}$, then $\mathfrak{h}_F$ is monotone under taking quotients.

["Continuity" under direct limits] If F is covariant and sends direct limits to direct limits, then $\mathfrak{h}_F(\phi) = \sup \mathfrak{h}_F(\phi|_{X_i})$ whenever $X = \varinjlim X_i$ and each X_i is a ϕ-invariant subobject of X

Further properties of $\mathfrak{h}_F$ depend on properties of the functor F. We start by monotonicity under taking invariant subobjects or factor flows.

[Monotonicity under taking invariant subobjects] If F sends subobject embeddings in $\mathfrak{X}$ to embeddings in $\mathfrak{S}$ or to surjective maps in $\mathfrak{S}$, then $\mathfrak{h}_F$ is monotone under taking invariant subobjects (i.e., if Y is a ϕ-invariant subobject of X, then $\mathfrak{h}_F(\phi\restriction_Y) \leq \mathfrak{h}_F(\phi)$).

[Monotonicity under taking quotients] If F sends quotients in $\mathfrak{X}$ to surjective maps in $\mathfrak{S}$ or to embeddings in $\mathfrak{S}$, then $\mathfrak{h}_F$ is monotone under taking quotients.

["Continuity" under direct limits] If F is covariant and sends direct limits to direct limits, then $\mathfrak{h}_F(\phi) = \sup \mathfrak{h}_F(\phi|_{X_i})$ whenever $X = \varinjlim X_i$ and each X_i is a ϕ-invariant subobject of X

Further properties of $\mathfrak{h}_F$ depend on properties of the functor F. We start by monotonicity under taking invariant subobjects or factor flows.

[Monotonicity under taking invariant subobjects] If F sends subobject embeddings in $\mathcal{X}$ to embeddings in $\mathfrak{S}$ or to surjective maps in $\mathfrak{S}$, then $\mathfrak{h}_F$ is monotone under taking invariant subobjects (i.e., if Y is a ϕ-invariant subobject of X, then $\mathfrak{h}_F(\phi \restriction_Y) \leq \mathfrak{h}_F(\phi)$).

[Monotonicity under taking quotients] If F sends quotients in $\mathcal{X}$ to surjective maps in $\mathfrak{S}$ or to embeddings in $\mathfrak{S}$, then $\mathfrak{h}_F$ is monotone under taking quotients.

[“Continuity” under direct limits] If F is covariant and sends direct limits to direct limits, then $\mathfrak{h}_F(\phi) = \sup \mathfrak{h}_F(\phi|_{X_i})$ whenever $X = \varinjlim X_i$ and each X_i is a ϕ-invariant subobject of X

Further properties of $\mathfrak{h}_F$ depend on properties of the functor F. We start by monotonicity under taking invariant subobjects or factor flows.

[Monotonicity under taking invariant subobjects] If F sends subobject embeddings in $\mathcal{X}$ to embeddings in $\mathfrak{S}$ or to surjective maps in $\mathfrak{S}$, then $\mathfrak{h}_F$ is monotone under taking invariant subobjects (i.e., if Y is a ϕ-invariant subobject of X, then $\mathfrak{h}_F(\phi\restriction_Y) \leq \mathfrak{h}_F(\phi)$).

[Monotonicity under taking quotients] If F sends quotients in $\mathcal{X}$ to surjective maps in $\mathfrak{S}$ or to embeddings in $\mathfrak{S}$, then $\mathfrak{h}_F$ is monotone under taking quotients.

["Continuity" under direct limits] If F is covariant and sends direct limits to direct limits, then $\mathfrak{h}_F(\phi) = \sup \mathfrak{h}_F(\phi|_{X_i})$ whenever $X = \varinjlim X_i$ and each X_i is a ϕ-invariant subobject of X

Further properties of $\mathfrak{h}_F$ depend on properties of the functor F. We start by monotonicity under taking invariant subobjects or factor flows.

[Monotonicity under taking invariant subobjects] If F sends subobject embeddings in $\mathcal{X}$ to embeddings in $\mathfrak{S}$ or to surjective maps in $\mathfrak{S}$, then $\mathfrak{h}_F$ is monotone under taking invariant subobjects (i.e., if Y is a ϕ-invariant subobject of X, then $\mathfrak{h}_F(\phi\restriction_Y) \leq \mathfrak{h}_F(\phi)$).

[Monotonicity under taking quotients] If F sends quotients in $\mathcal{X}$ to surjective maps in $\mathfrak{S}$ or to embeddings in $\mathfrak{S}$, then $\mathfrak{h}_F$ is monotone under taking quotients.

["Continuity" under direct limits] If F is covariant and sends direct limits to direct limits, then $\mathfrak{h}_F(\phi) = \sup \mathfrak{h}_F(\phi|_{X_i})$ whenever $X = \varinjlim X_i$ and each X_i is a ϕ-invariant subobject of X

Further properties of $\mathfrak{h}_F$ depend on properties of the functor F. We start by monotonicity under taking invariant subobjects or factor flows.

[Monotonicity under taking invariant subobjects] If F sends subobject embeddings in $\mathcal{X}$ to embeddings in $\mathfrak{S}$ or to surjective maps in $\mathfrak{S}$, then $\mathfrak{h}_F$ is monotone under taking invariant subobjects (i.e., if Y is a ϕ-invariant subobject of X, then $\mathfrak{h}_F(\phi\restriction_Y) \leq \mathfrak{h}_F(\phi)$).

[Monotonicity under taking quotients] If F sends quotients in $\mathcal{X}$ to surjective maps in $\mathfrak{S}$ or to embeddings in $\mathfrak{S}$, then $\mathfrak{h}_F$ is monotone under taking quotients.

["Continuity" under direct limits] If F is covariant and sends direct limits to direct limits, then $\mathfrak{h}_F(\phi) = \sup \mathfrak{h}_F(\phi|_{X_i})$ whenever $X = \varinjlim X_i$ and each X_i is a ϕ-invariant subobject of X

Further properties of $\mathfrak{h}_F$ depend on properties of the functor F. We start by monotonicity under taking invariant subobjects or factor flows.

[Monotonicity under taking invariant subobjects] If F sends subobject embeddings in $\mathcal{X}$ to embeddings in $\mathfrak{S}$ or to surjective maps in $\mathfrak{S}$, then $\mathfrak{h}_F$ is monotone under taking invariant subobjects (i.e., if Y is a ϕ-invariant subobject of X, then $\mathfrak{h}_F(\phi\restriction_Y) \leq \mathfrak{h}_F(\phi)$).

[Monotonicity under taking quotients] If F sends quotients in $\mathcal{X}$ to surjective maps in $\mathfrak{S}$ or to embeddings in $\mathfrak{S}$, then $\mathfrak{h}_F$ is monotone under taking quotients.

["Continuity" under direct limits] If F is covariant and sends direct limits to direct limits, then $\mathfrak{h}_F(\phi) = \sup \mathfrak{h}_F(\phi|_{X_i})$ whenever $X = \varinjlim X_i$ and each X_i is a ϕ-invariant subobject of X

Further properties of $\mathfrak{h}_F$ depend on properties of the functor F. We start by monotonicity under taking invariant subobjects or factor flows.

[Monotonicity under taking invariant subobjects] If F sends subobject embeddings in $\mathcal{X}$ to embeddings in $\mathfrak{S}$ or to surjective maps in $\mathfrak{S}$, then $\mathfrak{h}_F$ is monotone under taking invariant subobjects (i.e., if Y is a ϕ-invariant subobject of X, then $\mathfrak{h}_F(\phi\restriction_Y) \leq \mathfrak{h}_F(\phi)$).

[Monotonicity under taking quotients] If F sends quotients in $\mathcal{X}$ to surjective maps in $\mathfrak{S}$ or to embeddings in $\mathfrak{S}$, then $\mathfrak{h}_F$ is monotone under taking quotients.

[“Continuity” under direct limits] If F is covariant and sends direct limits to direct limits, then $\mathfrak{h}_F(\phi) = \sup \mathfrak{h}_F(\phi|_{X_i})$ whenever $X = \varinjlim X_i$ and each X_i is a ϕ-invariant subobject of X

Further properties of $\mathfrak{h}_F$ depend on properties of the functor F. We start by monotonicity under taking invariant subobjects or factor flows.

[Monotonicity under taking invariant subobjects] If F sends subobject embeddings in $\mathcal{X}$ to embeddings in $\mathfrak{S}$ or to surjective maps in $\mathfrak{S}$, then $\mathfrak{h}_F$ is monotone under taking invariant subobjects (i.e., if Y is a ϕ-invariant subobject of X, then $\mathfrak{h}_F(\phi\restriction_Y) \leq \mathfrak{h}_F(\phi)$).

[Monotonicity under taking quotients] If F sends quotients in $\mathcal{X}$ to surjective maps in $\mathfrak{S}$ or to embeddings in $\mathfrak{S}$, then $\mathfrak{h}_F$ is monotone under taking quotients.

[“Continuity” under direct limits] If F is covariant and sends direct limits to direct limits, then $\mathfrak{h}_F(\phi) = \sup \mathfrak{h}_F(\phi|_{X_i})$ whenever $X = \varinjlim X_i$ and each X_i is a ϕ-invariant subobject of X

Obtaining the topological entropy h_{top} as $\mathfrak{h}_{\mathfrak{cov}}$

For the contravariant functor $\mathfrak{cov} : \mathbf{CTop} \to \mathfrak{S}$ the entropy $\mathfrak{h}_{\mathfrak{cov}} : \mathbf{CTop} \to \mathbb{R}_+$ coincides with the topological entropy h_{top} defined by Adler et al.

Since the functor $\mathfrak{cov}$,

- takes factors in **CTop** to embeddings in $\mathfrak{S}$,
- takes embeddings in **CTop** to surjective morphisms in $\mathfrak{S}$, and
- takes inverse limits in **CTop** to direct limits in $\mathfrak{S}$

the topological entropy h_{top}

- is monotone w.r.t. taking factors or restrictions to invariant subspaces,
- is continuous w.r.t. inverse limits;
- satisfies the invariance under conjugation and inversions and the logarithmic laws (in particular, always $h_{top}(id_X) = 0$),
- satisfies the weak Addition Theorem: $h_{top}(\phi_1 \times \phi_2) = h_{top}(\phi_1) + h_{top}(\phi_2)$.

Obtaining the topological entropy h_{top} as $\mathfrak{h}_{\mathfrak{cov}}$

For the contravariant functor $\mathfrak{cov} : \mathbf{CTop} \to \mathfrak{S}$ the entropy $\mathfrak{h}_{\mathfrak{cov}} : \mathbf{CTop} \to \mathbb{R}_+$ coincides with the topological entropy h_{top} defined by Adler et al.

Since the functor $\mathfrak{cov}$,

- takes factors in **CTop** to embeddings in $\mathfrak{S}$,
- takes embeddings in **CTop** to surjective morphisms in $\mathfrak{S}$, and
- takes inverse limits in **CTop** to direct limits in $\mathfrak{S}$

the topological entropy h_{top}

- is monotone w.r.t. taking factors or restrictions to invariant subspaces,
- is continuous w.r.t. inverse limits;
- satisfies the invariance under conjugation and inversions and the logarithmic laws (in particular, always $h_{top}(id_X) = 0$),
- satisfies the weak Addition Theorem: $h_{top}(\phi_1 \times \phi_2) = h_{top}(\phi_1) + h_{top}(\phi_2)$.

Obtaining the topological entropy h_{top} as $\mathfrak{h}_{\mathfrak{cov}}$

For the contravariant functor $\mathfrak{cov}$: **CTop** $\to \mathfrak{S}$ the entropy $\mathfrak{h}_{\mathfrak{cov}}$: **CTop** $\to \mathbb{R}_+$ coincides with the topological entropy h_{top} defined by Adler et al.

Since the functor $\mathfrak{cov}$,

- takes factors in **CTop** to embeddings in $\mathfrak{S}$,
- takes embeddings in **CTop** to surjective morphisms in $\mathfrak{S}$, and
- takes inverse limits in **CTop** to direct limits in $\mathfrak{S}$

the topological entropy h_{top}

- is monotone w.r.t. taking factors or restrictions to invariant subspaces,
- is continuous w.r.t. inverse limits;
- satisfies the invariance under conjugation and inversions and the logarithmic laws (in particular, always $h_{top}(id_X) = 0$),
- satisfies the weak Addition Theorem: $h_{top}(\phi_1 \times \phi_2) = h_{top}(\phi_1) + h_{top}(\phi_2)$.

Obtaining the topological entropy h_{top} as $\mathfrak{h}_{\mathfrak{cov}}$

For the contravariant functor $\mathfrak{cov} : \mathbf{CTop} \to \mathfrak{S}$ the entropy $\mathfrak{h}_{\mathfrak{cov}} : \mathbf{CTop} \to \mathbb{R}_+$ coincides with the topological entropy h_{top} defined by Adler et al.

Since the functor $\mathfrak{cov}$,

- takes factors in **CTop** to embeddings in $\mathfrak{S}$,
- takes embeddings in **CTop** to surjective morphisms in $\mathfrak{S}$, and
- takes inverse limits in **CTop** to direct limits in $\mathfrak{S}$

the topological entropy h_{top}

- is monotone w.r.t. taking factors or restrictions to invariant subspaces,
- is continuous w.r.t. inverse limits;
- satisfies the invariance under conjugation and inversions and the logarithmic laws (in particular, always $h_{top}(id_X) = 0$),
- satisfies the weak Addition Theorem: $h_{top}(\phi_1 \times \phi_2) = h_{top}(\phi_1) + h_{top}(\phi_2)$.

Obtaining the topological entropy h_{top} as $\mathfrak{h}_{\mathfrak{cov}}$

For the contravariant functor $\mathfrak{cov}$: **CTop** $\to \mathfrak{S}$ the entropy $\mathfrak{h}_{\mathfrak{cov}}$: **CTop** $\to \mathbb{R}_+$ coincides with the topological entropy h_{top} defined by Adler et al.

Since the functor $\mathfrak{cov}$,

- takes factors in **CTop** to embeddings in $\mathfrak{S}$,
- takes embeddings in **CTop** to surjective morphisms in $\mathfrak{S}$, and
- takes inverse limits in **CTop** to direct limits in $\mathfrak{S}$

the topological entropy h_{top}

- is monotone w.r.t. taking factors or restrictions to invariant subspaces,
- is continuous w.r.t. inverse limits;
- satisfies the invariance under conjugation and inversions and the logarithmic laws (in particular, always $h_{top}(id_X) = 0$),
- satisfies the weak Addition Theorem: $h_{top}(\phi_1 \times \phi_2) = h_{top}(\phi_1) + h_{top}(\phi_2)$.

Obtaining the topological entropy h_{top} as $\mathfrak{h}_{\mathfrak{cov}}$

For the contravariant functor $\mathfrak{cov} : \mathbf{CTop} \to \mathfrak{S}$ the entropy $\mathfrak{h}_{\mathfrak{cov}} : \mathbf{CTop} \to \mathbb{R}_+$ coincides with the topological entropy h_{top} defined by Adler et al.

Since the functor $\mathfrak{cov}$,

- takes factors in **CTop** to embeddings in $\mathfrak{S}$,
- takes embeddings in **CTop** to surjective morphisms in $\mathfrak{S}$, and
- takes inverse limits in **CTop** to direct limits in $\mathfrak{S}$

the topological entropy h_{top}

- is monotone w.r.t. taking factors or restrictions to invariant subspaces,
- is continuous w.r.t. inverse limits;
- satisfies the invariance under conjugation and inversions and the logarithmic laws (in particular, always $h_{top}(id_X) = 0$),
- satisfies the weak Addition Theorem: $h_{top}(\phi_1 \times \phi_2) = h_{top}(\phi_1) + h_{top}(\phi_2)$.

Obtaining the topological entropy h_{top} as $\mathfrak{h}_{\mathfrak{cov}}$

For the contravariant functor $\mathfrak{cov} : \mathbf{CTop} \to \mathfrak{S}$ the entropy $\mathfrak{h}_{\mathfrak{cov}} : \mathbf{CTop} \to \mathbb{R}_+$ coincides with the topological entropy h_{top} defined by Adler et al.

Since the functor $\mathfrak{cov}$,

- takes factors in **CTop** to embeddings in $\mathfrak{S}$,
- takes embeddings in **CTop** to surjective morphisms in $\mathfrak{S}$, and
- takes inverse limits in **CTop** to direct limits in $\mathfrak{S}$

the topological entropy h_{top}

- is monotone w.r.t. taking factors or restrictions to invariant subspaces,
- is continuous w.r.t. inverse limits;
- satisfies the invariance under conjugation and inversions and the logarithmic laws (in particular, always $h_{top}(id_X) = 0$),
- satisfies the weak Addition Theorem: $h_{top}(\phi_1 \times \phi_2) = h_{top}(\phi_1) + h_{top}(\phi_2)$.

Obtaining the topological entropy h_{top} as $\mathfrak{h}_{\mathfrak{cov}}$

For the contravariant functor $\mathfrak{cov} : \mathbf{CTop} \to \mathfrak{S}$ the entropy $\mathfrak{h}_{\mathfrak{cov}} : \mathbf{CTop} \to \mathbb{R}_+$ coincides with the topological entropy h_{top} defined by Adler et al.

Since the functor $\mathfrak{cov}$,

- takes factors in **CTop** to embeddings in $\mathfrak{S}$,
- takes embeddings in **CTop** to surjective morphisms in $\mathfrak{S}$, and
- takes inverse limits in **CTop** to direct limits in $\mathfrak{S}$

the topological entropy h_{top}

- is monotone w.r.t. taking factors or restrictions to invariant subspaces,
- is continuous w.r.t. inverse limits;
- satisfies the invariance under conjugation and inversions and the logarithmic laws (in particular, always $h_{top}(id_X) = 0$),
- satisfies the weak Addition Theorem: $h_{top}(\phi_1 \times \phi_2) = h_{top}(\phi_1) + h_{top}(\phi_2)$.

Obtaining the topological entropy h_{top} as $\mathfrak{h}_{\mathfrak{cov}}$

For the contravariant functor $\mathfrak{cov} : \mathbf{CTop} \to \mathfrak{S}$ the entropy $\mathfrak{h}_{\mathfrak{cov}} : \mathbf{CTop} \to \mathbb{R}_+$ coincides with the topological entropy h_{top} defined by Adler et al.

Since the functor $\mathfrak{cov}$,

- takes factors in **CTop** to embeddings in $\mathfrak{S}$,
- takes embeddings in **CTop** to surjective morphisms in $\mathfrak{S}$, and
- takes inverse limits in **CTop** to direct limits in $\mathfrak{S}$

the topological entropy h_{top}

- is monotone w.r.t. taking factors or restrictions to invariant subspaces,
- is continuous w.r.t. inverse limits;
- satisfies the invariance under conjugation and inversions and the logarithmic laws (in particular, always $h_{top}(id_X) = 0$),
- satisfies the weak Addition Theorem: $h_{top}(\phi_1 \times \phi_2) = h_{top}(\phi_1) + h_{top}(\phi_2)$.

Obtaining the topological entropy h_{top} as $\mathfrak{h}_{\mathfrak{cov}}$

For the contravariant functor $\mathfrak{cov} : \mathbf{CTop} \to \mathfrak{S}$ the entropy $\mathfrak{h}_{\mathfrak{cov}} : \mathbf{CTop} \to \mathbb{R}_+$ coincides with the topological entropy h_{top} defined by Adler et al.

Since the functor $\mathfrak{cov}$,

- takes factors in **CTop** to embeddings in $\mathfrak{S}$,
- takes embeddings in **CTop** to surjective morphisms in $\mathfrak{S}$, and
- takes inverse limits in **CTop** to direct limits in $\mathfrak{S}$

the topological entropy h_{top}

- is monotone w.r.t. taking factors or restrictions to invariant subspaces,
- is continuous w.r.t. inverse limits;
- satisfies the invariance under conjugation and inversions and the logarithmic laws (in particular, always $h_{top}(id_X) = 0$),
- satisfies the weak Addition Theorem: $h_{top}(\phi_1 \times \phi_2) = h_{top}(\phi_1) + h_{top}(\phi_2)$.

Obtaining the topological entropy h_{top} as $\mathfrak{h}_{\mathfrak{cov}}$

For the contravariant functor $\mathfrak{cov}$: **CTop** $\to \mathfrak{S}$ the entropy $\mathfrak{h}_{\mathfrak{cov}}$: **CTop** $\to \mathbb{R}_+$ coincides with the topological entropy h_{top} defined by Adler et al.

Since the functor $\mathfrak{cov}$,

- takes factors in **CTop** to embeddings in $\mathfrak{S}$,
- takes embeddings in **CTop** to surjective morphisms in $\mathfrak{S}$, and
- takes inverse limits in **CTop** to direct limits in $\mathfrak{S}$

the topological entropy h_{top}

- is monotone w.r.t. taking factors or restrictions to invariant subspaces,
- is continuous w.r.t. inverse limits;
- satisfies the invariance under conjugation and inversions and the logarithmic laws (in particular, always $h_{top}(id_X) = 0$),
- satisfies the weak Addition Theorem: $h_{top}(\phi_1 \times \phi_2) = h_{top}(\phi_1) + h_{top}(\phi_2)$.

For a measure space $(X, \mathfrak{B}, \mu)$ let $\mathfrak{P}(X)$ be the family of all measurable partitions $\xi = \{A_1, A_2, \ldots, A_k\}$ of X. For $\xi, \eta \in \mathfrak{P}(X)$ let $\xi \vee \eta = \{U \cap V : U \in \xi, V \in \eta\}$. Then $(\mathfrak{P}(X), \vee)$ becomes a semilattice (as $\xi \vee \xi = \xi$) with zero (the cover $\xi_0 = \{X\}$). For $\xi = \{A_1, A_2, \ldots, A_k\} \in \mathfrak{P}(X)$ of X define the entropy of ξ by

$$v(\xi) = -\sum_{i=1}^{k} \mu(A_k) \log \mu(A_k) \quad \text{(Shannon entropy)}$$

This is a monotone norm making $\mathfrak{P}(X)$ a normed semilattice with 0. For a measure preserving $T : X \to Y$ and $\xi = \{A_i\}_{i=1}^k \in \mathfrak{P}(Y)$ let $T^{-1}(\xi) = \{T^{-1}(A_i)\}_{i=1}^k$. Since T is measure preserving, one has $T^{-1}(\xi) \in \mathfrak{P}(X)$ and $\mu(T^{-1}(A_i)) = \mu(A_i)$ for all i. Hence, $v(T^{-1}(\xi)) = v(\xi)$. The assignment $X \mapsto \mathfrak{P}(X)$ defines a contravariant functor

$$\mathfrak{P} : \mathbf{MesSp} \longrightarrow \mathfrak{L}.$$

For a measure space $(X, \mathfrak{B}, \mu)$ let $\mathfrak{P}(X)$ be the family of all measurable partitions $\xi = \{A_1, A_2, \ldots, A_k\}$ of X. For $\xi, \eta \in \mathfrak{P}(X)$ let $\xi \vee \eta = \{U \cap V : U \in \xi, V \in \eta\}$. Then $(\mathfrak{P}(X), \vee)$ becomes a semilattice (as $\xi \vee \xi = \xi$) with zero (the cover $\xi_0 = \{X\}$). For $\xi = \{A_1, A_2, \ldots, A_k\} \in \mathfrak{P}(X)$ of X define the entropy of ξ by

$$v(\xi) = -\sum_{i=1}^{k} \mu(A_k) \log \mu(A_k) \qquad \text{(Shannon entropy)}$$

This is a monotone norm making $\mathfrak{P}(X)$ a normed semilattice with 0. For a measure preserving $T : X \to Y$ and $\xi = \{A_i\}_{i=1}^{k} \in \mathfrak{P}(Y)$ let $T^{-1}(\xi) = \{T^{-1}(A_i)\}_{i=1}^{k}$. Since T is measure preserving, one has $T^{-1}(\xi) \in \mathfrak{P}(X)$ and $\mu(T^{-1}(A_i)) = \mu(A_i)$ for all i. Hence, $v(T^{-1}(\xi)) = v(\xi)$. The assignment $X \mapsto \mathfrak{P}(X)$ defines a contravariant functor

$$\mathfrak{P} : \mathbf{MesSp} \longrightarrow \mathfrak{L}.$$

For a measure space $(X, \mathfrak{B}, \mu)$ let $\mathfrak{P}(X)$ be the family of all measurable partitions $\xi = \{A_1, A_2, \dots, A_k\}$ of X. For $\xi, \eta \in \mathfrak{P}(X)$ let $\xi \vee \eta = \{U \cap V : U \in \xi, V \in \eta\}$. Then $(\mathfrak{P}(X), \vee)$ becomes a semilattice (as $\xi \vee \xi = \xi$) with zero (the cover $\xi_0 = \{X\}$). For $\xi = \{A_1, A_2, \dots, A_k\} \in \mathfrak{P}(X)$ of X define the entropy of ξ by

$$v(\xi) = -\sum_{i=1}^{k} \mu(A_k) \log \mu(A_k) \quad \text{(Shannon entropy)}$$

This is a monotone norm making $\mathfrak{P}(X)$ a normed semilattice with 0. For a measure preserving $T : X \to Y$ and $\xi = \{A_i\}_{i=1}^{k} \in \mathfrak{P}(Y)$ let $T^{-1}(\xi) = \{T^{-1}(A_i)\}_{i=1}^{k}$. Since T is measure preserving, one has $T^{-1}(\xi) \in \mathfrak{P}(X)$ and $\mu(T^{-1}(A_i)) = \mu(A_i)$ for all i. Hence, $v(T^{-1}(\xi)) = v(\xi)$. The assignment $X \mapsto \mathfrak{P}(X)$ defines a contravariant functor

$$\mathfrak{P} : \mathbf{MesSp} \longrightarrow \mathfrak{L}.$$

For a measure space $(X, \mathfrak{B}, \mu)$ let $\mathfrak{P}(X)$ be the family of all measurable partitions $\xi = \{A_1, A_2, \ldots, A_k\}$ of X. For $\xi, \eta \in \mathfrak{P}(X)$ let $\xi \vee \eta = \{U \cap V : U \in \xi, V \in \eta\}$. Then $(\mathfrak{P}(X), \vee)$ becomes a semilattice (as $\xi \vee \xi = \xi$) with zero (the cover $\xi_0 = \{X\}$). For $\xi = \{A_1, A_2, \ldots, A_k\} \in \mathfrak{P}(X)$ of X define the entropy of ξ by

$$v(\xi) = -\sum_{i=1}^{k} \mu(A_k) \log \mu(A_k) \quad \text{(Shannon entropy)}$$

This is a monotone norm making $\mathfrak{P}(X)$ a normed semilattice with 0. For a measure preserving $T : X \to Y$ and $\xi = \{A_i\}_{i=1}^k \in \mathfrak{P}(Y)$ let $T^{-1}(\xi) = \{T^{-1}(A_i)\}_{i=1}^k$. Since T is measure preserving, one has $T^{-1}(\xi) \in \mathfrak{P}(X)$ and $\mu(T^{-1}(A_i)) = \mu(A_i)$ for all i. Hence, $v(T^{-1}(\xi)) = v(\xi)$. The assignment $X \mapsto \mathfrak{P}(X)$ defines a contravariant functor

$$\mathfrak{P} : \mathbf{MesSp} \longrightarrow \mathfrak{L}.$$

For a measure space $(X, \mathfrak{B}, \mu)$ let $\mathfrak{P}(X)$ be the family of all measurable partitions $\xi = \{A_1, A_2, \dots, A_k\}$ of X. For $\xi, \eta \in \mathfrak{P}(X)$ let $\xi \vee \eta = \{U \cap V : U \in \xi, V \in \eta\}$. Then $(\mathfrak{P}(X), \vee)$ becomes a semilattice (as $\xi \vee \xi = \xi$) with zero (the cover $\xi_0 = \{X\}$). For $\xi = \{A_1, A_2, \dots, A_k\} \in \mathfrak{P}(X)$ of X define the entropy of ξ by

$$v(\xi) = -\sum_{i=1}^{k} \mu(A_k) \log \mu(A_k) \qquad \text{(Shannon entropy)}$$

This is a monotone norm making $\mathfrak{P}(X)$ a normed semilattice with 0. For a measure preserving $T : X \to Y$ and $\xi = \{A_i\}_{i=1}^k \in \mathfrak{P}(Y)$ let $T^{-1}(\xi) = \{T^{-1}(A_i)\}_{i=1}^k$. Since T is measure preserving, one has $T^{-1}(\xi) \in \mathfrak{P}(X)$ and $\mu(T^{-1}(A_i)) = \mu(A_i)$ for all i. Hence, $v(T^{-1}(\xi)) = v(\xi)$. The assignment $X \mapsto \mathfrak{P}(X)$ defines a contravariant functor

$$\mathfrak{P} : \mathbf{MesSp} \longrightarrow \mathfrak{L}.$$

For a measure space $(X, \mathfrak{B}, \mu)$ let $\mathfrak{P}(X)$ be the family of all measurable partitions $\xi = \{A_1, A_2, \ldots, A_k\}$ of X. For $\xi, \eta \in \mathfrak{P}(X)$ let $\xi \vee \eta = \{U \cap V : U \in \xi, V \in \eta\}$. Then $(\mathfrak{P}(X), \vee)$ becomes a semilattice (as $\xi \vee \xi = \xi$) with zero (the cover $\xi_0 = \{X\}$). For $\xi = \{A_1, A_2, \ldots, A_k\} \in \mathfrak{P}(X)$ of X define the entropy of ξ by

$$v(\xi) = -\sum_{i=1}^{k} \mu(A_k) \log \mu(A_k) \qquad \text{(Shannon entropy)}$$

This is a monotone norm making $\mathfrak{P}(X)$ a normed semilattice with 0. For a measure preserving $T : X \to Y$ and $\xi = \{A_i\}_{i=1}^k \in \mathfrak{P}(Y)$ let $T^{-1}(\xi) = \{T^{-1}(A_i)\}_{i=1}^k$. Since T is measure preserving, one has $T^{-1}(\xi) \in \mathfrak{P}(X)$ and $\mu(T^{-1}(A_i)) = \mu(A_i)$ for all i. Hence, $v(T^{-1}(\xi)) = v(\xi)$. The assignment $X \mapsto \mathfrak{P}(X)$ defines a contravariant functor

$$\mathfrak{P} : \mathbf{MesSp} \longrightarrow \mathfrak{L}.$$

For a measure space $(X, \mathfrak{B}, \mu)$ let $\mathfrak{P}(X)$ be the family of all measurable partitions $\xi = \{A_1, A_2, \dots, A_k\}$ of X. For $\xi, \eta \in \mathfrak{P}(X)$ let $\xi \vee \eta = \{U \cap V : U \in \xi, V \in \eta\}$. Then $(\mathfrak{P}(X), \vee)$ becomes a semilattice (as $\xi \vee \xi = \xi$) with zero (the cover $\xi_0 = \{X\}$). For $\xi = \{A_1, A_2, \dots, A_k\} \in \mathfrak{P}(X)$ of X define the entropy of ξ by

$$v(\xi) = -\sum_{i=1}^{k} \mu(A_k) \log \mu(A_k) \qquad \text{(Shannon entropy)}$$

This is a monotone norm making $\mathfrak{P}(X)$ a normed semilattice with 0. For a measure preserving $T : X \to Y$ and $\xi = \{A_i\}_{i=1}^k \in \mathfrak{P}(Y)$ let $T^{-1}(\xi) = \{T^{-1}(A_i)\}_{i=1}^k$. Since T is measure preserving, one has $T^{-1}(\xi) \in \mathfrak{P}(X)$ and $\mu(T^{-1}(A_i)) = \mu(A_i)$ for all i. Hence, $v(T^{-1}(\xi)) = v(\xi)$. The assignment $X \mapsto \mathfrak{P}(X)$ defines a contravariant functor

$$\mathfrak{P} : \mathbf{MesSp} \longrightarrow \mathfrak{L}.$$

For a measure space $(X, \mathfrak{B}, \mu)$ let $\mathfrak{P}(X)$ be the family of all measurable partitions $\xi = \{A_1, A_2, \dots, A_k\}$ of X. For $\xi, \eta \in \mathfrak{P}(X)$ let $\xi \vee \eta = \{U \cap V : U \in \xi, V \in \eta\}$. Then $(\mathfrak{P}(X), \vee)$ becomes a semilattice (as $\xi \vee \xi = \xi$) with zero (the cover $\xi_0 = \{X\}$). For $\xi = \{A_1, A_2, \dots, A_k\} \in \mathfrak{P}(X)$ of X define the entropy of ξ by

$$v(\xi) = -\sum_{i=1}^{k} \mu(A_k) \log \mu(A_k) \qquad \text{(Shannon entropy)}$$

This is a monotone norm making $\mathfrak{P}(X)$ a normed semilattice with 0. For a measure preserving $T : X \to Y$ and $\xi = \{A_i\}_{i=1}^k \in \mathfrak{P}(Y)$ let $T^{-1}(\xi) = \{T^{-1}(A_i)\}_{i=1}^k$. Since T is measure preserving, one has $T^{-1}(\xi) \in \mathfrak{P}(X)$ and $\mu(T^{-1}(A_i)) = \mu(A_i)$ for all i. Hence, $v(T^{-1}(\xi)) = v(\xi)$. The assignment $X \mapsto \mathfrak{P}(X)$ defines a contravariant functor

$$\mathfrak{P} : \mathbf{MesSp} \longrightarrow \mathfrak{L}.$$

For a measure space $(X, \mathfrak{B}, \mu)$ let $\mathfrak{P}(X)$ be the family of all measurable partitions $\xi = \{A_1, A_2, \dots, A_k\}$ of X. For $\xi, \eta \in \mathfrak{P}(X)$ let $\xi \vee \eta = \{U \cap V : U \in \xi, V \in \eta\}$. Then $(\mathfrak{P}(X), \vee)$ becomes a semilattice (as $\xi \vee \xi = \xi$) with zero (the cover $\xi_0 = \{X\}$). For $\xi = \{A_1, A_2, \dots, A_k\} \in \mathfrak{P}(X)$ of X define the entropy of ξ by

$$v(\xi) = -\sum_{i=1}^{k} \mu(A_k) \log \mu(A_k) \qquad (\text{Shannon entropy})$$

This is a monotone norm making $\mathfrak{P}(X)$ a normed semilattice with 0. For a measure preserving $T : X \to Y$ and $\xi = \{A_i\}_{i=1}^k \in \mathfrak{P}(Y)$ let $T^{-1}(\xi) = \{T^{-1}(A_i)\}_{i=1}^k$. Since T is measure preserving, one has $T^{-1}(\xi) \in \mathfrak{P}(X)$ and $\mu(T^{-1}(A_i)) = \mu(A_i)$ for all i. Hence, $v(T^{-1}(\xi)) = v(\xi)$. The assignment $X \mapsto \mathfrak{P}(X)$ defines a contravariant functor

$$\mathfrak{P} : \mathbf{MesSp} \longrightarrow \mathfrak{L}.$$

Obtaining the measure entropy h_{mes} as $\mathfrak{h}_{\mathfrak{P}}$

For the contravariant functor $\mathfrak{P} : \mathbf{MesSp} \to \mathfrak{L}$ the entropy $\mathfrak{h}_{\mathfrak{P}} = h_{\mathfrak{S}} \circ \mathfrak{P} : \mathbf{MesSp} \to \mathbb{R}_+$ coincides with measure-theoretic entropy h_m defined by Kolmogorov and Sinai in ergodic theory in the fifties.
This is why, similarly to h_{top}, also the measure-theoretic entropy h_{mes} is monotone w.r.t. taking quotients or restrictions to invariant subspaces, is continuous w.r.t. inverse limits, etc.

Example (measure entropy vs topological entropy)

Let X be a compact topological group, let μ be its Haar measure and let $\phi : G \to G$ be continuous endomorphism.

(a) [Halmos] ϕ is measure preserving iff ϕ is surjective.

(b) [Aoki] if ϕ is surjective, then $h_{mes}(\phi) = h_{top}(\phi)$.

(c) [variational priciple] if X is a compact space and $f : X \to X$ a continuous map, then
$h_{top}(\phi) = \sup\{h_{mes}(\phi_\mu) : \mu \text{ is an } f\text{-invariant measure on } X\}$.

Obtaining the measure entropy h_{mes} as $\mathfrak{h}_{\mathfrak{P}}$

For the contravariant functor $\mathfrak{P} : \mathbf{MesSp} \to \mathfrak{L}$ the entropy $\mathfrak{h}_{\mathfrak{P}} = h_{\mathfrak{S}} \circ \mathfrak{P} : \mathbf{MesSp} \to \mathbb{R}_+$ coincides with measure-theoretic entropy h_m defined by Kolmogorov and Sinai in ergodic theory in the fifties.

This is why, similarly to h_{top}, also the measure-theoretic entropy h_{mes} is monotone w.r.t. taking quotients or restrictions to invariant subspaces, is continuous w.r.t. inverse limits, etc.

Example (measure entropy vs topological entropy)

Let X be a compact topological group, let μ be its Haar measure and let $\phi : G \to G$ be continuous endomorphism.

(a) [Halmos] ϕ is measure preserving iff ϕ is surjective.

(b) [Aoki] if ϕ is surjective, then $h_{mes}(\phi) = h_{top}(\phi)$.

(c) [variational priciple] if X is a compact space and $f : X \to X$ a continuous map, then $h_{top}(\phi) = \sup\{h_{mes}(\phi_\mu) : \mu \text{ is an } f\text{-invariant measure on } X\}$.

Obtaining the measure entropy h_{mes} as $\mathfrak{h}_{\mathfrak{P}}$

For the contravariant functor $\mathfrak{P}$: **MesSp** $\to \mathfrak{L}$ the entropy $\mathfrak{h}_{\mathfrak{P}} = h_{\mathfrak{S}} \circ \mathfrak{P}$: **MesSp** $\to \mathbb{R}_+$ coincides with measure-theoretic entropy h_m defined by Kolmogorov and Sinai in ergodic theory in the fifties.

This is why, similarly to h_{top}, also the measure-theoretic entropy h_{mes} is monotone w.r.t. taking quotients or restrictions to invariant subspaces, is continuous w.r.t. inverse limits, etc.

Example (measure entropy vs topological entropy)

Let X be a compact topological group, let μ be its Haar measure and let $\phi : G \to G$ be continuous endomorphism.

(a) [Halmos] ϕ is measure preserving iff ϕ is surjective.

(b) [Aoki] if ϕ is surjective, then $h_{mes}(\phi) = h_{top}(\phi)$.

(c) [variational priciple] if X is a compact space and $f : X \to X$ a continuous map, then $h_{top}(\phi) = \sup\{h_{mes}(\phi_\mu) : \mu \text{ is an } f\text{-invariant measure on } X\}$.

Obtaining the measure entropy h_{mes} as $\mathfrak{h}_{\mathfrak{P}}$

For the contravariant functor $\mathfrak{P} : \mathbf{MesSp} \to \mathfrak{L}$ the entropy $\mathfrak{h}_{\mathfrak{P}} = h_{\mathfrak{S}} \circ \mathfrak{P} : \mathbf{MesSp} \to \mathbb{R}_+$ coincides with measure-theoretic entropy h_m defined by Kolmogorov and Sinai in ergodic theory in the fifties.

This is why, similarly to h_{top}, also the measure-theoretic entropy h_{mes} is monotone w.r.t. taking quotients or restrictions to invariant subspaces, is continuous w.r.t. inverse limits, etc.

Example (measure entropy vs topological entropy)

Let X be a compact topological group, let μ be its Haar measure and let $\phi : G \to G$ be continuous endomorphism.

(a) [Halmos] ϕ is measure preserving iff ϕ is surjective.

(b) [Aoki] if ϕ is surjective, then $h_{mes}(\phi) = h_{top}(\phi)$.

(c) [variational priciple] if X is a compact space and $f : X \to X$ a continuous map, then $h_{top}(\phi) = \sup\{h_{mes}(\phi_\mu) : \mu \text{ is an } f\text{-invariant measure on } X\}$.

Obtaining the measure entropy h_{mes} as $\mathfrak{h}_{\mathfrak{P}}$

For the contravariant functor $\mathfrak{P} : \mathbf{MesSp} \to \mathfrak{L}$ the entropy $\mathfrak{h}_{\mathfrak{P}} = h_{\mathfrak{S}} \circ \mathfrak{P} : \mathbf{MesSp} \to \mathbb{R}_+$ coincides with measure-theoretic entropy h_m defined by Kolmogorov and Sinai in ergodic theory in the fifties.

This is why, similarly to h_{top}, also the measure-theoretic entropy h_{mes} is monotone w.r.t. taking quotients or restrictions to invariant subspaces, is continuous w.r.t. inverse limits, etc.

Example (measure entropy vs topological entropy)

Let X be a compact topological group, let μ be its Haar measure and let $\phi : G \to G$ be continuous endomorphism.

(a) [Halmos] ϕ is measure preserving iff ϕ is surjective.

(b) [Aoki] if ϕ is surjective, then $h_{mes}(\phi) = h_{top}(\phi)$.

(c) [variational priciple] if X is a compact space and $f : X \to X$ a continuous map, then $h_{top}(\phi) = \sup\{h_{mes}(\phi_\mu) : \mu \text{ is an } f\text{-invariant measure on } X\}$.

Obtaining the measure entropy h_{mes} as $\mathfrak{h}_{\mathfrak{P}}$

For the contravariant functor $\mathfrak{P}$: **MesSp** $\to \mathfrak{L}$ the entropy $\mathfrak{h}_{\mathfrak{P}} = h_{\mathfrak{S}} \circ \mathfrak{P}$: **MesSp** $\to \mathbb{R}_+$ coincides with measure-theoretic entropy h_m defined by Kolmogorov and Sinai in ergodic theory in the fifties.

This is why, similarly to h_{top}, also the measure-theoretic entropy h_{mes} is monotone w.r.t. taking quotients or restrictions to invariant subspaces, is continuous w.r.t. inverse limits, etc.

Example (measure entropy vs topological entropy)

Let X be a compact topological group, let μ be its Haar measure and let $\phi : G \to G$ be continuous endomorphism.

(a) [Halmos] ϕ is measure preserving iff ϕ is surjective.

(b) [Aoki] if ϕ is surjective, then $h_{mes}(\phi) = h_{top}(\phi)$.

(c) [variational priciple] if X is a compact space and $f : X \to X$ a continuous map, then $h_{top}(\phi) = \sup\{h_{mes}(\phi_\mu) : \mu \text{ is an } f\text{-invariant measure on } X\}$.

Obtaining the measure entropy h_{mes} as $\mathfrak{h}_{\mathfrak{P}}$

For the contravariant functor $\mathfrak{P}$: **MesSp** $\to \mathfrak{L}$ the entropy $\mathfrak{h}_{\mathfrak{P}} = h_{\mathfrak{S}} \circ \mathfrak{P}$: **MesSp** $\to \mathbb{R}_+$ coincides with measure-theoretic entropy h_m defined by Kolmogorov and Sinai in ergodic theory in the fifties.
This is why, similarly to h_{top}, also the measure-theoretic entropy h_{mes} is monotone w.r.t. taking quotients or restrictions to invariant subspaces, is continuous w.r.t. inverse limits, etc.

Example (measure entropy vs topological entropy)

Let X be a compact topological group, let μ be its Haar measure and let $\phi : G \to G$ be continuous endomorphism.

(a) [Halmos] ϕ is measure preserving iff ϕ is surjective.

(b) [Aoki] if ϕ is surjective, then $h_{mes}(\phi) = h_{top}(\phi)$.

(c) [variational priciple] if X is a compact space and $f : X \to X$ a continuous map, then $h_{top}(\phi) = \sup\{h_{mes}(\phi_\mu) : \mu \text{ is an } f\text{-invariant measure on } X\}$.

Obtaining the measure entropy h_{mes} as $\mathfrak{h}_{\mathfrak{P}}$

For the contravariant functor $\mathfrak{P} : \mathbf{MesSp} \to \mathfrak{L}$ the entropy $\mathfrak{h}_{\mathfrak{P}} = h_{\mathfrak{S}} \circ \mathfrak{P} : \mathbf{MesSp} \to \mathbb{R}_+$ coincides with measure-theoretic entropy h_m defined by Kolmogorov and Sinai in ergodic theory in the fifties.

This is why, similarly to h_{top}, also the measure-theoretic entropy h_{mes} is monotone w.r.t. taking quotients or restrictions to invariant subspaces, is continuous w.r.t. inverse limits, etc.

Example (measure entropy vs topological entropy)

Let X be a compact topological group, let μ be its Haar measure and let $\phi : G \to G$ be continuous endomorphism.

(a) [Halmos] ϕ is measure preserving iff ϕ is surjective.

(b) [Aoki] if ϕ is surjective, then $h_{mes}(\phi) = h_{top}(\phi)$.

(c) [variational priciple] if X is a compact space and $f : X \to X$ a continuous map, then
$h_{top}(\phi) = \sup\{h_{mes}(\phi_\mu) : \mu \text{ is an } f\text{-invariant measure on } X\}$.

Obtaining the measure entropy h_{mes} as $\mathfrak{h}_{\mathfrak{P}}$

For the contravariant functor $\mathfrak{P} : \mathbf{MesSp} \to \mathfrak{L}$ the entropy $\mathfrak{h}_{\mathfrak{P}} = h_{\mathfrak{S}} \circ \mathfrak{P} : \mathbf{MesSp} \to \mathbb{R}_+$ coincides with measure-theoretic entropy h_m defined by Kolmogorov and Sinai in ergodic theory in the fifties.

This is why, similarly to h_{top}, also the measure-theoretic entropy h_{mes} is monotone w.r.t. taking quotients or restrictions to invariant subspaces, is continuous w.r.t. inverse limits, etc.

Example (measure entropy vs topological entropy)

Let X be a compact topological group, let μ be its Haar measure and let $\phi : G \to G$ be continuous endomorphism.

(a) [Halmos] ϕ is measure preserving iff ϕ is surjective.

(b) [Aoki] if ϕ is surjective, then $h_{mes}(\phi) = h_{top}(\phi)$.

(c) [variational priciple] if X is a compact space and $f : X \to X$ a continuous map, then
$h_{top}(\phi) = \sup\{h_{mes}(\phi_\mu) : \mu \text{ is an } f\text{-invariant measure on } X\}$.

Example (Adler, Konrad and McAndrew's algebraic entropy ent)

Let G be an Abelian group and let $(\mathcal{F}(G), +)$ be the semilattice of all finite subgroups of G. Letting $v(F) = \log|F|$ for $F \in \mathcal{F}(G)$, makes $\mathcal{F}(G)$ a normed semilattice with a monotone norm.
For every homomorphism $\phi : G \to H$ of Abelian groups the map $\mathcal{F}(\phi) : \mathcal{F}(G) \to \mathcal{F}(H)$ defined by $\mathcal{F}(\phi)(F) = \phi(F)$ for every $F \in \mathcal{F}(G)$ is a morphism in $\mathfrak{S}$. The assignments $G \mapsto \mathcal{F}(G)$, $\phi \mapsto \mathcal{F}(\phi)$ define a covariant functor

$$\mathcal{F} : \mathbf{AbGrp} \longrightarrow \mathfrak{S}.$$

The entropy $h_{\mathcal{F}} = h_{\mathfrak{S}} \circ \mathcal{F}$ coincides with the algebraic entropy ent defined by Adler, Konrad and McAndrew. So ent satisfies the invariance under conjugation and inversions as well as the logartmic law. Since $\mathcal{F}$ sends monomorphisms to embeddings, ent is also monotone w.r.t. taking invariant subgroups (not w.r.t. taking factors).

Example (Adler, Konrad and McAndrew's algebraic entropy ent)

Let G be an Abelian group and let $(\mathcal{F}(G), +)$ be the semilattice of all finite subgroups of G. Letting $v(F) = \log |F|$ for $F \in \mathcal{F}(G)$, makes $\mathcal{F}(G)$ a normed semilattice with a monotone norm.

For every homomorphism $\phi : G \to H$ of Abelian groups the map $\mathcal{F}(\phi) : \mathcal{F}(G) \to \mathcal{F}(H)$ defined by $\mathcal{F}(\phi)(F) = \phi(F)$ for every $F \in \mathcal{F}(G)$ is a morphism in $\mathfrak{S}$. The assignments $G \mapsto \mathcal{F}(G)$, $\phi \mapsto \mathcal{F}(\phi)$ define a covariant functor

$$\mathcal{F} : \mathbf{AbGrp} \longrightarrow \mathfrak{S}.$$

The entropy $h_{\mathcal{F}} = h_{\mathfrak{S}} \circ \mathcal{F}$ coincides with the algebraic entropy ent defined by Adler, Konrad and McAndrew. So ent satisfies the invariance under conjugation and inversions as well as the logartmic law. Since $\mathcal{F}$ sends monomorphisms to embeddings, ent is also monotone w.r.t. taking invariant subgroups (not w.r.t. taking factors).

Example (Adler, Konrad and McAndrew's algebraic entropy ent)

Let G be an Abelian group and let $(\mathcal{F}(G), +)$ be the semilattice of all finite subgroups of G. Letting $v(F) = \log|F|$ for $F \in \mathcal{F}(G)$, makes $\mathcal{F}(G)$ a normed semilattice with a monotone norm.
For every homomorphism $\phi : G \to H$ of Abelian groups the map $\mathcal{F}(\phi) : \mathcal{F}(G) \to \mathcal{F}(H)$ defined by $\mathcal{F}(\phi)(F) = \phi(F)$ for every $F \in \mathcal{F}(G)$ is a morphism in $\mathfrak{S}$. The assignments $G \mapsto \mathcal{F}(G)$, $\phi \mapsto \mathcal{F}(\phi)$ define a covariant functor

$$\mathcal{F} : \textbf{AbGrp} \longrightarrow \mathfrak{S}.$$

The entropy $h_{\mathcal{F}} = h_{\mathfrak{S}} \circ \mathcal{F}$ coincides with the algebraic entropy ent defined by Adler, Konrad and McAndrew. So ent satisfies the invariance under conjugation and inversions as well as the logartmic law. Since $\mathcal{F}$ sends monomorphisms to embeddings, ent is also monotone w.r.t. taking invariant subgroups (not w.r.t. taking factors).

Example (Adler, Konrad and McAndrew's algebraic entropy ent)

Let G be an Abelian group and let $(\mathcal{F}(G), +)$ be the semilattice of all finite subgroups of G. Letting $v(F) = \log|F|$ for $F \in \mathcal{F}(G)$, makes $\mathcal{F}(G)$ a normed semilattice with a monotone norm.
For every homomorphism $\phi : G \to H$ of Abelian groups the map $\mathcal{F}(\phi) : \mathcal{F}(G) \to \mathcal{F}(H)$ defined by $\mathcal{F}(\phi)(F) = \phi(F)$ for every $F \in \mathcal{F}(G)$ is a morphism in $\mathfrak{S}$. The assignments $G \mapsto \mathcal{F}(G)$, $\phi \mapsto \mathcal{F}(\phi)$ define a covariant functor

$$\mathcal{F} : \mathbf{AbGrp} \longrightarrow \mathfrak{S}.$$

The entropy $h_{\mathcal{F}} = h_{\mathfrak{S}} \circ \mathcal{F}$ coincides with the algebraic entropy ent defined by Adler, Konrad and McAndrew. So ent satisfies the invariance under conjugation and inversions as well as the logartmic law. Since $\mathcal{F}$ sends monomorphisms to embeddings, ent is also monotone w.r.t. taking invariant subgroups (not w.r.t. taking factors).

Example (The algebraic entropy h_{alg})

For $G \in$ **AbGrp** let $\mathcal{H}(G)$ be the family of all finite non-empty *subsets* of G. Then $(\mathcal{H}(G), +, \{0\})$ is a monoid. For every homomorphism $\phi : G \to H$ of Abelian groups, the map $\mathcal{H}(\phi) : \mathcal{H}(G) \to \mathcal{H}(H)$, defined by $\mathcal{H}(\phi)(F) = \phi(F)$ for every $F \in \mathcal{H}(G)$, is a semigroup morphism.
Letting $v(F) = \log |F|$ for $F \in \mathcal{H}(G)$ makes $\mathcal{H}(G)$ a normed semigroup. The assignements $G \mapsto (\mathcal{H}(G), v)$ and $\phi \mapsto \mathcal{H}(\phi)$ give a covariant functor

$$\mathcal{H} : \mathbf{AbGrp} \longrightarrow \mathfrak{S}.$$

Moreover, $(\mathcal{H}(G), \subseteq)$ is an ordered semigroup and the norm v is s-monotone. The entropy $h_{\mathcal{H}} = h_{\mathfrak{S}} \circ \mathcal{H}$ coincides with the algebraic entropy h_{alg}. So h_{alg} is invariant under conjugation and inversions, monotone w.r.t. taking invariant subgroups and satisfies the logartmic law.

Example (The algebraic entropy h_{alg})

For $G \in$ **AbGrp** let $\mathcal{H}(G)$ be the family of all finite non-empty *subsets* of G. Then $(\mathcal{H}(G), +, \{0\})$ is a monoid. For every homomorphism $\phi : G \to H$ of Abelian groups, the map $\mathcal{H}(\phi) : \mathcal{H}(G) \to \mathcal{H}(H)$, defined by $\mathcal{H}(\phi)(F) = \phi(F)$ for every $F \in \mathcal{H}(G)$, is a semigroup morphism.
Letting $v(F) = \log |F|$ for $F \in \mathcal{H}(G)$ makes $\mathcal{H}(G)$ a normed semigroup. The assignements $G \mapsto (\mathcal{H}(G), v)$ and $\phi \mapsto \mathcal{H}(\phi)$ give a covariant functor

$$\mathcal{H} : \mathbf{AbGrp} \longrightarrow \mathfrak{S}.$$

Moreover, $(\mathcal{H}(G), \subseteq)$ is an ordered semigroup and the norm v is s-monotone. The entropy $h_{\mathcal{H}} = h_{\mathfrak{S}} \circ \mathcal{H}$ coincides with the algebraic entropy h_{alg}. So h_{alg} is invariant under conjugation and inversions, monotone w.r.t. taking invariant subgroups and satisfies the logartmic law.

Example (The algebraic entropy h_{alg})

For $G \in$ **AbGrp** let $\mathcal{H}(G)$ be the family of all finite non-empty *subsets* of G. Then $(\mathcal{H}(G), +, \{0\})$ is a monoid. For every homomorphism $\phi : G \to H$ of Abelian groups, the map $\mathcal{H}(\phi) : \mathcal{H}(G) \to \mathcal{H}(H)$, defined by $\mathcal{H}(\phi)(F) = \phi(F)$ for every $F \in \mathcal{H}(G)$, is a semigroup morphism.
Letting $v(F) = \log |F|$ for $F \in \mathcal{H}(G)$ makes $\mathcal{H}(G)$ a normed semigroup. The assignements $G \mapsto (\mathcal{H}(G), v)$ and $\phi \mapsto \mathcal{H}(\phi)$ give a covariant functor

$$\mathcal{H} : \mathbf{AbGrp} \longrightarrow \mathfrak{S}.$$

Moreover, $(\mathcal{H}(G), \subseteq)$ is an ordered semigroup and the norm v is s-monotone. The entropy $h_{\mathcal{H}} = h_{\mathfrak{S}} \circ \mathcal{H}$ coincides with the algebraic entropy h_{alg}. So h_{alg} is invariant under conjugation and inversions, monotone w.r.t. taking invariant subgroups and satisfies the logartmic law.

Example (The algebraic entropy h_{alg})

For $G \in$ **AbGrp** let $\mathcal{H}(G)$ be the family of all finite non-empty *subsets* of G. Then $(\mathcal{H}(G), +, \{0\})$ is a monoid. For every homomorphism $\phi : G \to H$ of Abelian groups, the map $\mathcal{H}(\phi) : \mathcal{H}(G) \to \mathcal{H}(H)$, defined by $\mathcal{H}(\phi)(F) = \phi(F)$ for every $F \in \mathcal{H}(G)$, is a semigroup morphism.
Letting $v(F) = \log |F|$ for $F \in \mathcal{H}(G)$ makes $\mathcal{H}(G)$ a normed semigroup. The assignements $G \mapsto (\mathcal{H}(G), v)$ and $\phi \mapsto \mathcal{H}(\phi)$ give a covariant functor

$$\mathcal{H} : \mathbf{AbGrp} \longrightarrow \mathfrak{S}.$$

Moreover, $(\mathcal{H}(G), \subseteq)$ is an ordered semigroup and the norm v is s-monotone. The entropy $h_{\mathcal{H}} = h_{\mathfrak{S}} \circ \mathcal{H}$ coincides with the algebraic entropy h_{alg}. So h_{alg} is invariant under conjugation and inversions, monotone w.r.t. taking invariant subgroups and satisfies the logartmic law.

Example (The algebraic entropy h_{alg})

For $G \in$ **AbGrp** let $\mathcal{H}(G)$ be the family of all finite non-empty *subsets* of G. Then $(\mathcal{H}(G), +, \{0\})$ is a monoid. For every homomorphism $\phi : G \to H$ of Abelian groups, the map $\mathcal{H}(\phi) : \mathcal{H}(G) \to \mathcal{H}(H)$, defined by $\mathcal{H}(\phi)(F) = \phi(F)$ for every $F \in \mathcal{H}(G)$, is a semigroup morphism.
Letting $v(F) = \log |F|$ for $F \in \mathcal{H}(G)$ makes $\mathcal{H}(G)$ a normed semigroup. The assignements $G \mapsto (\mathcal{H}(G), v)$ and $\phi \mapsto \mathcal{H}(\phi)$ give a covariant functor

$$\mathcal{H} : \textbf{AbGrp} \longrightarrow \mathfrak{S}.$$

Moreover, $(\mathcal{H}(G), \subseteq)$ is an ordered semigroup and the norm v is s-monotone. The entropy $h_{\mathcal{H}} = h_{\mathfrak{S}} \circ \mathcal{H}$ coincides with the algebraic entropy h_{alg}. So h_{alg} is invariant under conjugation and inversions, monotone w.r.t. taking invariant subgroups and satisfies the logartmic law.

A discrete dynamical systems, namely a flow $T : X \to X$, can be considered also as an action $\mathbb{N} \overset{\alpha}{\curvearrowright} X$ of the semigroup $\mathbb{N}$ on X such that $\alpha(n)(x) = T^n(x)$ for $x \in X$ and $n \in \mathbb{N}$. This makes it natural to define entropy of d pairwise commuting endomorphisms of X, i.e., actions of $\mathbb{N}^d$. More generally, one may try to define entropy of arbitrary semigroup actions $S \overset{\alpha}{\curvearrowright} X$.

In this direction, the notion of entropy of actions of amenable groups on compact metrizable spaces ot measure spaces was introduced by Ornstein and Weiss [1987].

Hofmann and Stoyanov [1995] defined and studied topological entropy $h_\alpha(\gamma)$ of actions $S \overset{\gamma}{\curvearrowright} X$ of a locally compact semigroup S on a metric space X, depending on a countable system α of compact subsets $\alpha = (N_1, N_2, \ldots, N_n, \ldots)$ of S satisfying $N_i N_j \subseteq N_{i+j}$. If $S = \mathbb{N}$ is generated by a single map $f : X \to X$ and $N_n = [0, n-1]$, the entropy $h_\alpha(\gamma)$ coincides with Bowen's topological entropy $h_U(f)$.

A discrete dynamical systems, namely a flow $T : X \to X$, can be considered also as an action $\mathbb{N} \overset{\alpha}{\curvearrowright} X$ of the semigroup $\mathbb{N}$ on X such that $\alpha(n)(x) = T^n(x)$ for $x \in X$ and $n \in \mathbb{N}$. This makes it natural to define entropy of d pairwise commuting endomorphisms of X, i.e., actions of $\mathbb{N}^d$. More generally, one may try to define entropy of arbitrary semigroup actions $S \overset{\alpha}{\curvearrowright} X$.

In this direction, the notion of entropy of actions of amenable groups on compact metrizable spaces ot measure spaces was introduced by Ornstein and Weiss [1987].

Hofmann and Stoyanov [1995] defined and studied topological entropy $h_\alpha(\gamma)$ of actions $S \overset{\gamma}{\curvearrowright} X$ of a locally compact semigroup S on a metric space X, depending on a countable system α of compact subsets $\alpha = (N_1, N_2, \ldots, N_n, \ldots)$ of S satisfying $N_i N_j \subseteq N_{i+j}$. If $S = \mathbb{N}$ is generated by a single map $f : X \to X$ and $N_n = [0, n-1]$, the entropy $h_\alpha(\gamma)$ coincides with Bowen's topological entropy $h_U(f)$.

A discrete dynamical systems, namely a flow $T : X \to X$, can be considered also as an action $\mathbb{N} \overset{\alpha}{\curvearrowright} X$ of the semigroup $\mathbb{N}$ on X such that $\alpha(n)(x) = T^n(x)$ for $x \in X$ and $n \in \mathbb{N}$. This makes it natural to define entropy of d pairwise commuting endomorphisms of X, i.e., actions of $\mathbb{N}^d$. More generally, one may try to define entropy of arbitrary semigroup actions $S \overset{\alpha}{\curvearrowright} X$.

In this direction, the notion of entropy of actions of amenable groups on compact metrizable spaces ot measure spaces was introduced by Ornstein and Weiss [1987].

Hofmann and Stoyanov [1995] defined and studied topological entropy $h_\alpha(\gamma)$ of actions $S \overset{\gamma}{\curvearrowright} X$ of a locally compact semigroup S on a metric space X, depending on a countable system α of compact subsets $\alpha = (N_1, N_2, \ldots, N_n, \ldots)$ of S satisfying $N_i N_j \subseteq N_{i+j}$. If $S = \mathbb{N}$ is generated by a single map $f : X \to X$ and $N_n = [0, n-1]$, the entropy $h_\alpha(\gamma)$ coincides with Bowen's topological entropy $h_U(f)$.

A discrete dynamical systems, namely a flow $T : X \to X$, can be considered also as an action $\mathbb{N} \overset{\alpha}{\curvearrowright} X$ of the semigroup $\mathbb{N}$ on X such that $\alpha(n)(x) = T^n(x)$ for $x \in X$ and $n \in \mathbb{N}$. This makes it natural to define entropy of d pairwise commuting endomorphisms of X, i.e., actions of $\mathbb{N}^d$. More generally, one may try to define entropy of arbitrary semigroup actions $S \overset{\alpha}{\curvearrowright} X$.

In this direction, the notion of entropy of actions of amenable groups on compact metrizable spaces ot measure spaces was introduced by Ornstein and Weiss [1987].

Hofmann and Stoyanov [1995] defined and studied topological entropy $h_\alpha(\gamma)$ of actions $S \overset{\gamma}{\curvearrowright} X$ of a locally compact semigroup S on a metric space X, depending on a countable system α of compact subsets $\alpha = (N_1, N_2, \ldots, N_n, \ldots)$ of S satisfying $N_i N_j \subseteq N_{i+j}$. If $S = \mathbb{N}$ is generated by a single map $f : X \to X$ and $N_n = [0, n-1]$, the entropy $h_\alpha(\gamma)$ coincides with Bowen's topological entropy $h_U(f)$.

A discrete dynamical systems, namely a flow $T : X \to X$, can be considered also as an action $\mathbb{N} \overset{\alpha}{\curvearrowright} X$ of the semigroup $\mathbb{N}$ on X such that $\alpha(n)(x) = T^n(x)$ for $x \in X$ and $n \in \mathbb{N}$. This makes it natural to define entropy of d pairwise commuting endomorphisms of X, i.e., actions of $\mathbb{N}^d$. More generally, one may try to define entropy of arbitrary semigroup actions $S \overset{\alpha}{\curvearrowright} X$.

In this direction, the notion of entropy of actions of amenable groups on compact metrizable spaces ot measure spaces was introduced by Ornstein and Weiss [1987].

Hofmann and Stoyanov [1995] defined and studied topological entropy $h_\alpha(\gamma)$ of actions $S \overset{\gamma}{\curvearrowright} X$ of a locally compact semigroup S on a metric space X, depending on a countable system α of compact subsets $\alpha = (N_1, N_2, \ldots, N_n, \ldots)$ of S satisfying $N_i N_j \subseteq N_{i+j}$. If $S = \mathbb{N}$ is generated by a single map $f : X \to X$ and $N_n = [0, n-1]$, the entropy $h_\alpha(\gamma)$ coincides with Bowen's topological entropy $h_U(f)$.

Recently Ceccherini-Silberstein, Coornaert and Krieger extended the notions of entropy to actions of amenable semigroups, following Ornstein and Weiss's approach.
Let S be a semigroup and let $\mathcal{P}_{fin}(S)$ be the family of its non-empty subsets; S is right amenable, if for every $K \in \mathcal{P}_{fin}(S)$ and every $\varepsilon > 0$ there exists an $F \in \mathcal{P}_{fin}(S)$, such that $|Fx \setminus F| \leq \varepsilon |F|$ for every $x \in K$.
It is easy to see that a cancellative semigroup S is right-amenable iff S admits a right-Følner net, i.e., a net $(F_i)_{i\in I}$ in $\mathcal{P}_{fin}(S)$ such that for every $s \in S$

$$\lim_{i\in I} \frac{|F_i s \setminus F_i|}{|F_i|} = 0.$$

A map $f : \mathcal{P}_{fin}(S) \to \mathbb{R}$ is said to be:

1. subadditive if $f(F_1 \cup F_2) \leq f(F_1) + f(F_2)$ for every $F_1, F_2 \in \mathcal{P}_{fin}(S)$;
2. right-subinvariant if $f(Fs) \leq f(F)$ for every $s \in S$ and every $F \in \mathcal{P}_{fin}(S)$;

Recently Ceccherini-Silberstein, Coornaert and Krieger extended the notions of entropy to actions of amenable semigroups, following Ornstein and Weiss's approach.

Let S be a semigroup and let $\mathcal{P}_{fin}(S)$ be the family of its non-empty subsets; S is right amenable, if for every $K \in \mathcal{P}_{fin}(S)$ and every $\varepsilon > 0$ there exists an $F \in \mathcal{P}_{fin}(S)$, such that $|Fx \setminus F| \leq \varepsilon |F|$ for every $x \in K$.

It is easy to see that a cancellative semigroup S is right-amenable iff S admits a right-Følner net, i.e., a net $(F_i)_{i\in I}$ in $\mathcal{P}_{fin}(S)$ such that for every $s \in S$

$$\lim_{i \in I} \frac{|F_i s \setminus F_i|}{|F_i|} = 0.$$

A map $f : \mathcal{P}_{fin}(S) \to \mathbb{R}$ is said to be:

1. subadditive if $f(F_1 \cup F_2) \leq f(F_1) + f(F_2)$ for every $F_1, F_2 \in \mathcal{P}_{fin}(S)$;
2. right-subinvariant if $f(Fs) \leq f(F)$ for every $s \in S$ and every $F \in \mathcal{P}_{fin}(S)$;

Recently Ceccherini-Silberstein, Coornaert and Krieger extended the notions of entropy to actions of amenable semigroups, following Ornstein and Weiss's approach.
Let S be a semigroup and let $\mathcal{P}_{fin}(S)$ be the family of its non-empty subsets; S is right amenable, if for every $K \in \mathcal{P}_{fin}(S)$ and every $\varepsilon > 0$ there exists an $F \in \mathcal{P}_{fin}(S)$, such that $|Fx \setminus F| \leq \varepsilon |F|$ for every $x \in K$.
It is easy to see that a cancellative semigroup S is right-amenable iff S admits a right-Følner net, i.e., a net $(F_i)_{i\in I}$ in $\mathcal{P}_{fin}(S)$ such that for every $s \in S$

$$\lim_{i\in I} \frac{|F_i s \setminus F_i|}{|F_i|} = 0.$$

A map $f : \mathcal{P}_{fin}(S) \to \mathbb{R}$ is said to be:

1. subadditive if $f(F_1 \cup F_2) \leq f(F_1) + f(F_2)$ for every $F_1, F_2 \in \mathcal{P}_{fin}(S)$;
2. right-subinvariant if $f(Fs) \leq f(F)$ for every $s \in S$ and every $F \in \mathcal{P}_{fin}(S)$;

Recently Ceccherini-Silberstein, Coornaert and Krieger extended the notions of entropy to actions of amenable semigroups, following Ornstein and Weiss's approach.

Let S be a semigroup and let $\mathcal{P}_{fin}(S)$ be the family of its non-empty subsets; S is right amenable, if for every $K \in \mathcal{P}_{fin}(S)$ and every $\varepsilon > 0$ there exists an $F \in \mathcal{P}_{fin}(S)$, such that $|Fx \setminus F| \leq \varepsilon |F|$ for every $x \in K$.

It is easy to see that a cancellative semigroup S is right-amenable iff S admits a right-Følner net, i.e., a net $(F_i)_{i \in I}$ in $\mathcal{P}_{fin}(S)$ such that for every $s \in S$

$$\lim_{i \in I} \frac{|F_i s \setminus F_i|}{|F_i|} = 0.$$

A map $f : \mathcal{P}_{fin}(S) \to \mathbb{R}$ is said to be:

1. subadditive if $f(F_1 \cup F_2) \leq f(F_1) + f(F_2)$ for every $F_1, F_2 \in \mathcal{P}_{fin}(S)$;
2. right-subinvariant if $f(Fs) \leq f(F)$ for every $s \in S$ and every $F \in \mathcal{P}_{fin}(S)$;

Recently Ceccherini-Silberstein, Coornaert and Krieger extended the notions of entropy to actions of amenable semigroups, following Ornstein and Weiss's approach.

Let S be a semigroup and let $\mathcal{P}_{fin}(S)$ be the family of its non-empty subsets; S is right amenable, if for every $K \in \mathcal{P}_{fin}(S)$ and every $\varepsilon > 0$ there exists an $F \in \mathcal{P}_{fin}(S)$, such that $|Fx \setminus F| \leq \varepsilon |F|$ for every $x \in K$.

It is easy to see that a cancellative semigroup S is right-amenable iff S admits a right-Følner net, i.e., a net $(F_i)_{i \in I}$ in $\mathcal{P}_{fin}(S)$ such that for every $s \in S$

$$\lim_{i \in I} \frac{|F_i s \setminus F_i|}{|F_i|} = 0.$$

A map $f : \mathcal{P}_{fin}(S) \to \mathbb{R}$ is said to be:

1. subadditive if $f(F_1 \cup F_2) \leq f(F_1) + f(F_2)$ for every $F_1, F_2 \in \mathcal{P}_{fin}(S)$;
2. right-subinvariant if $f(Fs) \leq f(F)$ for every $s \in S$ and every $F \in \mathcal{P}_{fin}(S)$;

Recently Ceccherini-Silberstein, Coornaert and Krieger extended the notions of entropy to actions of amenable semigroups, following Ornstein and Weiss's approach.

Let S be a semigroup and let $\mathcal{P}_{fin}(S)$ be the family of its non-empty subsets; S is right amenable, if for every $K \in \mathcal{P}_{fin}(S)$ and every $\varepsilon > 0$ there exists an $F \in \mathcal{P}_{fin}(S)$, such that $|Fx \setminus F| \leq \varepsilon|F|$ for every $x \in K$.

It is easy to see that a cancellative semigroup S is right-amenable iff S admits a right-Følner net, i.e., a net $(F_i)_{i\in I}$ in $\mathcal{P}_{fin}(S)$ such that for every $s \in S$

$$\lim_{i\in I} \frac{|F_i s \setminus F_i|}{|F_i|} = 0.$$

A map $f : \mathcal{P}_{fin}(S) \to \mathbb{R}$ is said to be:

1. subadditive if $f(F_1 \cup F_2) \leq f(F_1) + f(F_2)$ for every $F_1, F_2 \in \mathcal{P}_{fin}(S)$;
2. right-subinvariant if $f(Fs) \leq f(F)$ for every $s \in S$ and every $F \in \mathcal{P}_{fin}(S)$;

The following theorem and definition, due to Ceccherini-Silberstein, Coornaert and Krieger, are inspired by Ornstein and Weiss' approach.

Theorem (Ceccherini-Silberstein, Coornaert and Krieger 2014)

Let S be a cancellative left amenable monoid and let $f : \mathcal{P}(S) \to \mathbb{R}$ be a subadditive, right-subinvariant map. Then there exists $\lambda \in \mathbb{R}_{\geq 0}$ such that, for every left-Følner net $(F_i)_{i \in I}$ of S,

$$\lim_{i \in I} \frac{f(F_i)}{|F_i|} = \lambda.$$

Let X be a compact topological space, let S be a cancellative left-amenable monoid and consider the left action $S \overset{\gamma}{\curvearrowright} X$ by continuous maps. For $\mathcal{U} \in \mathrm{cov}(X)$ and for every $F \in \mathcal{P}_{fin}(S)$, let

$$\mathcal{U}_{\gamma,F} = \bigvee_{s \in F} \gamma(s)^{-1}(\mathcal{U}).$$

The following theorem and definition, due to Ceccherini-Silberstein, Coornaert and Krieger, are inspired by Ornstein and Weiss' approach.

Theorem (Ceccherini-Silberstein, Coornaert and Krieger 2014)

Let S be a cancellative left amenable monoid and let $f : \mathcal{P}(S) \to \mathbb{R}$ be a subadditive, right-subinvariant map. Then there exists $\lambda \in \mathbb{R}_{\geq 0}$ such that, for every left-Følner net $(F_i)_{i\in I}$ of S,

$$\lim_{i\in I} \frac{f(F_i)}{|F_i|} = \lambda.$$

Let X be a compact topological space, let S be a cancellative left-amenable monoid and consider the left action $S \overset{\gamma}{\curvearrowright} X$ by continuous maps. For $\mathcal{U} \in \mathrm{cov}(X)$ and for every $F \in \mathcal{P}_{fin}(S)$, let

$$\mathcal{U}_{\gamma,F} = \bigvee_{s\in F} \gamma(s)^{-1}(\mathcal{U}).$$

The following theorem and definition, due to Ceccherini-Silberstein, Coornaert and Krieger, are inspired by Ornstein and Weiss' approach.

Theorem (Ceccherini-Silberstein, Coornaert and Krieger 2014)

Let S be a cancellative left amenable monoid and let $f : \mathcal{P}(S) \to \mathbb{R}$ be a subadditive, right-subinvariant map. Then there exists $\lambda \in \mathbb{R}_{\geq 0}$ such that, for every left-Følner net $(F_i)_{i\in I}$ of S,

$$\lim_{i\in I} \frac{f(F_i)}{|F_i|} = \lambda.$$

Let X be a compact topological space, let S be a cancellative left-amenable monoid and consider the left action $S \overset{\gamma}{\curvearrowright} X$ by continuous maps. For $\mathcal{U} \in \operatorname{cov}(X)$ and for every $F \in \mathcal{P}_{fin}(S)$, let

$$\mathcal{U}_{\gamma,F} = \bigvee_{s\in F} \gamma(s)^{-1}(\mathcal{U}).$$

The following theorem and definition, due to Ceccherini-Silberstein, Coornaert and Krieger, are inspired by Ornstein and Weiss' approach.

Theorem (Ceccherini-Silberstein, Coornaert and Krieger 2014)

Let S be a cancellative left amenable monoid and let $f : \mathcal{P}(S) \to \mathbb{R}$ be a subadditive, right-subinvariant map. Then there exists $\lambda \in \mathbb{R}_{\geq 0}$ such that, for every left-Følner net $(F_i)_{i\in I}$ of S,

$$\lim_{i\in I} \frac{f(F_i)}{|F_i|} = \lambda.$$

Let X be a compact topological space, let S be a cancellative left-amenable monoid and consider the left action $S \overset{\gamma}{\curvearrowright} X$ by continuous maps. For $\mathcal{U} \in \mathrm{cov}(X)$ and for every $F \in \mathcal{P}_{fin}(S)$, let

$$\mathcal{U}_{\gamma,F} = \bigvee_{s\in F} \gamma(s)^{-1}(\mathcal{U}).$$

For every $\mathcal{U} \in \mathrm{cov}(X)$ the function

$$f_{\mathcal{U}} : \mathcal{P}_{fin}(S) \to \mathbb{R}, \quad F \mapsto \log M(\mathcal{U}_{\gamma,F}).$$

is non-decreasing, subadditive and right-subinvariant. The above theorem gives the following:

Definition (Ceccherini-Silberstein, Coornaert and Krieger 2014)

Let S be a cancellative left-amenable semigroup acting $S \overset{\gamma}{\curvearrowright} X$ on a compact space X. For $\mathcal{U} \in \mathrm{cov}(X)$, the topological entropy of γ with respect to $\mathcal{U}$ is

$$H_{top}(\gamma, \mathcal{U}) = \lim_{i \in I} \frac{f_{\mathcal{U}}(F_i)}{|F_i|},$$

where $(F_i)_{i \in I}$ is a left-Følner net of S. The topological entropy of γ is

$$h_{top}(\gamma) = \sup\{H_{top}(\gamma, \mathcal{U}) \mid \mathcal{U} \in \mathrm{cov}(X)\}.$$

For every $\mathcal{U} \in \mathrm{cov}(X)$ the function

$$f_{\mathcal{U}} : \mathcal{P}_{fin}(S) \to \mathbb{R}, \quad F \mapsto \log M(\mathcal{U}_{\gamma,F}).$$

is non-decreasing, subadditive and right-subinvariant. The above theorem gives the following:

Definition (Ceccherini-Silberstein, Coornaert and Krieger 2014)

Let S be a cancellative left-amenable semigroup acting $S \overset{\gamma}{\curvearrowright} X$ on a compact space X. For $\mathcal{U} \in \mathrm{cov}(X)$, the topological entropy of γ with respect to $\mathcal{U}$ is

$$H_{top}(\gamma, \mathcal{U}) = \lim_{i \in I} \frac{f_{\mathcal{U}}(F_i)}{|F_i|},$$

where $(F_i)_{i \in I}$ is a left-Følner net of S. The topological entropy of γ is

$$h_{top}(\gamma) = \sup\{H_{top}(\gamma, \mathcal{U}) \mid \mathcal{U} \in \mathrm{cov}(X)\}.$$

For every $\mathcal{U} \in \mathrm{cov}(X)$ the function

$$f_{\mathcal{U}} : \mathcal{P}_{fin}(S) \to \mathbb{R}, \quad F \mapsto \log M(\mathcal{U}_{\gamma,F}).$$

is non-decreasing, subadditive and right-subinvariant. The above theorem gives the following:

Definition (Ceccherini-Silberstein, Coornaert and Krieger 2014)

Let S be a cancellative left-amenable semigroup acting $S \overset{\gamma}{\curvearrowright} X$ on a compact space X. For $\mathcal{U} \in \mathrm{cov}(X)$, the topological entropy of γ with respect to $\mathcal{U}$ is

$$H_{top}(\gamma, \mathcal{U}) = \lim_{i \in I} \frac{f_{\mathcal{U}}(F_i)}{|F_i|},$$

where $(F_i)_{i \in I}$ is a left-Følner net of S. The topological entropy of γ is

$$h_{top}(\gamma) = \sup\{H_{top}(\gamma, \mathcal{U}) \mid \mathcal{U} \in \mathrm{cov}(X)\}.$$

Let S be a cancellative right-amenable semigroup acting $S \overset{\alpha}{\curvearrowright} A$ on an abelian group A by endomorphisms. For $X \in \mathcal{P}_{fin}(A)$ and for every $F \in \mathcal{P}_{fin}(S)$, let

$$T_F(\alpha, X) = \sum_{s\in F} \alpha(s)(X) = \sum_{s\in F} s \cdot X \in \mathcal{P}_{fin}(A)$$

be the α-trajectory of X with respect to F.
The function

$$f_X : \mathcal{P}_{fin}(S) \to \mathbb{R}, \quad F \mapsto \log |T_F(\alpha, X)|.$$

is subadditive, left-subinvariant, so the limit

$$H_{alg}(\alpha, X) = \lim_{i\in I} \frac{\log |T_{F_i}(\alpha, X)|}{|F_i|},$$

exists for every right-Følner net $(F_i)_{i\in I}$ of S and does not depend on the choice of $(F_i)_{i\in I}$.

Let S be a cancellative right-amenable semigroup acting $S \overset{\alpha}{\curvearrowright} A$ on an abelian group A by endomorphisms. For $X \in \mathcal{P}_{fin}(A)$ and for every $F \in \mathcal{P}_{fin}(S)$, let

$$T_F(\alpha, X) = \sum_{s \in F} \alpha(s)(X) = \sum_{s \in F} s \cdot X \in \mathcal{P}_{fin}(A)$$

be the α-trajectory of X with respect to F.
The function

$$f_X : \mathcal{P}_{fin}(S) \to \mathbb{R}, \quad F \mapsto \log |T_F(\alpha, X)|.$$

is subadditive, left-subinvariant, so the limit

$$H_{alg}(\alpha, X) = \lim_{i \in I} \frac{\log |T_{F_i}(\alpha, X)|}{|F_i|},$$

exists for every right-Følner net $(F_i)_{i \in I}$ of S and does not depend on the choice of $(F_i)_{i \in I}$.

Let S be a cancellative right-amenable semigroup acting $S \overset{\alpha}{\curvearrowright} A$ on an abelian group A by endomorphisms. For $X \in \mathcal{P}_{fin}(A)$ and for every $F \in \mathcal{P}_{fin}(S)$, let

$$T_F(\alpha, X) = \sum_{s \in F} \alpha(s)(X) = \sum_{s \in F} s \cdot X \in \mathcal{P}_{fin}(A)$$

be the α-trajectory of X with respect to F.

The function

$$f_X : \mathcal{P}_{fin}(S) \to \mathbb{R}, \quad F \mapsto \log |T_F(\alpha, X)|.$$

is subadditive, left-subinvariant, so the limit

$$H_{alg}(\alpha, X) = \lim_{i \in I} \frac{\log |T_{F_i}(\alpha, X)|}{|F_i|},$$

exists for every right-Følner net $(F_i)_{i \in I}$ of S and does not depend on the choice of $(F_i)_{i \in I}$.

Let S be a cancellative right-amenable semigroup acting $S \overset{\alpha}{\curvearrowright} A$ on an abelian group A by endomorphisms. For $X \in \mathcal{P}_{fin}(A)$ and for every $F \in \mathcal{P}_{fin}(S)$, let

$$T_F(\alpha, X) = \sum_{s \in F} \alpha(s)(X) = \sum_{s \in F} s \cdot X \in \mathcal{P}_{fin}(A)$$

be the α-trajectory of X with respect to F.
The function

$$f_X : \mathcal{P}_{fin}(S) \to \mathbb{R}, \quad F \mapsto \log |T_F(\alpha, X)|.$$

is subadditive, left-subinvariant, so the limit

$$H_{alg}(\alpha, X) = \lim_{i \in I} \frac{\log |T_{F_i}(\alpha, X)|}{|F_i|},$$

exists for every right-Følner net $(F_i)_{i \in I}$ of S and does not depend on the choice of $(F_i)_{i \in I}$.

Definition (Fornasiero, Giordano Bruno, DD - 2017, Virili 2013 for amenable groups)

Let S be a cancellative right-amenable semigroup, A an abelian group and $S \overset{\alpha}{\curvearrowright} A$. For $X \in \mathcal{P}_{fin}(A)$, the algebraic entropy of α with respect to X is $H_{alg}(\alpha, X)$ and the algebraic entropies of α are

$$h_{alg}(\alpha) = \sup\{H_{alg}(\alpha, X) \mid X \in \mathcal{P}_{fin}(A)\}.$$

and

$$\mathrm{ent}(\alpha) = \sup\{H_{alg}(\alpha, X) \mid X \in \mathcal{F}(A)\}.$$

These entropies share many of the properties of the algebriac entropies h_{alg} and ent defined for single endomorphisms.

Moreover, if $f \in \mathrm{End}(A)$ and the action $\mathbb{N} \overset{\alpha}{\curvearrowright} A$ is defined by $\alpha(n)(x) = f^n(x)$ for $n \in \mathbb{N}$ and $x \in A$, then

$$h_{alg}(\mathbb{N} \overset{\alpha}{\curvearrowright} A) = h_{alg}(f) \quad \text{and} \quad \mathrm{ent}(\mathbb{N} \overset{\alpha}{\curvearrowright} A) = \mathrm{ent}(f)$$

Definition (Fornasiero, Giordano Bruno, DD - 2017, Virili 2013 for amenable groups)

Let S be a cancellative right-amenable semigroup, A an abelian group and $S \overset{\alpha}{\curvearrowright} A$. For $X \in \mathcal{P}_{fin}(A)$, the algebraic entropy of α with respect to X is $H_{alg}(\alpha, X)$ and the algebraic entropies of α are

$$h_{alg}(\alpha) = \sup\{H_{alg}(\alpha, X) \mid X \in \mathcal{P}_{fin}(A)\}.$$

and

$$\mathrm{ent}(\alpha) = \sup\{H_{alg}(\alpha, X) \mid X \in \mathcal{F}(A)\}.$$

These entropies share many of the properties of the algebriac entropies h_{alg} and ent defined for single endomorphisms.
Moreover, if $f \in \mathrm{End}(A)$ and the action $\mathbb{N} \overset{\alpha}{\curvearrowright} A$ is defined by $\alpha(n)(x) = f^n(x)$ for $n \in \mathbb{N}$ and $x \in A$, then

$$h_{alg}(\mathbb{N} \overset{\alpha}{\curvearrowright} A) = h_{alg}(f) \quad \text{and} \quad \mathrm{ent}(\mathbb{N} \overset{\alpha}{\curvearrowright} A) = \mathrm{ent}(f)$$

Definition (Fornasiero, Giordano Bruno, DD - 2017, Virili 2013 for amenable groups)

Let S be a cancellative right-amenable semigroup, A an abelian group and $S \overset{\alpha}{\curvearrowright} A$. For $X \in \mathcal{P}_{fin}(A)$, the algebraic entropy of α with respect to X is $H_{alg}(\alpha, X)$ and the algebraic entropies of α are

$$h_{alg}(\alpha) = \sup\{H_{alg}(\alpha, X) \mid X \in \mathcal{P}_{fin}(A)\}.$$

and

$$\mathrm{ent}(\alpha) = \sup\{H_{alg}(\alpha, X) \mid X \in \mathcal{F}(A)\}.$$

These entropies share many of the properties of the algebriac entropies h_{alg} and ent defined for single endomorphisms.

Moreover, if $f \in \mathrm{End}(A)$ and the action $\mathbb{N} \overset{\alpha}{\curvearrowright} A$ is defined by $\alpha(n)(x) = f^n(x)$ for $n \in \mathbb{N}$ and $x \in A$, then

$$h_{alg}(\mathbb{N} \overset{\alpha}{\curvearrowright} A) = h_{alg}(f) \quad \text{and} \quad \mathrm{ent}(\mathbb{N} \overset{\alpha}{\curvearrowright} A) = \mathrm{ent}(f)$$

Definition (Fornasiero, Giordano Bruno, DD - 2017, Virili 2013 for amenable groups)

Let S be a cancellative right-amenable semigroup, A an abelian group and $S \overset{\alpha}{\curvearrowright} A$. For $X \in \mathcal{P}_{fin}(A)$, the algebraic entropy of α with respect to X is $H_{alg}(\alpha, X)$ and the algebraic entropies of α are

$$h_{alg}(\alpha) = \sup\{H_{alg}(\alpha, X) \mid X \in \mathcal{P}_{fin}(A)\}.$$

and

$$\mathrm{ent}(\alpha) = \sup\{H_{alg}(\alpha, X) \mid X \in \mathcal{F}(A)\}.$$

These entropies share many of the properties of the algebriac entropies h_{alg} and ent defined for single endomorphisms.

Moreover, if $f \in \mathrm{End}(A)$ and the action $\mathbb{N} \overset{\alpha}{\curvearrowright} A$ is defined by $\alpha(n)(x) = f^n(x)$ for $n \in \mathbb{N}$ and $x \in A$, then

$$h_{alg}(\mathbb{N} \overset{\alpha}{\curvearrowright} A) = h_{alg}(f) \quad \text{and} \quad \mathrm{ent}(\mathbb{N} \overset{\alpha}{\curvearrowright} A) = \mathrm{ent}(f)$$

Definition (Fornasiero, Giordano Bruno, DD - 2017, Virili 2013 for amenable groups)

Let S be a cancellative right-amenable semigroup, A an abelian group and $S \overset{\alpha}{\curvearrowright} A$. For $X \in \mathcal{P}_{fin}(A)$, the algebraic entropy of α with respect to X is $H_{alg}(\alpha, X)$ and the algebraic entropies of α are

$$h_{alg}(\alpha) = \sup\{H_{alg}(\alpha, X) \mid X \in \mathcal{P}_{fin}(A)\}.$$

and

$$\mathrm{ent}(\alpha) = \sup\{H_{alg}(\alpha, X) \mid X \in \mathcal{F}(A)\}.$$

These entropies share many of the properties of the algebriac entropies h_{alg} and ent defined for single endomorphisms.

Moreover, if $f \in \mathrm{End}(A)$ and the action $\mathbb{N} \overset{\alpha}{\curvearrowright} A$ is defined by $\alpha(n)(x) = f^n(x)$ for $n \in \mathbb{N}$ and $x \in A$, then

$$h_{alg}(\mathbb{N} \overset{\alpha}{\curvearrowright} A) = h_{alg}(f) \quad \text{and} \quad \mathrm{ent}(\mathbb{N} \overset{\alpha}{\curvearrowright} A) = \mathrm{ent}(f)$$

Theorem (Continuity for direct limit)

Let S be a cancellative right-amenable semigroup, A an abelian group and consider $S \overset{\alpha}{\curvearrowright} A$. If A is a direct limit of α-invariant subgroups $\{A_i \mid i \in I\}$, then $h_{alg}(\alpha) = \sup_{i \in I} h_{alg}(\alpha_{A_i})$.

Theorem (Logarithmic Law)

Let G be an amenable group, A an abelian group and $G \overset{\alpha}{\curvearrowright} A$. If H is a subgroup of G of finite index $[G : H] = k \in \mathbb{N}$, then

$$h_{alg}(\alpha \restriction_H) = k \cdot h_{alg}(\alpha) \quad \textit{and} \quad \mathrm{ent}(\alpha \restriction_H) = k \cdot \mathrm{ent}(\alpha).$$

Theorem (Fornasiero, Giordano Bruno, DD - 2017)

Let A be a torsion abelian group, S be a right-amenable monoid, α be a left action of S on A, and B be an α-invariant subgroup of A. Then

$$\mathrm{ent}(\alpha) = \mathrm{ent}(\alpha_B) + \mathrm{ent}(\alpha_{A/B}).$$

Theorem (Continuity for direct limit)

Let S be a cancellative right-amenable semigroup, A an abelian group and consider $S \overset{\alpha}{\curvearrowright} A$. If A is a direct limit of α-invariant subgroups $\{A_i \mid i \in I\}$, then $h_{alg}(\alpha) = \sup_{i\in I} h_{alg}(\alpha_{A_i})$.

Theorem (Logarithmic Law)

Let G be an amenable group, A an abelian group and $G \overset{\alpha}{\curvearrowright} A$. If H is a subgroup of G of finite index $[G : H] = k \in \mathbb{N}$, then

$$h_{alg}(\alpha \restriction_H) = k \cdot h_{alg}(\alpha) \quad \text{and} \quad \mathrm{ent}(\alpha \restriction_H) = k \cdot \mathrm{ent}(\alpha).$$

Theorem (Fornasiero, Giordano Bruno, DD - 2017)

Let A be a torsion abelian group, S be a right-amenable monoid, α be a left action of S on A, and B be an α-invariant subgroup of A. Then

$$\mathrm{ent}(\alpha) = \mathrm{ent}(\alpha_B) + \mathrm{ent}(\alpha_{A/B}).$$

Theorem (Continuity for direct limit)

Let S be a cancellative right-amenable semigroup, A an abelian group and consider $S \overset{\alpha}{\curvearrowright} A$. If A is a direct limit of α-invariant subgroups $\{A_i \mid i \in I\}$, then $h_{alg}(\alpha) = \sup_{i \in I} h_{alg}(\alpha_{A_i})$.

Theorem (Logarithmic Law)

Let G be an amenable group, A an abelian group and $G \overset{\alpha}{\curvearrowright} A$. If H is a subgroup of G of finite index $[G : H] = k \in \mathbb{N}$, then

$$h_{alg}(\alpha \restriction_H) = k \cdot h_{alg}(\alpha) \quad \textit{and} \quad \mathrm{ent}(\alpha \restriction_H) = k \cdot \mathrm{ent}(\alpha).$$

Theorem (Fornasiero, Giordano Bruno, DD - 2017)

Let A be a torsion abelian group, S be a right-amenable monoid, α be a left action of S on A, and B be an α-invariant subgroup of A. Then

$$\mathrm{ent}(\alpha) = \mathrm{ent}(\alpha_B) + \mathrm{ent}(\alpha_{A/B}).$$

Theorem (Continuity for direct limit)

Let S be a cancellative right-amenable semigroup, A an abelian group and consider $S \overset{\alpha}{\curvearrowright} A$. If A is a direct limit of α-invariant subgroups $\{A_i \mid i \in I\}$, then $h_{alg}(\alpha) = \sup_{i\in I} h_{alg}(\alpha_{A_i})$.

Theorem (Logarithmic Law)

Let G be an amenable group, A an abelian group and $G \overset{\alpha}{\curvearrowright} A$. If H is a subgroup of G of finite index $[G : H] = k \in \mathbb{N}$, then

$$h_{alg}(\alpha \restriction_H) = k \cdot h_{alg}(\alpha) \quad \textit{and} \quad \mathrm{ent}(\alpha \restriction_H) = k \cdot \mathrm{ent}(\alpha).$$

Theorem (Fornasiero, Giordano Bruno, DD - 2017)

Let A be a torsion abelian group, S be a right-amenable monoid, α be a left action of S on A, and B be an α-invariant subgroup of A. Then

$$\mathrm{ent}(\alpha) = \mathrm{ent}(\alpha_B) + \mathrm{ent}(\alpha_{A/B}).$$

For a locally compact abelian group A and a continuous endomorphism $\phi : A \to A$ denote by $\widehat{A}$ the Pontryagin dual of A and $\widehat{\phi} : \widehat{A} \to \widehat{A}$ be the dual of ϕ, defined by $\widehat{\phi}(\chi) = \chi \circ \phi$.
A left action $S \overset{\gamma}{\curvearrowright} K$ of a cancellative left-amenable semigroup S on a compact abelian group K induces a right dual action $\widehat{K} \overset{\widehat{\gamma}}{\curvearrowleft} S$ on the discrete group $\widehat{K}$, defined by

$$\widehat{\gamma}(s) = \widehat{\gamma(s)} : \widehat{K} \to \widehat{K} \quad \text{for every } s \in S.$$

The Bridge theorem remains true in this much more general context (where $\widehat{\gamma}^{op}$ is the left action of S^{op} associated to $\widehat{\gamma}$):

Theorem (Fornasiero, Giordano Bruno, DD - 2017)

For a left action $S \overset{\gamma}{\curvearrowright} K$ of a cancellative left-amenable semigroup S on a compact totally disconnected abelian group K

$$h_{top}(\gamma) = h_{alg}(\widehat{\gamma}^{op}).$$

For a locally compact abelian group A and a continuous endomorphism $\phi : A \to A$ denote by $\widehat{A}$ the Pontryagin dual of A and $\widehat{\phi} : \widehat{A} \to \widehat{A}$ be the dual of ϕ, defined by $\widehat{\phi}(\chi) = \chi \circ \phi$.

A left action $S \overset{\gamma}{\curvearrowright} K$ of a cancellative left-amenable semigroup S on a compact abelian group K induces a right **dual action** $\widehat{K} \overset{\widehat{\gamma}}{\curvearrowleft} S$ on the discrete group $\widehat{K}$, defined by

$$\widehat{\gamma}(s) = \widehat{\gamma(s)} : \widehat{K} \to \widehat{K} \quad \text{for every } s \in S.$$

The Bridge theorem remains true in this much more general context (where $\widehat{\gamma}^{op}$ is the left action of S^{op} associated to $\widehat{\gamma}$):

Theorem (Fornasiero, Giordano Bruno, DD - 2017)

For a left action $S \overset{\gamma}{\curvearrowright} K$ of a cancellative left-amenable semigroup S on a compact totally disconnected abelian group K

$$h_{top}(\gamma) = h_{alg}(\widehat{\gamma}^{op}).$$

For a locally compact abelian group A and a continuous endomorphism $\phi : A \to A$ denote by $\widehat{A}$ the Pontryagin dual of A and $\widehat{\phi} : \widehat{A} \to \widehat{A}$ be the dual of ϕ, defined by $\widehat{\phi}(\chi) = \chi \circ \phi$.

A left action $S \overset{\gamma}{\curvearrowright} K$ of a cancellative left-amenable semigroup S on a compact abelian group K induces a right dual action $\widehat{K} \overset{\widehat{\gamma}}{\curvearrowleft} S$ on the discrete group $\widehat{K}$, defined by

$$\widehat{\gamma}(s) = \widehat{\gamma(s)} : \widehat{K} \to \widehat{K} \quad \text{for every } s \in S.$$

The Bridge theorem remains true in this much more general context (where $\widehat{\gamma}^{op}$ is the left action of S^{op} associated to $\widehat{\gamma}$):

Theorem (Fornasiero, Giordano Bruno, DD - 2017)

For a left action $S \overset{\gamma}{\curvearrowright} K$ of a cancellative left-amenable semigroup S on a compact totally disconnected abelian group K

$$h_{top}(\gamma) = h_{alg}(\widehat{\gamma}^{op}).$$

For a locally compact abelian group A and a continuous endomorphism $\phi : A \to A$ denote by $\widehat{A}$ the Pontryagin dual of A and $\widehat{\phi} : \widehat{A} \to \widehat{A}$ be the dual of ϕ, defined by $\widehat{\phi}(\chi) = \chi \circ \phi$.

A left action $S \overset{\gamma}{\curvearrowright} K$ of a cancellative left-amenable semigroup S on a compact abelian group K induces a right dual action $\widehat{K} \overset{\widehat{\gamma}}{\curvearrowleft} S$ on the discrete group $\widehat{K}$, defined by

$$\widehat{\gamma}(s) = \widehat{\gamma(s)} : \widehat{K} \to \widehat{K} \quad \text{for every } s \in S.$$

The Bridge theorem remains true in this much more general context (where $\widehat{\gamma}^{op}$ is the left action of S^{op} associated to $\widehat{\gamma}$):

Theorem (Fornasiero, Giordano Bruno, DD - 2017)

For a left action $S \overset{\gamma}{\curvearrowright} K$ of a cancellative left-amenable semigroup S on a compact totally disconnected abelian group K

$$h_{top}(\gamma) = h_{alg}(\widehat{\gamma}^{op}).$$

For a locally compact abelian group A and a continuous endomorphism $\phi : A \to A$ denote by $\widehat{A}$ the Pontryagin dual of A and $\widehat{\phi} : \widehat{A} \to \widehat{A}$ be the dual of ϕ, defined by $\widehat{\phi}(\chi) = \chi \circ \phi$.

A left action $S \overset{\gamma}{\curvearrowright} K$ of a cancellative left-amenable semigroup S on a compact abelian group K induces a right dual action $\widehat{K} \overset{\widehat{\gamma}}{\curvearrowleft} S$ on the discrete group $\widehat{K}$, defined by

$$\widehat{\gamma}(s) = \widehat{\gamma(s)} : \widehat{K} \to \widehat{K} \quad \text{for every } s \in S.$$

The Bridge theorem remains true in this much more general context (where $\widehat{\gamma}^{op}$ is the left action of S^{op} associated to $\widehat{\gamma}$):

Theorem (Fornasiero, Giordano Bruno, DD - 2017)

For a left action $S \overset{\gamma}{\curvearrowright} K$ of a cancellative left-amenable semigroup S on a compact totally disconnected abelian group K

$$h_{top}(\gamma) = h_{alg}(\widehat{\gamma}^{op}).$$