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Some Applications of the Point-Open Subbase Game

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The point-open subbase game

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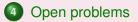
David Guerrero Sánchez The P-S game











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The point-open subbase game

Definition

Given a subbase S of a space X, the game PO(S, X) is defined for two players P and O who respectively pick, at the *n*-th move, a point $x_n \in X$ and a set $U_n \in S$ such that $x_n \in U_n$. The game stops after the moves $\{x_n, U_n : n \in \omega\}$ have been made and the player P wins if $\bigcup_{n \in \omega} U_n = X$; otherwise O is the winner.

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Definitions

A strategy for Player P in the point-open game PO(X) on a space X is a function σ with values in X defined on the initial segments of PO(X) called σ -admissible; it is inductively defined as follows. The empty segment is σ -admissible; if n > 0, then a segment $\{x_0, U_0, ..., x_n, U_n\}$ is σ -admissible if $\{x_0, U_0, \dots, x_{n-1}, U_{n-1}\}$ is σ -admissible and $x_n = \sigma(x_0, U_0, ..., x_{n-1}, U_{n-1}).$ The definition of a strategy s for Player O is analogous for s-admissible segments $\{x_0, U_0, \dots, x_{n-1}, U_{n-1}, x_n\}$. A play $\mathcal{P} = \{x_n, U_n : n \in \omega\}$ is called σ -admissible for a strategy σ of Player *P* if every initial segment of *P* is σ -admissible; in this case we will also say that P applies the strategy σ .

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Definitions

An *s*-admissible play for a strategy *s* of Player *O* is defined analogously. A strategy σ of Player *P* is winning on *X* if *P* wins in any σ -admissible play.

Analogously, a strategy s of Player O is winning on X if O is the winner in any s-admissible play.

A game PO(X) or PO(S, X) is undetermined on a space X if neither of the players P and O has a winning strategy in the respective game on X. If a game is considered on a space X and A is one of the players, then X is called A-favorable if A has a winning strategy on X.

The Player P

Theorem

If X is a space and S is a subbase in X, then the games PO(X) and PO(S, X) are equivalent for P, i.e., Player P has a winning strategy in the game PO(X) if and only if P has a winning strategy in the game PO(S, X).

Corollary

If PO(X) is undetermined on a space X, then so is PO(S, X) for any subbase S of the space X.

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Observations

Telgarsky constructed a Lindelöf *P* -space *X* on which PO(X) is undetermined. By the previous Corollary, on the same space *X* the game PO(S, X) is undetermined for any subbase *S*. It is also worth mentioning that Under Martin?s Axiom, if $M \subset \mathbb{R}$ and $\omega < |M| < c$, then the game PO(M) is undetermined on *M*; this was proved by Galvin. Applying our Corollary once again we conclude that PO(S, M) is undetermined on *M* for any subbase *S* of the space *M*.

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Observations

A complete characterization was given by Pawlikowski for the game PO(X) to be undetermined on a space X of countable pseudocharacter. In particular, the game PO(M) is undetermined on a set $M \subset \mathbb{R}$ if and only if $|M| > \omega$ and M is a C''-set, i.e., for every sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of *M*,there exists a sequence $\{U_n : n \in \omega\} \subset \tau(X)$ such that $U_n \in \mathcal{U}_n$ for each $n \in \omega$ and $\bigcup U_n = M$. Therefore the game PO(S, M) is undetermined on a set $M \subset \mathbb{R}$ for every subbase S of M if M is a C''-set. We will see later that the above implication cannot be reversed.

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More equivalences

Theorrem

Assume that a space X has a pseudocompact crowded subspace. Then Player O has a winning strategy in PO(S, X) for any subbase S in the space X.

Corollary

If X is a compact space and S is a subbase of X, then the following conditions are equivalent:

- X is scattered;
- 2 Player *P* has a winning strategy in the game PO(S, X);
- 3 Player *O* has no winning strategy in the game PO(S, X).

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Discrete spaces

Theorem

Suppose that $X \subset [0, 1]$ is a Bernstein set. Consider the families $S_0 = \{[0, x] \cap X : x \in X\}$ and $S_1 = \{[x, 1] \cap X : x \in X\}$; then $S = S_0 \cup S_1$ is a subbase for the discrete topology on X and neither of the players has a winning strategy in the game PO(S, X).

Corollary

There exists a space $X \subset [0, 1]$ such that PO(X) is determined on X but PO(S, X) is undetermined for some subbase S of X. In particular, Pawlikowski's characterization does not hold for the game PO(S, X).

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Measurable cardinals

Recall that a cardinal κ is called measurable if there exists a free σ -complete ultrafilter on κ .

Theorem

If κ is a measurable cardinal and X is a discrete space of cardinality κ , then Player O has a winning strategy in the game PO(S, X) for any subbase S of the space X.

Open problems

- Suppose that X is a discrete space such that Player O has a winning strategy in the game PO(S, X) for every subbase S in X. Must the cardinality of X be measurable?
- Suppose that X is a discrete space of cardinality 2^c. Does there exist a subbase S in X for which Player O has no winning strategy in the game PO(S, X)?
- Does there exist a pseudocompact space X such that the games PO(X) and PO(S, X) are not equivalent for Player O for some subbase S in the space X?

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