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Compactness via Adherence Dominators

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Outline

Introduction

This talk is based on a joint work by T. A. Edwards, J. E. Joseph, M. H. Kwack and B. M. P. Nayar and the paper apperared in the Journal of Advanced studies in Topology, Vol. 5 (4), 2014), 8 - 15.

Introduction

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In [1] the following questions were stated as open:

1. Problem 14.[B] Is a regular space in which every closed subset is regular-closed compact?
2. Problem 15. Is a Urysohn-space in which every closed subset is Urysohn-closed compact?

Introduction (Contd.)

Similar questin in the case of Hausdorff spaces was answered affirmatively in 1937 by Stone [17] using Boolean rings and in 1940 by Katětov [14] using topological methods. All of these questions were answered in the affirmative in [13] using topological methods and more generalizations of these were given in [2] and [3]. Generilzations of theorems in [4] and [5] are also given in this article.

Here, the concept of adherence dominator is employed to subsume results in [2], [3] and [13] and also to provide generalizations of results in [4] and [5].

Preliminaries

An adherence dominator on a topological space X is a function π from the collection of filterbases on X to the family of closed subsets of X satisfying $\mathcal{A}\Omega \subset \pi\Omega$ where $\mathcal{A}\Omega$ is the adherence of Ω [11].

Preliminaries

An adherence dominator on a topological space X is a function π from the collection of filterbases on X to the family of closed subsets of X satisfying $\mathcal{A}\Omega \subset \pi\Omega$ where $\mathcal{A}\Omega$ is the adherence of Ω [11]. In [11] the concept of an adherence dominator was used to provide a frame work under which many characerizations of minimal P-spaces and P-closed spaces were subsumed, for P =Hausdorff, Urysohn, regular. This concept was also used to provide several new charecterizations of such spaces as well as charecterizations of compact spaces. Here are presented generalizations of the affirmative answers to questions raised in [1] and solved in [2], [3] and [13].

Preliminaries (Contd.)

The collection of open subsets of a space X containing $A \subseteq X$ is denoted by $\Sigma(A), \Sigma(x)$, if $A = \{x\}$. The bountary of A in a space X is denoted by $bd(A)$. An element x is in θ- closure of A, denoted as $cl_\theta(A)$, [18] if $cl(V) \cap A \neq \emptyset$ for each $V \in \sum(x)$. A set A is θ-closed if $cl_\theta(A) = A$.

Preliminaries (Contd.)

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Preliminaries (Contd.)

Herrington [8] defined a point $x \in X$ to be in the s-adherence of a filterbase $\mathcal{F}$, denoted as $x \in adh_s\mathcal{F}$, if for each shrinkable family $\mathcal{G}$ of open sets about x and $F \in \mathcal{F}$, there is a $V \in \mathcal{G}$ such that $F \cap V \neq \emptyset$, where a family of open sets, $\mathcal{G}$, is a shrinkable family of open sets about a point $x \in X$ if for each $U \in \mathcal{G}$, there is a $V \in \mathcal{G}$ such that $x \in U \subseteq clU \subseteq V$. Thus, $adh_s\Omega = \bigcap_{\Omega} cl_s(F) \neq \emptyset$ [8].

Preliminaries (Contd.)

Herrington [8] defined a point $x \in X$ to be in the s-adherence of a filterbase $\mathcal{F}$, denoted as $x \in adh_s\mathcal{F}$, if for each shrinkable family $\mathcal{G}$ of open sets about x and $F \in \mathcal{F}$, there is a $V \in \mathcal{G}$ such that $F \cap V \neq \emptyset$, where a family of open sets, $\mathcal{G}$, is a shrinkable family of open sets about a point $x \in X$ if for each $U \in \mathcal{G}$, there is a $V \in \mathcal{G}$ such that $x \in U \subseteq clU \subseteq V$. Thus, $adh_s\Omega = \bigcap_\Omega cl_s(F) \neq \emptyset$ [8]. It was then proved that a regular space is regular-closed if and only if each filterbase on the space has non-empty s-adherence. See [4], [5]. A subset A of a space X is an H-set (U-set) [R-set] if each open filterbase Ω on A satisfies $A \cap adh\Omega (A \cap adh_u\Omega)[A \cap adh_s\Omega] \neq \emptyset$.

Preliminaries (Contd.)

The following Theorem will be used throughout.

Theorem 1

If Ω is an ultrafilter on a space X and $\mathcal{O} = \{V \subset X : V \text{open}, F \subset V$ for some $F \in \Omega\}$, then $\mathcal{O}$ is an open ultrafilter on X and $\pi\Omega = \pi\mathcal{O}$.

Proof. Clearly $\pi\Omega \subset \pi\mathcal{O}$. For the reverse inclusion suppose $x \notin \pi\Omega$. Then there is an open set V with $F \subset V$ such that $x \in X - \pi V$. Therefore, $x \notin \pi\mathcal{O}$. Hence $\pi\Omega = \pi\mathcal{O}$. □

Preliminaries (Contd.)

We now define the concepts of a π-space, a π-closed subset and a π-closed space. These definitions are motivated by the nature and role of θ-closure (u-closure) [s-closure] of a set and p-adherence of a filterbase in a p-closed space, where p represents the property of being Hausdorff or Urysohn or regular, etc.

Preliminaries (Contd.)

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Definition 1. A nonempty subset A of a space X is π-closed if $\pi A = A$. A space X is a π-space if $\{x\}$ is π-closed for every $x \in X$.

Definition 2. A space is π-closed if $\pi\Omega \neq \emptyset$ for every filterbase Ω on the space.

Preliminaries (Contd.)

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Definition 2. A space is π-closed if $\pi\Omega \neq \emptyset$ for every filterbase Ω on the space.

Theorem 2

If $\mathcal{O}$ is an open ultrafilter in a π-space, $\pi\mathcal{O}$ has fewer than two elements.

Main Resilts

Theorem 3

A π-closed π-space X is compact if and only if each closed subset is π-closed.

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Proof. Clearly, every compact subset of a space is π-closed. Let X be π-closed and let $\mathcal{U}$ be an ultrafilter on X. Then there is $x \in X$ such that $\pi\mathcal{U} = \{x\}$. If $V \in \sum(x), F - V = \emptyset$ for some $F \in \mathcal{U}$. So $F \subseteq V$, for some $F \in \mathcal{U}$. Therefore $\mathcal{U} \rightarrow x$.
$\square$

Main Results (Contd.)

Theorem 4

A π-closed π-space X is compact if and only if $bd(V)$ is π-closed for every open subset of X.

Main Results (Contd.)

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Proof. If X is compact, $bd(V)$, being compact, is π-closed. Let X be π-closed and let $\mathcal{U}$ be an ultrafilter on X. There is $x \in X$ such that $\pi\mathcal{U} = \{x\}$. If $V \in \sum(x), F \cap bd(V) = \emptyset$ for some $F \in \mathcal{U}$. Otherwise, $bd(V) \in \mathcal{U}$, which contradicts the fact that $\pi\mathcal{U} = \{x\}$. Hence $\mathcal{U} \to x$. $\square$

Main Results (Contd.)

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Theorem 5

A π-closed π-space X is compact if and only if X is rim π-closed.

Proof. Let $\mathcal{U}$ be an ultrafilter on X. There is an $x \in X$ such that $\pi\mathcal{U} = \{x\}$. If $V \in \sum(x), F \cap bd(V) = \emptyset$ for some $F \in \mathcal{U}$ and hence $\mathcal{U} \to x$. □

Main Results (Contd.)

Definition 4. A subset A of a space is a π-set if each filterbase Ω on A satisfies $\pi\Omega \cap A \neq \emptyset$.

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Definition 5. A subset A of a space X is π-rigid if for any open filterbase Ω on the space $X, F \cap \pi(V) \neq \emptyset$ for each $F \in \Omega,\ V \in \sum(A)$ implies $\pi\Omega \cap A \neq \emptyset$.

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Definition 6. A space is locally π-closed (LπC) if each point has a π-closed neighborhood.

Main Results (Contd.)

Corollary 1

Let X be a $L\pi C$ π-space. The following are equivalent:

1. *(1) X is compact;*
2. *(2) Each closed subset of X is π-closed ;*
3. *(3) Each closed subset of X is a π-set ;*
4. *(4) X is rim π-set;*
5. *(5) The boundary $bd(V)$ is π-rigid, for each open subset V;*
6. *(6) The boundary $bd(V)$ is π-closed for each open subset V;*
7. *(7) The boundary $bd(V)$ is a π-set for each open subset V;*
8. *(8) X is rim π-closed.*

Main Results (Contd.)

Proof. (1) $\Rightarrow$ (2). In a compact space, every closed set is π-closed.

(2) $\Rightarrow$ (1). Let $\mathcal{U}$ be an ultrafilter on X. By Theorem 2, $\pi\mathcal{U}$ is a singleton, and therefore $\mathcal{U}$ converges.

(2) $\Rightarrow$ (3). Every π-closed set is a π-set.

(3) $\Rightarrow$ (4). Follows from Theorem 5 and (2) $\Rightarrow$ (3).

Main Results (Contd.)

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(3) $\Rightarrow$ (4). Follows from Theorem 5 and (2) $\Rightarrow$ (3).

(4) $\Rightarrow$ (5). Let X be rim π-set and let an open filterbase Ω on the space X satisfy $F \cap \pi(A) \neq \emptyset$ for each $F \in \Omega, A \in \sum(bd(V))$. Then $\Gamma = \{\text{bd(V)}\} \cup \Omega$ is a fiterbase and $\pi\Gamma \neq \emptyset$.

(5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (8) Follows clearly.

(8) $-$ (3) $\Rightarrow$ (1). As in the proof of (2) $\Rightarrow$ (1) find an $x \in X$ such that $\pi\mathcal{U} = \{x\}$. If $V \in \sum(x), \mathcal{U}$ an ultrafilter on X, there exists $F \in \mathcal{U} such that F \cap bd(V) = \emptyset$. So $\mathcal{U} \to x$. □

Main Results (Contd.)

A Hausdorff (Urysohn) [regular] space is called locally H-closed (LHC) (locally Urysohn-closed (LUC)) [locally regular-closed (LRC)] if each point has an H-closed (Urysohn-closed) [regular-closed] neighborhood.

Information on Hausdorff-closed, Urysohn-closed, regular-closed spaces may be found in [1],[3], [13]. The following result can be found in [2].

Main Results (Contd.)

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Information on Hausdorff-closed, Urysohn-closed, regular-closed spaces may be found in [1],[3], [13]. The following result can be found in [2].

Theorem 6

Let X be a LHC (LUC) [LRC] space. The following are equivalent:

Main Results (Contd.)

1. (1) X is compact;
2. (2) Each closed subset of X is θ-closed (u-closed) [s-closed];
3. (3) Each closed subset of X is an H-set (a U-set) [an R-set] ;
4. (4) X is rim θ-closed (rim u-closed) [rim s-closed];
5. (5) X is rim H-set(rim U-set) [rim R-set];
6. (6) The boundary $bd(V)$ is θ-rigid (u-rigid) [s-rigid] for each open subset V;
7. (7) The boundary $bd(V)$ is θ-closed (u-closed) [s-closed] for each open subset V;
8. (8) The boundary $bd(V)$ is an H-set (a U-set) [an R-set] for each open subset V;
9. (9) X is rim Hausdorff-closed (rim Urysohn-closed) [rim regular-closed].

Main Results (Contd.)

Definition 7. A space X is a π(i) space if every filterbase Ω on X satisfies that $\pi\Omega \neq \emptyset$.

Note that a π(i) space which is a π-space is a π-closed space.

Theorem 7

Let X be a $\pi(i)$ space. Then $adh_\theta(\Omega)(adh_u(\Omega)) \cap \pi\Gamma \neq \emptyset$ for every open filterbase Γ on $adh_\theta(\Omega)(adh_u(\Omega))$.

Proof. Let Γ be an open filterbase on $adh_\theta(\Omega)(adh_u(\Omega))$.
Then $\Gamma^* = \{V \cap W : V \in \bigcup_\Omega \Sigma(F)(V \in \bigcup_\Omega \Lambda(F)), W \in \Gamma\}$ is an open filterbase on X. So,
$\emptyset \neq \pi\Gamma^* \subset \pi\Gamma \cap adh_\theta\Omega(adh_u\Omega)$. □

Main Results (Contd.)

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Then $\Gamma^* = \{V \cap W : V \in \bigcup_\Omega \Sigma(F)(V \in \bigcup_\Omega \Lambda(F)), W \in \Gamma\}$ is an open filterbase on X. So,
$\emptyset \neq \pi\Gamma^* \subset \pi\Gamma \cap adh_\theta\Omega(adh_u\Omega)$. □

Corollary 2

In a π(i) space X, $adh_\theta\Omega(adh_u\Omega)$ is a π-set for a filterbase Ω on X.

Main Results (Contd.)

The following corollaries follow easily, since a π-space which is a π(i) space is π-closed.

Corollary 3

In a π-closed space X, $adh_\theta\Omega(adh_u\Omega)$ is a π-set for every filterbase Ω on X.

Main Results (Contd.)

The following corollaries follow easily, since a π-space which is a π(i) space is π-closed.

Corollary 3

In a π-closed space X, $adh_\theta\Omega(adh_u\Omega)$ is a π-set for every filterbase Ω on X.

The operator π taking the role of θ-closure and u-closure, the next results follow:

Corollary 4

Let X be an H(i) space. Then $adh_\theta(\Omega) \cap adh\Gamma \neq \emptyset$ for every open filterbase Γ on $adh_\theta(\Omega)$.

Corollary 5

Let X be an H-closed space. Then $adh_\theta(\Omega)$ is an H-set for every filterbase Ω on X.

Main Results (Contd.)

Corollary 6

Let X be an H(i) space. Then $adh_u(\Omega)$ is an π-set for every filterbase Ω on X.

Corollary 7

Let X be an Hausdorff-closed space. Then $adh_u(\Omega)$ is an H-set for every filterbase Ω on X.

Corollary 8

If X is a π-closed π-space and Ω is a filterbase on X, the following are equivalent:

Main Results (Contd.)

1. (1) The subset $adh_u(\Omega)(adh_\theta(\Omega)$ is compact;
2. (2) A closed subset of $adh_u(\Omega)(adh_\theta(\Omega)$ is π-closed;
3. (3) A closed subset of $adh_u(\Omega)(adh_\theta(\Omega)$ is a π-set;
4. (4) A closed subset of $adh_u(\Omega)(adh_\theta(\Omega)$ is π-closed;
5. (5) A closed subset of $adh_u(\Omega)(adh_\theta(\Omega)$ is π-rigid;
6. (6) The subset $adh_u(\Omega)(adh_\theta(\Omega)$ is rim π-set;
7. (7) The subset $adh_u(\Omega)(adh_\theta(\Omega)$ is rim π-closed;
8. (8) The boundary $bd(W)$ is a π-closed subset of $adh_u(\Omega)(adh_\theta(\Omega)$ for every open subset W of $adh_u(\Omega)(adh_\theta(\Omega)$;
9. (9) The boundary $bd(W)$ is a π-rigid subset of $adh_u(\Omega)(adh_\theta(\Omega))$ for every open subset W of $adh_u(\Omega)(adh_\theta(\Omega)$.

Main Results (Contd.)

Proof. Follows from the foregoing results. □

Corollary 9

If X is a Hausdorff-closed space and Ω is a filterbase on X, the following are equivalent:

1. *(1) The subset $adh_u(\Omega)(adh_\theta(\Omega)$ is compact;*
2. *(2) A closed subset of $adh_u(\Omega)(adh_\theta(\Omega)$ is Hausdorff-closed;*
3. *(3) A closed subset of $adh_u(\Omega)(adh_\theta(\Omega)$ is an H-set;*
4. *(4) A closed subset of $adh_u(\Omega)(adh_\theta(\Omega)$ is θ-closed;*

Main Results (Contd.)

1. (5) A closed subset of $adh_u(\Omega)(adh_\theta(\Omega)$ is θ-rigid;
2. (6) The subset $adh_u(\Omega)(adh_\theta(\Omega)$ is rim H-set;
3. (7) The subset $adh_u(\Omega)(adh_\theta(\Omega)$ is rim Hausdorff-closed;
4. (8) The boundary $bd(W)$ is a θ-closed subset of $adh_u(\Omega)(adh_\theta(\Omega)$ for every open subset W of $adh_u(\Omega)(adh_\theta(\Omega)$;
5. (9) The boundary $bd(W)$ is a θ-rigid subset of $adh_u(\Omega)(adh_\theta(\Omega))$ for every open subset W of $adh_u(\Omega)(adh_\theta(\Omega)$.

Main Results (Contd.)

Definition 8. A filterbase Ω on a space is point dominating (p.d.) [12] if each point is a member of all but finitely many elements of Ω; a filterbase Ω on a space is neighborhood dominating (n.d.) [15] if each point has a neighborhood contained in all but finitely many elements of Ω.

Main Results (Contd.)

Definition 8. A filterbase Ω on a space is point dominating (p.d.) [12] if each point is a member of all but finitely many elements of Ω; a filterbase Ω on a space is neighborhood dominating (n.d.) [15] if each point has a neighborhood contained in all but finitely many elements of Ω. The above concepts of a p. d. family and a n. d. family were used to give the following filterbase characterizations for metacompact spaces and for paracompact spaces. A space X is metacompact [12] (paracompact [15]) if each filterbase on X, with the property that each p.d (n.d.) subcollection has non-empty adherence, has non-empty adherence.

Main Results (Contd.)

Theorem 8

Let X be a π-closed π-space. The following are equivalent:

1. *(1) X is compact;*
2. *(2) Each closed subset of X is paracompact.*
3. *(3) The boundary $bd(V)$ is paracompact for each open subset V;*
4. *(4) X is rim paracompact;*
5. *(5) The boundary $bd(V)$ is metacompact for each open subset V;*
6. *(6) X is rim metacompact;*
7. *(7) Each closed subset of X is metacompact.*

Main Results (Contd.)

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6). Obvious.

(6) $\Rightarrow$ (1). Let $\mathcal{U}$ be an ultrafilter on X. There is $x \in X$ such that $\pi\mathcal{U} = \{x\}$. Let V be a basic open set about x. Then $bd(V)$ is metacompact. If $V \notin \mathcal{U}, \Gamma = \{bd(V) \cap F | F \in \mathcal{U}\}$ is a filter base in $bd(V)$ and there is a p.d. subcollection of Γ with empty adherence. Hence $\mathcal{A}\Gamma = \emptyset$. Therefore, $\mathcal{U} \rightarrow x$.

Main Results (Contd.)

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6). Obvious.

(6) $\Rightarrow$ (1). Let $\mathcal{U}$ be an ultrafilter on X. There is $x \in X$ such that $\pi\mathcal{U} = \{x\}$. Let V be a basic open set about x. Then $bd(V)$ is metacompact. If $V \notin \mathcal{U}, \Gamma = \{bd(V) \cap F | F \in \mathcal{U}\}$ is a filter base in $bd(V)$ and there is a p.d. subcollection of Γ with empty adherence. Hence $\mathcal{A}\Gamma = \emptyset$. Therefore, $\mathcal{U} \to x$.

(7) $\Rightarrow$ (1). Let $\mathcal{U}$ be an ultrafilter on X. Then there exists $x \in X$ such that $\pi\mathcal{U} = \{x\}$. Let $V \in \sum(x)$. Then $X - V$ is metacompact. If $V \notin \mathcal{U}$, then $\{F \cap (X - V) : F \in \mathcal{U}\}$ is a filterbase in $X - V$ and there exists a p.d. subcollection of $\{F \cap (X - V) : F \in \mathcal{U}\}$ with empty adherence. Thus $\pi\Omega \cap (X - V) = \emptyset, \mathcal{U} \to x$.

(1) $\Rightarrow$ (7) is obvious. □

Main Results (Contd.)

Utilizing π to be θ -closure, u -closure, s-closure, the following Corollary for Hausdorff-closed, Urysohn-closed or regular-closed spaces follows readily:

Main Results (Contd.)

Corollary 10

Let X be a Hausdorff-closed, (Urysohn-closed) [regular-closed] space. The following are equivalent:

1. *(1) X is compact;*
2. *(2) Each closed subset of X is paracompact.*
3. *(3) The boundary $bd(V)$ is paracompact for each open subset V;*
4. *(4) X is rim paracompact;*
5. *(5) The boundary $bd(V)$ is metacompact for each open subset V;*
6. *(6) X is rim metacompact;*
7. *(7) Each closed subset of X is metacompact.*

Main Results (Contd.)

Corollary 11

A Hausdorff-closed (Urysohn-closed) [regular-closed] paracompact space X is compact

Proof. Every closed subspace of a paracompact space is paracompact. Therefore, the result follows from the equivalence of (1) and (2) of Corollary 10. □

Corollary 12

A Hausdorff-closed (Urysohn-closed) [regular-closed] metacompact space X is compact

Proof. Every closed subspace of a metacompact space is metacompact. Therefore, the result follows from the equivalence of (1) and (7) of Corollary 10. □

Main Results (Contd.)

Definition 9. A topological space is called a U(i) (R(i)) space if every filterbase Ω on the space satisfies $adh_u\Omega(adh_s\Omega) \neq \emptyset$. A space is locally U(i) (locally R(i)) if each point has a U(i) (R(i)) neighborhood.

Theorem 9

Let X be a Hausdorff locally U(i) (R(i)) space. The following are equivalent:

Main Results (Contd.)

1. X is compact;
2. Each closed subset of X is U(i) (R(i));
3. Each closed subset of X is u-closed (s-closed);
4. Each closed subset of X is a U-set (an R-set) ;
5. X is rim u-closed (rim s-closed) ;
6. X is rim U-set (rim R-set);
7. The boundary $bd(V)$ is u-rigid (s-rigid) for each open subset V;
8. The boundary $bd(V)$ is θ-closed u-closed (s-closed) for each open subset V;
9. The boundary $bd(V)$ is a U-set (an R-set) for each open subset V.

Main Results (Contd.)

Proof. Clearly, $(1) \Rightarrow (2)-(9)$. To see that $(2) \Rightarrow (1)$, let $\mathcal{U}$ be an ultrafilter on X. Since X is Hausdorff , there is an $x \in X$ such that $adh_\theta \mathcal{U} = \{x\}$. If $V \in \sum(x), F \cap (X - V) = \emptyset$ for some $F \in \mathcal{U}$ since $X - V$ is U(i) (R(i)). Therefore, $\mathcal{U} \to x$.

$(9)-(3) \Rightarrow (1)$. As in the proof of $(2) \Rightarrow (1)$, there esists an $x \in X, adh_\theta \mathcal{U} = \{x\}$. If $V \in \sum(x), \mathcal{U}$ an ultrafillter on X there exists $F \in \mathcal{U}$ such that $F \cap bd(V) = \emptyset$. So $\mathcal{U} \to x$.
□

Main Results (Contd.)

Definition 10. A relation μ from X to Y is a function $\mu : X \to 2^Y - \{\emptyset\}$; a relation μ from a space X to a space Y is upper semicontinuous (u.s.c.) if for every $W \in \sum \mu(x)$ there is a $V \in \sum(x)$ such that $\mu(V) \subset W$; μ from a space X to a space Y has a π-strongly closed graph if $\pi\mu(\sum(x)) = \mu(x)$ for each $x \in X$.

Theorem 10

If μ is an u.s.c. relation from X to Y and π is an adherence dominator then $\pi\mu(\sum(x)) = \pi\mu(x)$.

Main Result (Contd.)

Proof. Clearly $\pi\mu(x) \subset \pi\mu(\sum(x))$. Since μ is u.s.c., for each $W \in \sum(\mu(x))$, some $V \in \sum(x)$ satisfies $\mu(V) \subset W$. Thus $\pi\mu\sum(x) \subset \pi\mu(x)$. Therefore, $\pi\mu(\sum(x)) = \pi\mu(x)$. □

Corollary 13

An u.s.c. relation μ has a π-strongly-closed graph if and only if μ has π- closed point images.

Main Result (Contd.)

Proof. Clearly $\pi\mu(x) \subset \pi\mu(\sum(x))$. Since μ is u.s.c., for each $W \in \sum(\mu(x))$, some $V \in \sum(x)$ satisfies $\mu(V) \subset W$. Thus $\pi\mu\sum(x) \subset \pi\mu(x)$. Therefore, $\pi\mu(\sum(x)) = \pi\mu(x)$. □

Corollary 13

An u.s.c. relation μ has a π-strongly-closed graph if and only if μ has π- closed point images.

Proof. Clearly, since μ is u.s.c. and π is an adherence dominator, for $x \in X, \pi\mu(\sum(x)) = \pi\mu(x)$. Also since μ has a π-strongly-closed graph $\pi\mu(\sum(x)) = \mu(x)$ for each $x \in X$. Therefore, for each $x \in X$, $\pi\mu(x) = \mu(x)$. Hence, μ has π-closed point images. □

Main Result (Contd.)

Theorem 11

The following statements are equivalent:

1. *The space X is compact;*
2. *For each u.s.c. relation λ on X the relation defined by $\mu(x)=\pi(\lambda(x))$ assumes a maximal value under set inclusion;*
3. *Each u.s.c. relation λ on X with π- closed point images assumes a maximal value under set inclusion;*
4. *Each u.s.c. relation λ on X with π strongly closed graphs assumes a maximal value under set inclusion.*

Main Result (Contd.)

Proof. The equivalence of (3) and (4) follows from Corollary 13 and (3) is obviously implied by (2).

(1) $\Rightarrow$ (2). Assume that X is compact. Let λ be an u.s.c. relation on X and let $\Omega = \{\mu(x) : \mu(x) = \pi(\lambda(x)), x \in X\}$ and be ordered by set inclusion. Let Ω_0 be a nonempty chain in Ω.

Main Result (Contd.)

Proof. The equivalence of (3) and (4) follows from Corollary 13 and (3) is obviously implied by (2).

(1) $\Rightarrow$ (2). Assume that X is compact. Let λ be an u.s.c. relation on X and let $\Omega = \{\mu(x) : \mu(x) = \pi(\lambda(x)), x \in X\}$ and be ordered by set inclusion. Let Ω_0 be a nonempty chain in Ω. For each $y \in X$ such that $\mu(y) \in \Omega_0$, define

$$F(y) = \{x \in X : \mu(y) \subset \mu(x)\}.$$

Then $F(y)$ is a filterbase on the compact space$X$. For such y, let $v \in cl(F(y)$ and let $W \in \sum(\lambda(v))$. Since λ is u.s.c., there is a $V \in \sum(v)$ such that $\lambda(V) \subset W$. Let $q \in V \cap F(y)$. Then, from the definition of $F(y)$ and since $\lambda(V) \subset W$,

$$\mu(y) \subset \mu(q) = \pi(\lambda(q)) \subset \pi(\lambda(V)) \subset \pi W.$$

Main Result (Contd.)

Thus $\mu(y) \subset \mu(v)$ and hence $v \in F(y)$. Therefore, $F(y)$ is closed. Let $q \in \bigcap_{\mu(y)\in\Omega_0} F(y)$. Then $\mu(q)$ is an upper bound for Ω_0. By Zorn's Lemma, Ω has a maximal element. That is, $\mu(\mathrm{x})=\pi(\lambda(x))$ assumes a maximal value under set inclusion. Therefore (1) implies (2).

(3) $\Rightarrow$ (1). If X is not compact, there is a net g in X with an ordinal $\mathcal{G}$ as its index set and with no convergent subnet. Let $\mathcal{G}$ have the order topology and, for each $k \in \mathcal{G}$, define

$$V(k) = X - \pi\{g(j) : j \geq k\}.$$

Main Result (Contd.)

Then $\{V(k) : k \in \mathcal{G}\}$ is an increasing open cover of X with no finite subcover. Define a multifunction $\lambda : X \to \mathcal{G}$ as $\lambda(x) = \{j \in \mathcal{G} : j \leq k\}$ where k is the first element of $\mathcal{G}$ with $x \in V(k)$. Since $\mathcal{G}$, with the order topology, is regular and $\lambda(x)$ is π-closed for each x, $\mu(x) = \pi(\lambda(x)) = \lambda(x)$ for each $x \in X$. To show that λ is u.s.c., let $W \in \sum(\lambda(x))$ and let $y \in V(k_x)$. Then $k_y \leq k_x$ so that $\lambda(y) \subset \lambda(x) \subset W$. Hence $\lambda(V(k_x)) \subset W$ and λ is u.s.c. Since μ clearly assumes no maximal value with respect to set inclusion, (3) does not hold. $\square$

Main Result (Contd.)

Definition 11. A relation μ from a space X to a space Y has a u-strongly closed (strongly closed) [s-strongly closed] graph if
$cl_u(\mu(\sum(x))(cl_\theta(\mu(\sum(x))[cl_s(\mu(\sum(x))] = \mu(x)$ for each $x \in X$.

Theorem 12

If μ is an u.s.c. relation from X to Y then $cl_u(\sum(x))(cl_\theta(\sum(x))[cl_s(\sum(x)] = cl_u(\mu(x))(cl_\theta(\mu(x)))[cl_s(\mu(x))]$.

Proof. Similar to the proof of Theorem 10. □

Main Result (Contd.)

Corollary 14

A u.s.c. relation μ has a u-strongly-closed (strongly-closed) [s-strongly closed] graph if and only if μ has $u(\theta)[s]$- closed point images.

Proof. Similar to the proof of Corollary 13. □

The operator π taking the role of θ-closure (u-closure) [s-closure], the following is a Corollary to Theorem 11.

Corollary 15

The following statements are equivalent:

Main Result (Contd.)

(1) The space X is compact;

(2) For each u.s.c. relation λ on X the relation defined by $\mu(x) = cl_u(\lambda(x))(\mu(x) = cl_\theta(\lambda(x))[\mu(x) = cl_s(\lambda(x)]$ assumes a maximal value under set inclusion;

(3) For each u.s.c.relation λ on X with u-closed (θ-closed) [s-closed] point images, the relation defined by $\mu(x) = cl_u(\lambda(x))(cl_\theta(\lambda(x)))$ $[cl_s(\lambda(x))]$ assumes a maximal value under set inclusion;

(4) For each u.s.c. relation λ on X with a u- strongly closed graph (strongly closed graph) [s- strongly closed graph] relation defined by $\mu(\mathrm{x}) = cl_u(\lambda(x))(\mu(x) = cl_\theta(\lambda(x))[\mu(\mathrm{x}) = cl_s(\lambda(x))]$ assumes a maximal value under set inclusion.

Conclusion

Taking the π-adherence as adherence, θ-adherence [18], u-adherence [5], [7], [9], s-adherence [8], [10], f-adherence [6], [11] δ-adherence [16], etc., of a filterbase, many of the theorems in [2], [3] and [13] on Hausdorff-closed, Urysohn-closed, and regular-closed spaces are subsumed in this. It is also shown that a space X is compact if and only if for each upper-semicontinuous λ on X with π-strongly closed graph, the relation μ on X defined by $\mu = \pi\lambda$ has a maximal value with respect to set inclusion, generalizing results in [4], [5].

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THANK YOU