6-2017

Some Applications of the Point-Open Subbase Game

David Guerrero Sanchez
Universidad Autonoma Metropolitana - Iztapalapa, dgs@ciencias.unam.mx

The point-open subbase game

David Guerrero Sánchez, V. V. Tkachuk

Summer Conference on Topology

Dayton 2017

Contents

1. Defnitions
2. First equivalences
3. The player O
4. Open problems

The point-open subbase game

Definition

Given a subbase $\mathcal{S}$ of a space X, the game $PO(\mathcal{S}, X)$ is defined for two players P and O who respectively pick, at the n-th move, a point $x_n \in X$ and a set $U_n \in S$ such that $x_n \in U_n$. The game stops after the moves $\{x_n, U_n : n \in \omega\}$ have been made and the player P wins if $\bigcup_{n\in\omega} U_n = X$; otherwise O is the winner.

Definitions

A strategy for Player P in the point-open game $PO(X)$ on a space X is a function σ with values in X defined on the initial segments of $PO(X)$ called σ-admissible; it is inductively defined as follows. The empty segment is σ-admissible; if $n > 0$, then a segment $\{x_0, U_0, ..., x_n, U_n\}$ is σ-admissible if $\{x_0, U_0, ..., x_{n-1}, U_{n-1}\}$ is σ-admissible and $x_n = \sigma(x_0, U_0, ..., x_{n-1}, U_{n-1})$.
The definition of a strategy s for Player O is analogous for s-admissible segments $\{x_0, U_0, ..., x_{n-1}, U_{n-1}, x_n\}$. A play $\mathcal{P} = \{x_n, U_n : n \in \omega\}$ is called σ-admissible for a strategy σ of Player P if every initial segment of P is σ-admissible; in this case we will also say that P applies the strategy σ.

Definitions

An s-admissible play for a strategy s of Player O is defined analogously. A strategy σ of Player P is winning on X if P wins in any σ-admissible play.
Analogously, a strategy s of Player O is winning on X if O is the winner in any s-admissible play.
A game $PO(X)$ or $PO(\mathcal{S}, X)$ is undetermined on a space X if neither of the players P and O has a winning strategy in the respective game on X. If a game is considered on a space X and A is one of the players, then X is called A-favorable if A has a winning strategy on X.

The Player P

Theorem

If X is a space and $\mathcal{S}$ is a subbase in X, then the games $PO(X)$ and $PO(\mathcal{S}, X)$ are equivalent for P, i.e., Player P has a winning strategy in the game $PO(X)$ if and only if P has a winning strategy in the game $PO(\mathcal{S}, X)$.

Corollary

If $PO(X)$ is undetermined on a space X, then so is $PO(S, X)$ for any subbase $\mathcal{S}$ of the space X.

Observations

Telgarsky constructed a Lindelöf P -space X on which $PO(X)$ is undetermined. By the previous Corollary, on the same space X the game $PO(\mathcal{S}, X)$ is undetermined for any subbase $\mathcal{S}$. It is also worth mentioning that Under Martin?s Axiom, if $M \subset \mathbb{R}$ and $\omega < |M| < \mathfrak{c}$, then the game $PO(M)$ is undetermined on M; this was proved by Galvin. Applying our Corollary once again we conclude that $PO(\mathcal{S}, M)$ is undetermined on M for any subbase $\mathcal{S}$ of the space M.

Observations

A complete characterization was given by Pawlikowski for the game $PO(X)$ to be undetermined on a space X of countable pseudocharacter. In particular, the game $PO(M)$ is undetermined on a set $M \subset \mathbb{R}$ if and only if $|M| > \omega$ and M is a C''-set, i.e., for every sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of M,there exists a sequence $\{U_n : n \in \omega\} \subset \tau(X)$ such that $U_n \in \mathcal{U}_n$ for each $n \in \omega$ and $\bigcup_{n \in \omega} U_n = M$. Therefore the game $PO(\mathcal{S}, M)$ is undetermined on a set $M \subset \mathbb{R}$ for every subbase $\mathcal{S}$ of M if M is a C''-set. We will see later that the above implication cannot be reversed.

More equivalences

Theorrem

Assume that a space X has a pseudocompact crowded subspace. Then Player O has a winning strategy in $PO(\mathcal{S}, X)$ for any subbase $\mathcal{S}$ in the space X.

Corollary

If X is a compact space and $\mathcal{S}$ is a subbase of X, then the following conditions are equivalent:

1. X is scattered;
2. Player P has a winning strategy in the game $PO(\mathcal{S}, X)$;
3. Player O has no winning strategy in the game $PO(\mathcal{S}, X)$.

Discrete spaces

Theorem

Suppose that $X \subset [0,1]$ is a Bernstein set. Consider the families $\mathcal{S}_0 = \{[0,x] \cap X : x \in X\}$ and $\mathcal{S}_1 = \{[x,1] \cap X : x \in X\}$; then $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$ is a subbase for the discrete topology on X and neither of the players has a winning strategy in the game $PO(\mathcal{S}, X)$.

Corollary

There exists a space $X \subset [0,1]$ such that $PO(X)$ is determined on X but $PO(\mathcal{S}, X)$ is undetermined for some subbase $\mathcal{S}$ of X. In particular, Pawlikowski's characterization does not hold for the game $PO(\mathcal{S}, X)$.

Measurable cardinals

Recall that a cardinal κ is called measurable if there exists a free σ-complete ultrafilter on κ.

Theorem

If κ is a measurable cardinal and X is a discrete space of cardinality κ, then Player O has a winning strategy in the game $PO(\mathcal{S}, X)$ for any subbase $\mathcal{S}$ of the space X.

Open problems

1. Suppose that X is a discrete space such that Player O has a winning strategy in the game $PO(\mathcal{S}, X)$ for every subbase $\mathcal{S}$ in X. Must the cardinality of X be measurable?
2. Suppose that X is a discrete space of cardinality $2^{\mathfrak{c}}$. Does there exist a subbase $\mathcal{S}$ in X for which Player O has no winning strategy in the game $PO(\mathcal{S}, X)$?
3. Does there exist a pseudocompact space X such that the games $PO(X)$ and $PO(\mathcal{S}, X)$ are not equivalent for Player O for some subbase $\mathcal{S}$ in the space X?