2010

Parity Periodicity: An Eliminative Approach to the Collatz Conjecture

Austin J. Phillips
Ouachita Baptist University

SENIOR THESIS APPROVAL

This Honors thesis entitled

Parity Periodicity: An Eliminative Approach to the Collatz Conjecture

written by

Austin J. Phillips

and submitted in partial fulfillment of the requirements for completion of the Carl Goodson Honors Program meets the criteria for acceptance and has been approved by the undersigned readers.

(Name) thesis director

(Name) second reader

(Name) third reader

honors program director

April 21, 2010

PARITY PERIODICITY: AN ELIMINATIVE APPROACH TO THE COLLATZ CONJECTURE

AUSTIN PHILLIPS

ABSTRACT. The $3n+1$ Conjecture states that when the Collatz function is applied repeatedly to an initial value, the sequence of values generated always converges to 1, regardless of the starting value. This paper strengthens the claim that all such sequences are convergent by showing that certain types of nonconvergent sequences cannot exist. Specifically, no sequence with parity-periodic values can exist. This eliminates all possible nontrivially periodic sequences and all divergent sequences with periodic parity. Therefore, if a counterexample to the conjecture exists, it must be a divergent sequence whose values display no parity periodicity.

1. INTRODUCTION

Many of the most alluring topics and problems in mathematics consist of a simple process that generates complex or unexpected results. The $3n+1$ problem, also known as the Collatz Conjecture (among other names), is one such problem [1]. The process involved is simple: start with any positive integer n and apply the *Collatz function*:

$$C(n) = \begin{cases} 3n+1 & \text{if } n \text{ is odd} \\ \dfrac{n}{2} & \text{if } n \text{ is even,} \end{cases}$$

producing another positive integer. Repeating this algorithm with each successive output generates a sequence $H(n)$, the *Hailstone Sequence* of n, where

$$\begin{aligned} H(n) &= \{n, C(n), C(C(n)), C(C(C(n))), \dots\} \\ &= \{n, C(n), C^2(n), C^3(n), \dots\}. \end{aligned}$$

The process stops as soon as it reaches the value 1. For example,

$$H(5) = \{5, 16, 8, 4, 2, 1\}$$

$$\text{and} \qquad H(7) = \{7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1\}.$$

Observing hailstone sequences over a broad range of starting values leads to the main conjecture: for all starting values, the process reaches 1 after a finite number of steps. Formally,

Conjecture 1 (3n + 1 Conjecture).

For all $n \in \mathbb{Z}^+$, there exists some $k \in \mathbb{Z}^+$ such that $C^k(n) = 1$.

While the conjecture has been verified for roughly all $n < 10^{17}$, the general case remains unproven [2]. Statistical and heuristic arguments provide some evidence that the conjecture is true [1]. What complicates the $3n+1$ problem is that even though the Collatz function is simple, each hailstone sequence follows a seemingly random path, and there are few immediately discernible patterns among hailstone sequences ranging over a large set of starting values (*see Figures 1, 2, & 3*). These seemingly random paths are reminiscent of the way hailstones rise and fall unpredictably in storm clouds before inevitably falling to the ground, inspiring the name of the sequences. Essentially, the task of this problem is to show that all hailstones do indeed fall eventually.

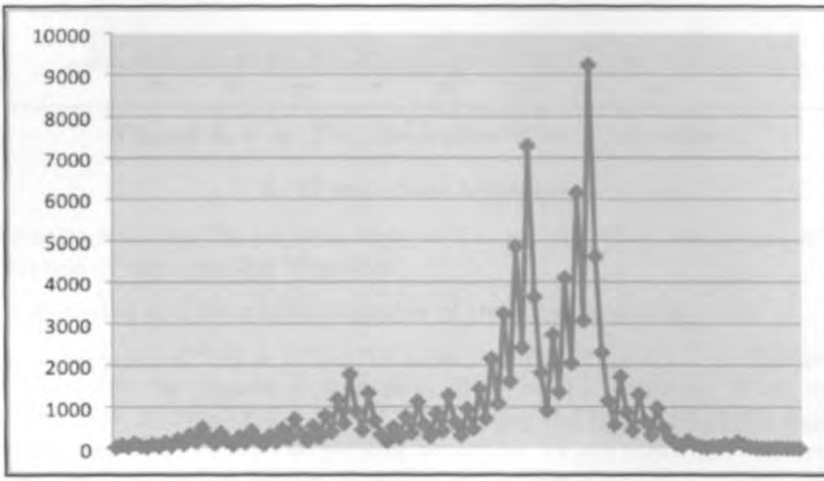

Figure 1: Graph of $H(n)$ for $n = 27$.

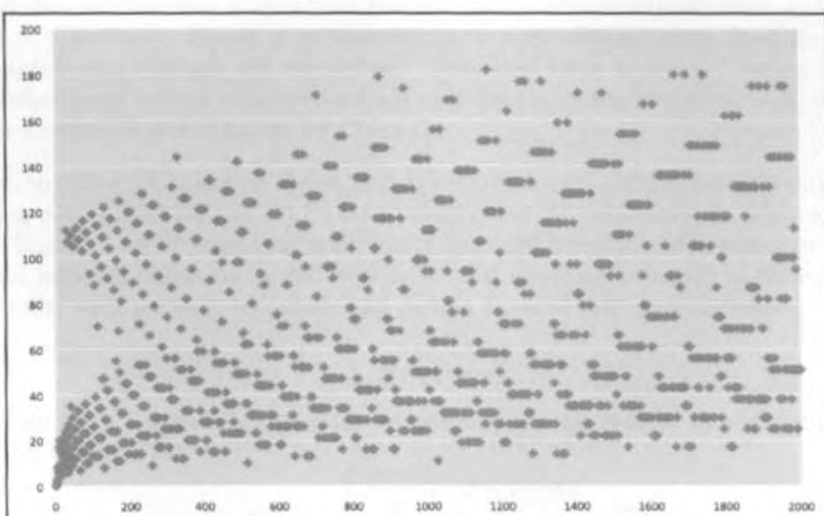

Figure 2: n vs. $L(n)$, the length of $H(n)$.

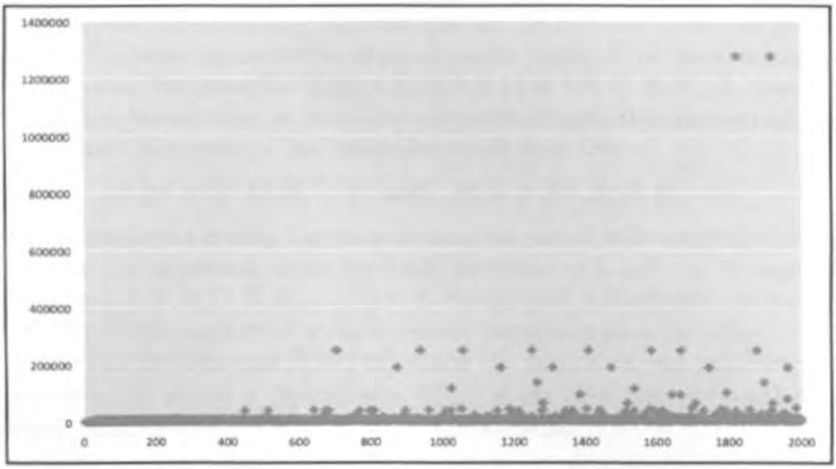

Figure 3: n vs. $P(n)$, the highest value $H(n)$ attains.

2. Eliminative Approach

To begin analyzing the problem, observe that any given hailstone sequence must exhibit one of three possible behaviors:

(1) It *converges* to 1 after a finite number of steps (notated $H(n) \to 1$).

(2) It is *periodic*—$C^a(n) = C^b(n)$ for some positive integers $a > b$. Notice that since we define the process to stop at 1, $H(1) = \{1\}$. However, if we remove that condition, $H(1) = \{1, 4, 2, 1, 4, 2, \ldots\}$, forming the *trivially periodic sequence*. From now on, when we refer to periodic sequences, we are referring to *nontrivially periodic sequences*, which neither contain nor converge to 1.

(3) It *diverges* ($H(n) \to \infty$). For any given value x a sequence attains, there are a finite number of positive integers less than x. So eventually the sequence must either exceed x or reach a value that has already occurred in the sequence, in which case it is periodic. Hence, a nonconverging, nonperiodic sequence must increase without bound (though not strictly so). Note that since $H(n) \to 1$ means $H(n)$ actually equals 1 eventually, we need not consider the case where $H(n) \to a$, $a \neq 1$. Such a sequence would fall under (2) or (3).

Since the $3n+1$ Conjecture states that (1) is true for all positive integers n, eliminating possibilities (2) and (3) for all n would prove the conjecture. In other words, showing that no divergent or nontrivially periodic hailstone sequences can exist would force all sequences to converge to 1. Our approach succeeds in eliminating all nontrivially periodic sequences and certain types of divergent sequences.

3. Notation

Since $3n+1$ is even if n is odd, we can shorten the process by redefining the Collatz function as

$$C(n) = \begin{cases} \dfrac{3n+1}{2} & \text{if } n \text{ is odd} \\ \dfrac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

We will use this definition of $C(n)$ from now on. In addition, it will be important to express hailstone sequences in terms of parity (using E for even values and O for odd values). For example, $H(5) = \{5, 8, 4, 2, 1\} \equiv \{\text{O, E, E, E, 1}\}$. Our process of elimination focuses first on sequences of infinite length that contain odd values spaced periodically apart. A few examples might look like

$$H(n) = \{O, O, O, \dots\} \quad \text{and} \quad H(n) = \{O, E, O, E, \dots\}.$$

Define the *odd period* p of such sequences to be the period with which the odd values occur. The two sequences above have odd periods $p = 1$ and $p = 2$, respectively. For $H(n) = \{O, E, E, O, E, E, \dots\}$, $p = 3$. Notice that p itself does not have to be odd; we are simply concerned with sequences containing periodic odds.
Finally, define the sequence $F_p^k(n)$ to denote the successive odd numbers in a sequence with odd period p. Specifically, $F_p^k(n) \equiv$ the $(k+1)$st odd number in the sequence, starting with $F_p^0(n) = n$. So in the sequence

$$H(n) = \{n, E, O_1, E, O_2, E, O_3, E, \dots\}$$

we have

$$F_2^0(n) = n, \quad F_2^1(n) = O_1, \quad F_2^2(n) = O_2, \quad F_2^3(n) = O_3,$$

and so on.

4. Applying the Eliminative Approach

Notice that if $H(n) \to 1$, then $H(n)$ looks something like

$$H(n) = \{n, \dots, E_1, E_2, \dots, E_k, 1\}, \tag{1}$$

where $E_1, \dots, E_k$ is a sequence of powers of 2. Once $H(n)$ reaches a power of 2, it immediately collapses to 1, and $H(n)$ can only reach 1 by going through 8, 4, and 2. (If we consider the *reverse mapping* of n—the values m for which $C(m) = n$—then 1 only maps to 2, which only maps to 4, which only maps to 8.) As a result, the problem of eliminating nontrivially periodic and divergent sequences is equivalent to showing that there exist no sequences $H(n)$ that cannot be expressed in the form of (1), for some $E_1, \dots, E_k$.

A logical class of sequences to consider first would be those with some odd period p. Apart from the trivial cycle, they must either be nontrivially periodic or divergent since they are infinite in length. For now, we will assume that these sequences begin with odd n values; by eliminating them, we also eliminate all sequences containing a tail of periodically spaced odds. (For example, eliminating $H(n) = \{O, E, O, E, \dots\}$ would also eliminate $H(n) = \{E, O, E, O, \dots\}$.) In addition, we are only considering sequences with a single odd period. Later, we will discuss sequences containing a composition of odd periods, such as

$$H(n) = \{O, E, O, O, E, O, O, E, O, \dots\}.$$

We now assume that such a hailstone sequence with odd period p exists and show it leads to a contradiction. To begin, consider the sequence

$$H(n) = \{O, O, O, \dots\},$$

which contains an infinite sequence of odds, starting with n. Obviously, it is not convergent. Since each term is odd, we have the recurrence relation

$$\begin{aligned} F_1^{k+1}(n) &= C(F_1^k(n)) \\ &= \frac{3F_1^k(n)+1}{2}, \qquad F_1^0(n) = n. \end{aligned}$$

By iterating this sequence, it appears that, in general,

$$\begin{aligned} F_1^k(n) &= \frac{3^k n + (3^{k-1} + 3^{k-2}2^1 + 3^{k-3}2^2 + \cdots + 3^1 2^{k-2} + 2^{k-1})}{2^k} \\ &= \frac{3^k n + \frac{3^k - 2^k}{3-2}}{2^k} \\ &= \frac{3^k(n+1) - 2^k}{2^k} \\ &= \left(\frac{3}{2}\right)^k (n+1) - 1 \qquad \text{for } k = 0, 1, 2, \ldots \end{aligned}$$

We can conduct a similar process for $p = 2$ ($H(n) = \{O, E, O, E, \ldots\}$), since

$$\begin{aligned} F_2^{k+1}(n) &= C(C(F_2^k(n))) \\ &= \frac{C(F_2^k(n))}{2} \\ &= \frac{3F_2^k(n)+1}{4}. \end{aligned}$$

By iterating again, it appears that:

$$F_2^k(n) = \left(\frac{3}{4}\right)^k (n-1) + 1 \qquad \text{for } k = 0, 1, 2, \ldots$$

And similarly for $p = 3$ ($H(n) = \{O, E, E, O, E, E, \ldots\}$), it appears

$$F_3^k(n) = \left(\frac{3}{8}\right)^k \left(n - \frac{1}{5}\right) + \frac{1}{5}.$$

Notice that there is a pattern between p and the generalized form for $F_p^k(n)$, which leads us to our first theorem.

Theorem 1. *For a sequence $H(n)$ with odd period $p \in \mathbb{Z}^+$,*

$$F_p^k(n) = \left(\frac{3}{2^p}\right)^k \left(n - \frac{1}{2^p - 3}\right) + \frac{1}{2^p - 3}, \qquad k = 0, 1, 2, \ldots$$

Proof. Notice that for $p = 1$ and $k = 0$,

$$F_1^0(n) = 1 \cdot \left(n + \frac{1}{2}\right) - \frac{1}{2} = n$$

in agreement with our definition.

We now induct on k, showing that, for a constant p,

$$F_p^k(n) = \left(\frac{3}{2^p}\right)^k \left(n - \frac{1}{2^p - 3}\right) + \frac{1}{2^p - 3}$$

$$\text{implies} \qquad F_p^{k+1}(n) = \left(\frac{3}{2^p}\right)^{k+1} \left(n - \frac{1}{2^p - 3}\right) + \frac{1}{2^p - 3}.$$

Assume, for a given p and k, that

$$F_p^k(n) = \left(\frac{3}{2}\right)^k \left(n - \frac{1}{2^p - 3}\right) + \frac{1}{2^p - 3}.$$

Since $H(n)$ has odd period p, the next odd after $F_p^k(n)$ is

$$F_p^{k+1}(n) = \frac{\left(\frac{3F_p^k(n)+1}{2}\right)}{2^{p-1}} = \frac{3F_p^k(n) + 1}{2^p},$$

since we apply $\frac{3n+1}{2}$ to the odd $F_p^k(n)$ and then $\frac{n}{2}$ to the $(p-1)$ subsequent even values. Expanding this, we find that

$$\begin{aligned}
F_p^{k+1}(n) &= \frac{3\left[\left(\frac{3}{2^p}\right)^k \left(n - \frac{1}{2^p-3}\right) + \frac{1}{2^p-3}\right] + 1}{2^p} \\
&= \frac{\left(\frac{3^{k+1}}{2^{pk}}\right)\left(n - \frac{1}{2^p-3}\right) + \left(\frac{3}{2^p-3} + 1\right)}{2^p} \\
&= \left(\frac{3^{k+1}}{2^{pk+p}}\right)\left(n - \frac{1}{2^p-3}\right) + \frac{2^p}{2^p(2^p-3)} \\
&= \left(\frac{3^{k+1}}{2^{p(k+1)}}\right)\left(n - \frac{1}{2^p-3}\right) + \frac{1}{2^p-3} \\
&= \left(\frac{3}{2^p}\right)^{k+1}\left(n - \frac{1}{2^p-3}\right) + \frac{1}{2^p-3}.
\end{aligned}$$

Therefore, for a constant p,

$$F_p^k(n) = \left(\frac{3}{2^p}\right)^k \left(n - \frac{1}{2^p - 3}\right) + \frac{1}{2^p - 3}, \quad k = 0, 1, 2, \ldots$$

Finally, notice that for all p,

$$F_p^0(n) = \left(\frac{3}{2^p}\right)^0 \left(n - \frac{1}{2^p - 3}\right) + \frac{1}{2^p - 3} = n,$$

in accordance with our definition.

We have shown that $F_p^0(n)$ is true for all p, and that $F_p^k(n)$ implies $F_p^{k+1}(n)$ for all k. Since the base case with $k = 0$, $p = 1$ is true, we conclude that in general

$$F_p^k(n) = \left(\frac{3}{2^p}\right)^k \left(n - \frac{1}{2^p - 3}\right) + \frac{1}{2^p - 3} \quad \text{for all } k, p.$$

□

We now show that this formulation of $F_p^k(n)$ excludes the existence of hailstone sequences with any odd period p.

Theorem 2. *There exists no positive integer n such that $H(n)$ contains an infinite sequence with any odd period p.*

Proof. Assume for contradiction that there does exist some n where $H(n)$ contains some odd period p. Then by Theorem 1,

$$F_p^k(n) = \left(\frac{3}{2^p}\right)^k \left(n - \frac{1}{2^p-3}\right) + \frac{1}{2^p-3} \qquad \text{for all } k.$$

If $H(n)$ is truly a hailstone sequence, then for each $k = 0, 1, 2, \ldots,$ $F_p^k(n)$ must equal some positive integer, denoted m_k (which varies with k). In other words, for every k there must exist some m_k such that

$$\left(\frac{3}{2^p}\right)^k \left(n - \frac{1}{2^p-3}\right) + \frac{1}{2^p-3} = m_k,$$

or

$$3^k \left(n - \frac{1}{2^p-3}\right) = 2^{pk} \left(m_k - \frac{1}{2^p-3}\right).$$

Multiplying through by $(2^p - 3)$,

$$3^k \left[(2^p-3)\, n - 1\right] = 2^{pk} \left[(2^p-3)\, m_k - 1\right].$$

Substituting $a = (2^p - 3)\, n - 1$ and $b = (2^p - 3)\, m_k - 1$,

$$3^k a = 2^{pk} b.$$

Remember that a depends only on p and n and is therefore constant, whereas b depends on p and m_k, which varies with k.

Thus, it must be true that:

Given a, for every k there exists some b such that $3^k a = 2^{pk} b$.

Now let $a = 2^\ell r$, where 2^ℓ is the highest power of 2 that divides a ($\ell = 0, 1, 2, \ldots$). Then $2 \nmid r$.

Now we have:

For every k there exists some b such that $3^k 2^\ell r = 2^{pk} b$.

However, once $k > \frac{\ell}{p}$, we require:

There exists some b such that $3^k r = 2^{pk-\ell} b$.

Since $pk - \ell > 0$, 2 divides the right side of the equation but not the left side, and we have reached a contradiction. Therefore, it is false that there exists some

integer b for every k, and consequently it is false that there exists some integer m_k for every k.
Hence, the assumption that there exists some $H(n)$ with odd period p (for *any* p) is false. □

Note that the problem isn't that the above equation can't be satisfied; the problem is that it can't be satisfied with positive integers. Interestingly, this argument is compatible with the existence of the trivial sequence $H(1) = \{1, 2, 1, 2, \dots\}$. The equation

$$3^k a = 2^{pk} b$$

is satisfiable with integers only if $a = 0$ (just let $b = 0$ for all k). Since we assigned $a = (2^p - 3)n - 1$, we have:

$$0 = (2^p - 3)n - 1,$$

$$\text{or} \qquad n = \frac{1}{2^p - 3},$$

and n is an integer only if $(2^p - 3) \mid 1$, giving $p = 1$ or $p = 2$ (corresponding to $n = -1$ and $n = 1$, respectively). Since the parameters of the problem do not allow $n = -1$, we conclude that the trivial sequence with $n = 1$ and $p = 2$ is the only periodic sequence of this kind.

5. Generalizing the Approach

Theorems 1 & 2 show that no such $H(n)$ sequence exists with a *single* odd period (e.g., $H(n) = \{O, E, E, E, O, E, E, E, \dots\}$), but there still remain a number of possible sequences with more complex periodicity. For example, the above theorems would not apply to sequences containing a composition of two or more odd periods such as

$$H(n) = \{O, E, O, E, E, O, E, O, E, E, \dots\}$$

$$\text{or} \quad H(n) = \{O, O, E, O, O, E, \dots\}$$

Fortunately, we can use the same techniques to disprove the existence of *any* periodic hailstone sequence. Moreover, we will show that no sequences with periodic *parity* can exist, which also rules out divergent sequences with a recurring sequence of evens and odds.

First, suppose there exists some parity-periodic hailstone sequence (i.e., $H(n)$ contains some sequence of even and odd parities that repeats indefinitely). Redefine $F^k(n)$ in the following way: $F^0(n) = n$, the beginning of the repeated parity sequence; $F^1(n)$ is the next value at which the parity sequence begins to repeat; and so on. For example, in the sequence

$$H(n) = \{\underbrace{\mathbf{n}, O, O, E, O, E, O, O, E,}_{\text{parity sequence}} \underbrace{\mathbf{O_1}, O, O, E, O, E, O, O, E,} \mathbf{O_2}, \dots\},$$

$F^0(n) = n$, $F^1(n) = O_1$, $F^2(n) = O_2$, etc. While in this case the starting value n is odd, we no longer require that to be true. Nor do we suppose any kind of pattern within each parity sequence; the only requirement is that the parity sequence repeats indefinitely.

Notice that the process of iterating from $F^k(n)$ to $F^{k+1}(n)$ reduces to some composition of the functions $f(n) = \frac{3n+1}{2}$ and $g(n) = \frac{n}{2}$. This fact enables us to develop another recurrence relation among the $F^k(n)$ values.

Lemma 1.

For all k, $F^{k+1}(n) = \dfrac{3^a F^k(n) + b}{2^c}$ *for some nonnegative integers* a, b, c.

Proof. It will suffice to show an even stronger statement: that every element in a hailstone sequence can be expressed in the form

$$C^k(n) = \frac{3^a n + b}{2^c} \qquad \text{for some } a, b, c.$$

For $k = 0$, we have $\langle a, b, c\rangle = \langle 0, 0, 0\rangle$; for $k = 1$, we have

$$C^1(n) = \frac{3^a n + b}{2^c}$$

so $\langle a, b, c\rangle = \langle 1, 1, 1\rangle$ if n is odd and $\langle 0, 0, 1\rangle$ if n is even.

Now assume for some k that

$$C^k(n) = \frac{3^a n + b}{2^c}.$$

If $C^k(n)$ odd, then

$$C^{k+1}(n) = \frac{3C^k(n) + 1}{2} = \frac{3^{a+1} n + (3b + 2^c)}{2^{c+1}},$$

which is in the required form. If $C^k(n)$ is even, then

$$C^{k+1}(n) = \frac{C^k(n)}{2} = \frac{3^a n + b}{2^{c+1}},$$

also in the required form. Thus, every element in a given hailstone sequence can be expressed in this form. In particular (going back to our parity-periodic sequence), $F^{k+1}(n)$ is an element of $H(F^k(n))$, and therefore can be expressed in the given form, proving Lemma 1. □

Given this convenient recurrence relation among the $F^k(n)$ values, we simply iterate the $F^k(n)$ sequence starting with $F^0(n) = n$. Since we are assuming the sequence is parity-periodic, the same values of a, b, and c will describe the iteration from any $F^k(n)$ to the next, and therefore a, b, and c will remain constant. Through iteration, we can derive a general form for $F^k(n)$.

Theorem 3. *If $H(n)$ is a parity-periodic hailstone sequence where*

$$F^{k+1}(n) = \frac{3^a F^k(n) + b}{2^c} \quad \textit{for some } a,\ b,\ c,$$

then

$$F^k(n) = \left(\frac{3^a}{2^c}\right)^k \left(n + \frac{b}{3^a - 2^c}\right) - \frac{b}{3^a - 2^c}, \quad k = 0, 1, 2, \ldots$$

Note the similarity to Theorem 1.

Proof. For $k=0$, $F^0(n)=n$. Now assume for some k that

$$F^k(n)=\left(\frac{3^a}{2^c}\right)^k\left(n+\frac{b}{3^a-2^c}\right)-\frac{b}{3^a-2^c}.$$

Then

$$\begin{aligned}
F^{k+1}(n) &= \frac{3^aF^k(n)+b}{2^c}\\
&= \frac{3^a\left[\left(\frac{3^a}{2^c}\right)^k\left(n+\frac{b}{3^a-2^c}\right)-\frac{b}{3^a-2^c}\right]+b}{2^c}\\
&= \frac{3^{ak+a}}{2^{ck+c}}\left(n+\frac{b}{3^a-2^c}\right)+\frac{\left(\frac{-3^ab}{3^a-2^c}+b\right)}{2^c}\\
&= \frac{3^{a(k+1)}}{2^{c(k+1)}}\left(n+\frac{b}{3^a-2^c}\right)+\left(\frac{-3^ab+3^ab-2^cb}{2^c(3^a-2^c)}\right)\\
&= \left(\frac{3^a}{2^c}\right)^{k+1}\left(n+\frac{b}{3^a-2^c}\right)-\frac{b}{3^a-2^c}.
\end{aligned}$$

□

Now that we have a generalized form for $F^k(n)$, we can use the same technique to eliminate the existence of any parity-periodic sequence.

Theorem 4. *There exists no positive integer n such that $H(n)$ contains an infinite parity-periodic sequence.*

Proof. Suppose for contradiction that there exists some n such that $H(n)$ has periodic parity. By Theorem 3,

$$F^k(n)=\left(\frac{3^a}{2^c}\right)^k\left(n+\frac{b}{3^a-2^c}\right)-\frac{b}{3^a-2^c},\quad k=0,1,2,\ldots$$

for some nonnegative integers a, b, and c. If $H(n)$ is a true hailstone sequence, then for each k, $F^k(n)$ must equal some integer m_k. Formally,

For each k there exists some m_k such that $\left(\frac{3^a}{2^c}\right)^k\left(n+\frac{b}{3^a-2^c}\right)-\frac{b}{3^a-2^c}=m_k$.

Rearranging, we have

$$3^{ak}\left(n+\frac{b}{3^a-2^c}\right)=2^{ck}\left(m_k+\frac{b}{3^a-2^c}\right).$$

And multiplying through by (3^a-2^c),

$$3^{ak}[(3^a-2^c)n+b]=2^{ck}[(3^a-2^c)m_k+b].$$

Substituting $x = (3^a - 2^c)n + b$ (which is constant) and $y = (3^a - 2^c)m_k$ (which is variable), we have:

Given x, for every k there exists some y such that $3^{ak}x = 2^{ck}y$.

Up to this point, the reasoning has been similar to that in Theorem 2. However, we must now divide the argument into two cases. First, assume that $x \neq 0$. Now let $x = 2^\ell r$, where 2^ℓ is the highest power of 2 that divides x, and $2 \nmid r$.

We require

$$3^{ak}2^\ell r = 2^{ck}y.$$

However, once $k > \frac{\ell}{c}$,

$$3^{ak}r = 2^{ck-\ell}y.$$

Since $ck - \ell > 0$, 2 divides the right side of the equation but not the left side, and we have reached a contradiction. Therefore, it is false that there exists some integer y for every k, and consequently it is false that there exists some integer m_k for every k. Hence, the assumption that there exists any such $H(n)$ is false, proving *there exist no hailstone sequences with periodic parity.* □

6. Future Work

This proof works perfectly under the condition that $x \neq 0$. Unfortunately, it is not clear that x is always nonzero. If $x = 0$, then the statement

Given x, for every k there exists some y such that $3^{ak}x = 2^{ck}y$

is satisfiable with integers (just let $y = 0$ for all k). Since $x = (3^a - 2^c)n + b$, we must have:

$$0 = (3^a - 2^c)n + b,$$

or

$$n = \frac{b}{2^c - 3^a}.$$

Proving Theorem 4 in general depends on this formulation of n being impossible or contradictory. Since n must be a positive integer, we could try to reach a contradiction by showing that the expression

$$\frac{b}{2^c - 3^a}$$

is either negative or not an integer. For example, by our definition of the parity-periodic sequence as

$$F^{k+1}(n) = \frac{3^a F^k(n) + b}{2^c},$$

the constant b must be positive since each $F^k(n)$ comes from some composition of $\frac{3n+1}{2}$ and $\frac{n}{2}$.

Now we can establish conditions for which the denominator $2^c - 3^a$ is negative, forcing n to be negative.

$$2^c - 3^a < 0$$

if

$$2^c < 3^a,$$

or

$$c < a \log_2 3.$$

Moreover, we can express a and c in terms of the number of even and odd iterations within a parity sequence. Specifically, if a given parity sequence consists of applying $\frac{3n+1}{2}$ a total of α times and $\frac{n}{2}$ a total of β times, then $a = \alpha$ and $c = \alpha + \beta$. Thus,

$$\alpha + \beta < \alpha \log_2 3$$

or

$$\beta < \alpha(\log_2 3 - 1).$$

This result gives us conditions under which a contradiction is reached. If a repeating parity sequence consists of α odds and β evens and $\beta < \alpha(\log_2 3 - 1)$, then the starting value n must be negative, which is impossible. We have therefore eliminated all parity-periodic hailstone sequences *except* those for which $\beta > \alpha(\log_2 3 - 1)$. The task remains to show that this latter inequality also produces a contradiction.

Proving Theorem 4 without having to assume $x \neq 0$ would be immensely helpful since it would eliminate all sequences whose actual values are periodic (i. e., all nontrivially periodic sequences) as well as divergent sequences whose values display periodic parity. If Theorem 4 is true, then any possible counterexample to the $3n+1$ Conjecture would have to be a divergent sequence whose values display random or nonperiodic parity.

References

[1] Jeffrey C. Lagarias, *The $3x + 1$ Problem and its Generalizations*, Amer. Math. Monthly **92** (1985), 3–23.

[2] Jeffrey C. Lagarias, *The $3x + 1$ Problem: An Annotated Bibliography, II* (2009), preprint available at `arxi.org/abs/math/0608208`.