2004

Pebbling on Directed Graphs

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Electronic Proceedings of Undergraduate Mathematics Day
2004, No. 1, http://www.udayton.edu/∽mathdept/EPUMD/EPMUD.htm

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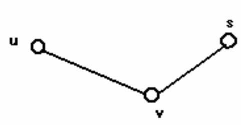
Submitted January 25th, 2004
Revised and submitted April 16th, 2004

Abstract

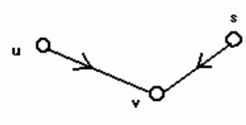
Consider a finite connected graph *G* whose vertices are labeled with non-negative integers representing the number of pebbles on each vertex. A pebbling move on a graph *G* is defined as the removal of two pebbles from one vertex and the addition of one pebble to an adjacent vertex. The pebbling number *f*(*G*) of a connected graph is the least number of pebbles such that any distribution of *f*(*G*) pebbles on *G* allows one pebble to be moved to any specified but arbitrary vertex. We consider pebbling on directed graphs and study what configurations of directed graphs allow for pebbling to be meaningful. We also obtain the pebbling numbers of certain orientations of directed wheel graphs (W_n) with odd order where $n \geq 6$ and directed complete graphs (K_n) with odd order where $n \geq 5$. *G* is said to be demonic if $f(G) = n$ where *n* is the order of *G*. We demonstrate the existence of demonic directed graphs and establish that the sharp upper bound and sharp lower bound of the pebbling numbers of the directed graphs is the same as that of the undirected graphs: $n \leq f(G) \leq 2^{n-1}$.

Pebbling

A *graph* is an ordered pair of sets (Vertices[V], Edges[E]) where V is non-empty and E is a set of pairs of elements of V. The number of vertices of a graph is called its *order*. The graph is said to be an *undirected graph* if the elements of E are unordered pairs and a *directed graph* if they are ordered pairs. Figure A below is an undirected graph and Figure B is a directed graph, both with *3* vertices and *2* edges.



A

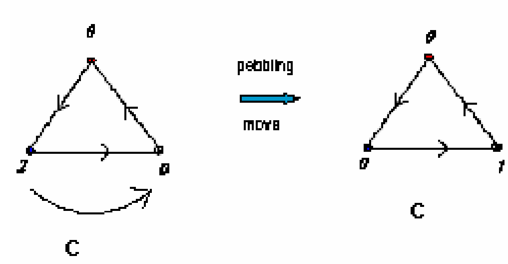


B

The edge or the ordered pair (*u*, *v*) in figure B *goes from u to v* and we also say that *u is adjacent to v*, or *v is adjacent from u*.

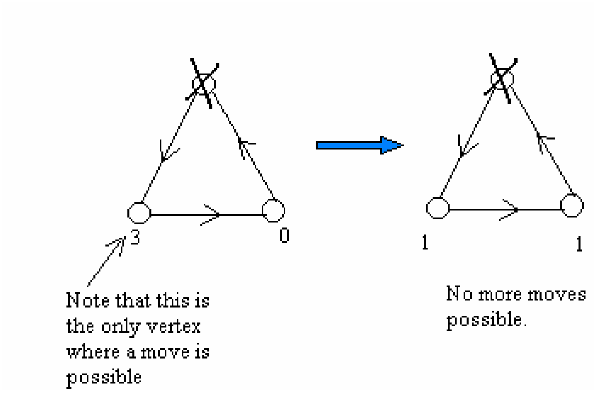
Consider a graph *G*. Assign nonnegative integers to the vertices of G. If the vertex *v* is associated with the integer label *m*, we say that *m pebbles are placed on v*. If the sum of all the integer labels on *G* is *n*, we say *n pebbles are distributed on G*.

A *pebbling move* is an operation of subtracting *2* from the label of a vertex, and adding *1* to the label of an adjacent vertex. Note that a pebbling move can only be made on a vertex with a label of *2* or more. Also note that a pebbling move on a graph with *n* pebbles distributed on it results in a distribution of *n-1* pebbles.

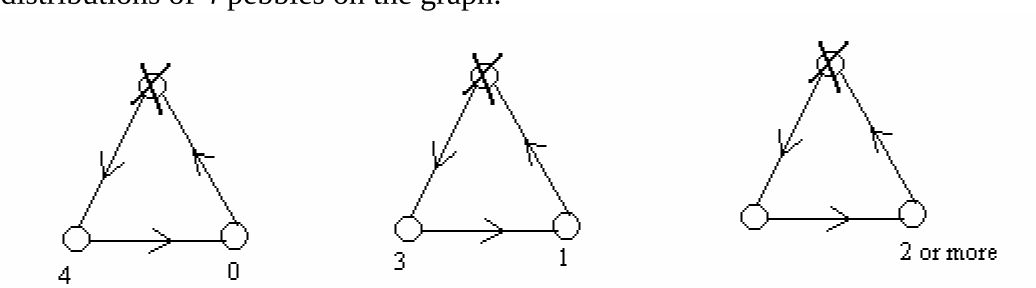


The *pebbling number f(v, G) of a vertex v* is the least number *m* such that if *m* pebbles are randomly distributed over the graph *G*, there is some series of pebbling moves by which a pebble can be placed on *v*.

The *pebbling number f(G) of a graph G* is the maximum of the pebbling numbers of all the vertices *v* in *G*. For example, one can determine the pebbling number of graph *C* to be *4*. It can be seen that every rotation of the graph results in the same graph and therefore every vertex of the graph is the same as any other vertex. It is then clear that every vertex would have the same pebbling number, which would then also be *f(G)*. Without loss of generality, assume that any one of them is the target vertex (marked with an X in the figure below). It can be seen easily that the following distribution of *3* pebbles does not allow one pebble to reach the target.



Therefore, *f(G)* has to be greater than *3*. To see if *f(G) =4*, consider the following possible distributions of *4* pebbles on the graph:



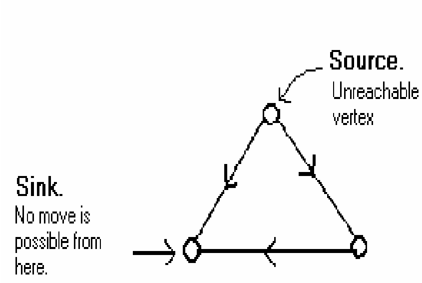
In each case, it is easy to see that a pebble can reach the target with a finite number of pebbling moves. Therefore, *f(G) =4.*

The number of pebbles on any vertex *u* is represented by *p(u)*.

Strong Graphs

A *(directed) walk* in a directed graph *D* is an alternating sequence of vertices and edges v_o, x_1, v_1,…, x_n, v_n in which each edge x_i is $v_{i-1}v_i$. A *path* is a walk in which all vertices are distinct. If there is a path from *u* to *v*, then *v* is said to be *reachable from u*. A directed graph is *strongly connected or strong* if every two vertices are mutually reachable (see ref[1], 206).

Pebbling on a directed graph is meaningful only if every vertex can be reached from any other vertex. Therefore, a directed graph must be strongly connected in order to do any pebbling. A *source* is a vertex which is only adjacent to other vertices; a *sink* is a vertex that is only adjacent from other vertices. It is easy to see that any graphs with sinks or sources are not strong and therefore pebbling cannot be done on such graphs.



An *orientation of a graph G* is any directed graph that results from an assignment of directions to the edges of *G*. It is important for us to know whether or not a particular orientation of a graph is strong before we start pebbling on it. The following results whose proofs can be found in [1] are useful in classifying graphs as strong or not strong.

Theorem 1

1. A graph *G* has an orientation that is strong if and only if *G* is connected and has no bridges. Therefore every edge must lie on a cycle.
2. A vertex is called a *leaf* if there is only one other vertex either adjacent to it or from it. A graph that contains any leaves is not strong.
3. A *closed walk* has the same first and last vertices and a *spanning walk* contains all the nodes of the graph. A directed graph is strong if and only if it has a closed spanning walk. A directed graph is *hamiltonian* if it has a closed spanning path. Every hamiltonian directed graph is strong. (The converse is not always true.)
4. Let *D* be a non-trivial directed graph of order *p*. If every pair of distinct vertices *u* and *v* with *u* not adjacent to *v* satisfies $od(u) + id(v) \geq p$, where $od(u)$ is the number of vertices adjacent from *u* and $id(v)$ is the number of vertices adjacent to *v*, then *D* is hamiltonian and therefore strong.

Before we look at the pebbling numbers of certain class of strongly connected directed graphs, we need to examine some previously found results regarding the pebbling properties of undirected graphs to see if they apply to directed graphs as well.

Pebbling On Undirected Graphs

There is a growing literature on pebbling on undirected graphs (see [4]). Let *G* be any undirected graph and *u*, *v* and *w* be vertices of *G*. One immediate consequence of the definition of pebbling number is that $p \leq f(G)$, where *G* has order *p*, the number of vertices of *G*. To see this, consider the distribution where $p-1$ pebbles are distributed on the vertices of *G* with no pebbles on the target, and exactly one pebble on every vertex other than the target. Then no pebbling moves are possible since no vertex has two or more pebbles. So $p-1 < f(G)$. A graph whose pebbling number is equal to its number of vertices is called a *demonic graph*.

If u is a distance d from v, and $2^d - 1$ pebbles are placed on u, and these are all the pebbles on the graph, then no pebble can be moved to v. A shortest u-v path is called *a u-v geodesic*. The *diameter of G* is the length of any longest geodesic. So it is clear that $f(G) \geq \max \{|V(G)|, 2^d\}$, where $|V(G)|$ is the number of the vertices of G, and d is the diameter of the graph G. It is also easy to see that $f(G) \leq 2^{|V(G)|-1}$.

We can conclude that there is a range of values that the pebbling number of an undirected graph G, with order p, takes on: $p \leq f(G) \leq 2^{p-1}$. In [5], Jessup shows the existence of several demonic graphs, so we know that p is the greatest lower bound. Similarly, it has been shown that $f(P_n)=2^{n-1}$ where P_n is a path graph on n vertices, so 2^{p-1} is the least upper bound on the range of $f(G)$.

Pebbling on Directed Graphs

Let G denote a graph and let G_D denote a strongly connected directed graph with the same number of vertices and the same edges as G. Following the arguments similar to the case of an undirected graph, $p \leq f(G_D) \leq 2^{p-1}$, where p is the number of vertices of G_D.

A natural question that arises is what are the greatest lower bounds and least upper bounds for the pebbling number of any directed graph? In other words, are there any directed graphs of order p whose pebbling numbers are equal to p or 2^{p-1}? By showing directed graphs whose pebbling numbers are equal to p, we would be demonstrating the existence of demonic directed graphs. Later in this paper, we show that a certain orientation of odd-ordered complete directed graphs are demonic.

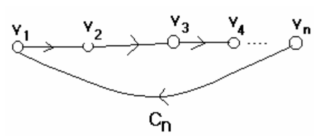
It should be noted that $f(G) \leq f(G_D)$ since if G_D can be pebbled with r pebbles then so can G by following the same pebbling moves.

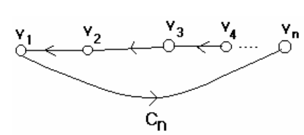
We now explore the pebbling numbers of various classes of directed graphs.

Cycle Graphs, C_n

A *cycle graph*, C_n, is a directed graph on n vertices $\{v_1, v_2, \ldots, v_n\}$ where every vertex v_i $(i<n)$ is either adjacent to v_{i+1} or adjacent from it and v_n is adjacent to v_1 or adjacent from it.

Note that the only two strong orientations on a cycle graph, C_n, are the following:





Note that for any vertex, v_i, indegree (v_i) = outdegree (v_i) = 1.

It is obvious that any orientation of C_n other than those shown above creates sources or sinks.

For undirected cycles, $f(C_{2k}) = 2^k$ and $f(C_{2k\_1}) = 2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 1$ (see [4]).

Theorem 2. Let C_n be a cycle graph of order n, with a strong orientation. Then $f(C_n) = 2^{n-1}$.

Proof. The longest path between any vertex and any other vertex in C_n has length $n-1$. Each pebbling move costs 2 pebbles at the vertex of origin. Therefore for a pebble to travel to the target vertex from any vertex, we need a minimum of 2^{n-1} pebbles. Since the pebbling number of any graph with n vertices cannot be greater than 2^{n-1}, then $f(C_n)$ must be equal to 2^{n-1}.

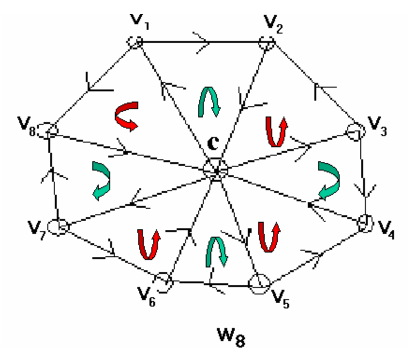
We just showed an example of a directed graph where the pebbling number is equal to 2^{n-1} where n is the order of the graph. We can conclude that the least upper bound of the pebbling number of any directed graph is 2^{n-1}.

Alternating Wheel Graphs, W_n

We define an *alternating wheel*, W_n to be a directed wheel graph whose order is $n+1$, where n is even and with the following properties:

- There is one central vertex, c.
- The remaining n vertices $v_1, v_2, \ldots, v_n$ are called the outer vertices and v_i is adjacent to c if i is even, while c is adjacent to v_i if i is odd.
- The only other edges are as follows:
  v_i is adjacent to both v_{i-1} and v_{i+1} if i is odd, for $3 \leq i \leq n-1$; and v_1 is adjacent to v_n.

The following diagram shows an example of an alternating wheel.

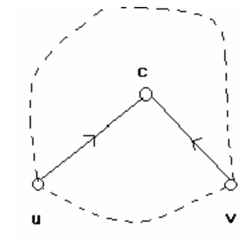


By definition, the number of outer vertices is even and therefore there are an even number of triangles that contain the vertex c. It should also be noted that each triangle forms a cycle and that these cycles alternate in direction as shown in the figure. For the central vertex c, $id(c) = od(c)$, where $od(c)$ is the number of vertices adjacent from c and $id(c)$ is the number of vertices adjacent to c. For any outer vertex v_i, $od(v_i) = id(v_i) \pm 1$, depending on whether v_i is adjacent from c or adjacent to c.

Another property to be noted is that the diameter of an alternating wheel is 4. The proofs that these properties hold are fairly straightforward and are left for the reader.

Lemma 1. Consider an alternating wheel W_n. Let u and v be any two outer vertices that are both adjacent to c. If $p(u) + p(v) \geq 2t+1$, then t pebbles can travel to the center c and there will be at least one pebble left on either u or v.

Proof. We use induction on t to prove this lemma. Consider the case when $t = 1$. This implies that $p(u) + p(v) \geq 2(1) + 1 = 3$. This means that either u or v has at least 2 pebbles (by the pigeonhole principle). Thus, one pebble can reach c since both u and v are adjacent to c.



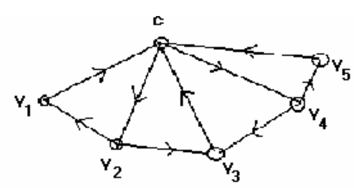
For the induction step, let us assume that the t^{th} case holds and show that the $(t+1)^{st}$ case works. We need to show that if $p(u) + p(v) \geq 2(t+1)+1 = 2t+3$, then $t+1$ pebbles can reach c with one pebble left on u or v. Since $2t+3 = (2t+1)+2$, by the induction hypothesis, the $2t+1$ pebbles guarantee us that t pebbles can reach c and one of these pebbles is left on u or v. So, we now have a total of 3 pebbles remaining on u and v and we still need to send one more pebble to c. But this is simply the basis case, so one more pebble can reach c and one pebble is left on u or v.

Alternating Fan Graphs, F_k

An *alternating fan*, F_k, is a directed fan graph whose order is $k+1$, where k is odd and with the following properties:

- There is one central vertex, c.
- The remaining k vertices $v_1, v_2, \ldots, v_k$ are called the outer vertices and v_i is adjacent to c if i is odd, while c is adjacent to v_i if i is even.
- The only other edges are as follows:
  v_i is adjacent to both v_{i+1} and v_{i-1} if i is even, for $2 \leq i \leq k-1$.

An example of a fan graph is shown below:



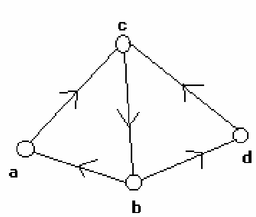
F_5

It is important to notice that an alternating fan graph is a subgraph of an alternating wheel graph. This fact will play an important role later in finding the pebbling number of an alternating wheel.

Lemma 2. In an alternating fan graph F_k, $k - 3 + 4t$ pebbles are sufficient for t pebbles to reach c for $k \geq 3$. In particular, $k+1$ pebbles are sufficient for one pebble to reach c in F_k.

Proof. We employ the principle of mathematical induction on k.

Consider the basis case when $k=3$. Let the three outer vertices in F_3 be a, b, d as shown in the figure below.



We need to show that $3-3+4t = 4t$ pebbles are sufficient to send t pebbles to c. The only four possible distributions of pebbles on a and d are:

(a) $p(a) + p(d) = 0$
(b) $0 < p(a) + p(d) \leq 2$
(c) $2 < p(a) + p(d) < 2t + 1$
(d) $2t+1 \leq p(a) + p(d) \leq 4t$

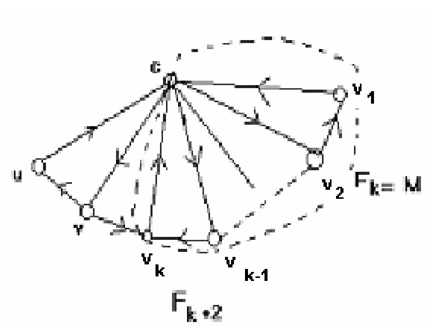
Case (a): If $p(a) + p(d) = 0$ then that means that $p(b) = 4t$. It is obvious that $4t$ pebbles are sufficient to send t pebbles to c via the path b-a-c.

Case (b): If $0 < p(a) + p(d) \leq 2$, then that means that either $p(a) \geq 1$ or $p(d) \geq 1$ (by pigeonhole principle). Without loss of generality, let us assume that $p(a) \geq 1$. Since $p(a) + p(d) \leq 2$, b must have at least $4t\text{-}2 = 2(2t\text{-}1)$ pebbles. This means that $(2t\text{-}1)$ pebbles can be sent to a; $p(a)$ is now at least $2t$, which is enough to send t pebbles to c.

Case (c): If $2 < p(a) + p(d) < 2t + 1$, then either $p(a) + p(d) = 2(t\text{-}m)+1$ or $p(a) + p(d) = 2(t\text{-}m)+2$ for some integer m such that $0<m<t$ (if $t=m$, we have case (b)). In either case, we know by Lemma 1 that $t\text{-}m$ pebbles can be sent to c. If a and d have either $2(t\text{-}m)+1$ or $2(t\text{-}m)+2$ pebbles, then b must have the remaining either $4t\text{-}2(t\text{-}m)\text{-}1$ or $4t\text{-}2(t\text{-}m)\text{-}2$ pebbles. Therefore either $p(b) = 2(t+m\text{-}1)+1$ or $p(b) = 2(t+m\text{-}1)$, and $p(b) \geq 2(t+m\text{-}1) \geq 4m$ (since $t>m$). So m pebbles can be sent to c.

Case (d): When $p(a) + p(d) \geq 2t + 1$, we know t pebbles can reach c by Lemma 1.

For the induction step, assume that the result holds for some positive integer k and prove it for $(k+2)$, since k must be odd. Note that it is not the $(k+1)^{st}$ case because of the fact that the number of outer vertices has to be odd by the definition of a fan. We need to show that $(k+2)\text{-}3+4t = k\text{-}1+4t$ pebbles are sufficient to send t pebbles to the center. The outer vertices of F_{k+2} can be partitioned into two sets: $\{v_1, v_2, \ldots, v_k\}$ and $\{v_{k+1}, v_{k+2}\}$. If v_{k+1} and v_{k+2} are removed along with their edges, it is easy to see that F_k is left, which means that F_k is embedded within F_{k+2}. This is where the induction step comes into play.



Let v denote v_{k+1} and u denote v_{k+2} and M denote the embedded F_k. Consider any distribution of $k\text{-}1+4t$ pebbles on the outer vertices of F_{k+2}. Focusing on the number of pebbles on u and v, the proof breaks down into two cases:

(1) $p(u) + p(v) \geq 4t$
(2) $p(u) + p(v) < 4t$

Case (1): When $p(u) + p(v) \geq 4t$ then t pebbles can reach c via the path v-u-c path.

Case (2): $p(u) + p(v) < 4t$ implies one of four possibilities:

(a) $p(u)+p(v) = 4(t\text{-}m)+1$
(b) $p(u)+p(v) = 4(t\text{-}m)+2$
(c) $p(u)+p(v) = 4(t\text{-}m)+3$
(d) $p(u)+p(v) = 4(t\text{-}m)+4$, where m is some positive integer, $t > m$.

In each of these possibilities, the number of pebbles on u and v is enough to send $t\text{-}m$ pebbles to c via the path v-u-c, with some extra pebbles left over on $\{u, v\}$. So, in each case, we need to show that m pebbles can be sent to c from the pebbles on $v_1, v_2, \ldots, v_k$.

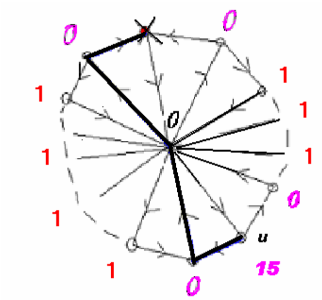
Case 2(a) & (b): If either $p(u)+p(v) = 4(t-m)+1$ or $p(u)+p(v) = 4(t-m)+2$, then M has at least $k+4m-3$ pebbles. By the induction hypothesis, m pebbles can be sent to c.

Case 2(c): If $p(u)+p(v) = 4(t-m)+3$, the number of pebbles on M is $k+4m-4 = k + 4(m-1) > k+4(m-1)-3$. We can send $(t-m)$ pebbles to c from the $4(t-m)+3$ pebbles on u and v, leaving at least 3 pebbles on $\{u, v\}$. So either u has at least 2 pebbles or v has at least 2 pebbles. If u has at least 2 pebbles, then one of those can be sent to c and by the induction hypothesis, $m-1$ pebbles from M can be sent to c. If v has at least 2 pebbles then one pebble can be sent to v_k.. M now has $(k+4m-4)+1= k+4m-3$ pebbles and by the induction hypothesis, we can send m pebbles to c.

Case 2(d): If $p(u)+p(v) = 4(t-m)+4 = 4(t-m+1)$, then the number of pebbles on M is $((k+2)-3+4t)-(4(t-m+1))= k-1+ 4t- 4t+ 4m- 4= k+ 4m- 5= k+4(m-1)-1 > k+4(m-1)-3$. So, $(t-m+1)$ pebbles can be sent from $\{u, v\}$ to c, while by the induction hypothesis, $m-1$ pebbles can be sent from M to c. Hence a total of t pebbles can be sent to c.

Theorem 3. Let W_n be an alternating wheel graph. Then $f(W_n)=10+n$ for $n \geq 6$.

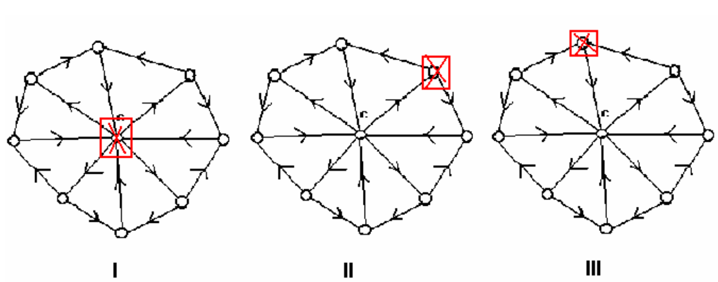
Proof. We demonstrate a distribution of $n+9$ pebbles that does not allow a pebble to reach a specified target. Let us consider the following distribution of $n+9$ pebbles on W_n:



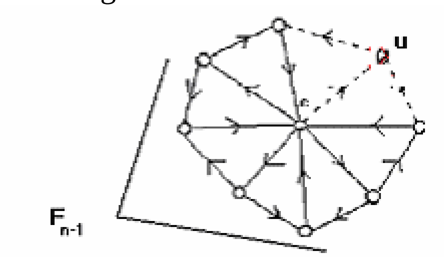
The chosen target is a vertex that is adjacent to c and is represented with an X in the figure. The vertices adjacent from the target vertex have no pebbles on them as indicated in the figure. A vertex, u, that is not adjacent to the target and is adjacent from the center has 15 pebbles. The central vertex c and the vertices adjacent from u have no pebbles on them. The remaining n-6 vertices each have one pebble on them.

The only vertex where a pebbling move is possible is u since all the other vertices have fewer than 2 pebbles. The only way to reach the target is via the path of length four shown in the figure. Fifteen pebbles will not suffice to send one pebble to the target.

Now let us consider any distribution of $10+n$ pebbles on W_n. There are only three types of possible target vertices: the central vertex and the two types of outer vertices, where one is adjacent from c and the other is adjacent to c. The three types of targets are illustrated below and are marked with an X.



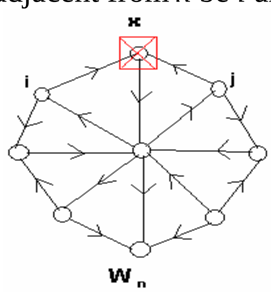
Case 1: Let c be the target. A distribution of $10+n$ pebbles implies that there is at least one outer vertex with 2 or more pebbles on it. If that outer vertex is adjacent to c, one pebble can reach c. If that outer vertex, say u, is adjacent from c, then there are two possibilities. If $p(u) \geq 4$ then one pebble can reach c. If $p(u) < 4$, then the rest of the graph has more than $6+n$ pebbles. Consider what happens when the vertex u and its edges are removed. The remaining graph is F_{n-1} as shown in the figure below.



Since $6+n > n$, by Lemma 2 we can send one pebble to c.

Case 2: Consider the case when the target is an outer vertex that is adjacent from c. Since the only way to get a pebble to the target is through c, we need at least 2 pebbles on c. Suppose there are fewer than 2 pebbles on c. Consider the subgraph where the target and its edges are removed like in case 1. The remaining graph is F_{n-1}. The outer vertices of the subgraph have at least $9+n$ pebbles. By Lemma 2, we need $n+4$ pebbles to send two pebbles to c.

Case 3: Consider the case when the target is an outer vertex, x, that is adjacent to c. Let the neighboring vertices that are adjacent from x be i and j, as shown in the figure below.



The last edge of any path to the target must be either i-x or j-x. If $p(i) \geq 2$ or $p(j) \geq 2$, then one pebble can be moved to x. We need only consider the case where $p(i)<2$ and $p(j)<2$.

If we delete i, j, and x and the edges associated with them, we are left with an alternating fan graph, F_{n-3}. By Lemma 2, if the number of pebbles on F_{n-3} is $(n-3)-3+4t$, then we can move t pebbles to c. The number of pebbles on this F_{n-3} is $(10+n)-p(i)-p(j)$.

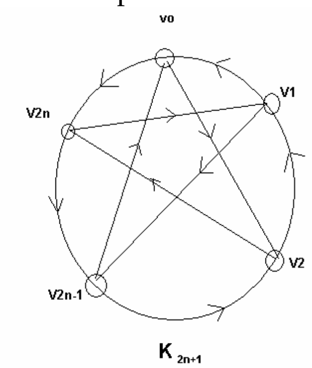
If $(10+n)-p(i)-p(j) \geq (n-3)-3+4t$, then we can move t pebbles to c. This condition simplifies to $16-p(i)-p(j) \geq 4t$. If $p(i)+p(j)=0$, then we can move four pebbles to c, and using the path c-i-x we can move one pebble to the target.

If $0<p(i)+p(j) \leq 2$, then either one or both of i and j have a pebble. Then we only need two pebbles to be moved to c to use c-j-x or c-i-x to put one pebble on x. So $16-p(i)-p(j) \geq 14 \geq 4(3)$, indicating that we can move three pebbles to c from the vertices of F_{n-3}.

It is a known fact that undirected wheel graphs (on any number of vertices) are demonic [3]. We have shown that $f(G) < f(G_D)$ where G is a wheel with more than 7 vertices. We now demonstrate a class of graphs for which $f(G) = f(G_D)$.

Alternating Complete Graphs, K_{2n+1}

We define an *alternating complete graph*, K_{2n+1}, to be a directed graph with an odd number of vertices, $\{v_o, v_1, v_2, \ldots, v_{2n}\}$ and we say that v_i is adjacent to v_j if and only if $(i-j)$ mod $2n+1$ is odd. An example of an alternating complete graph is shown below.



Note that at each vertex the direction of the edges incident with it alternate in direction as illustrated in the figure above. Note also that any rotation of the alternating complete graph gives us an isomorphic graph.

Theorem 4. $f(K_{2n+1}) = 2n+1$ for $n \geq 2$.

Proof. We use induction on n.

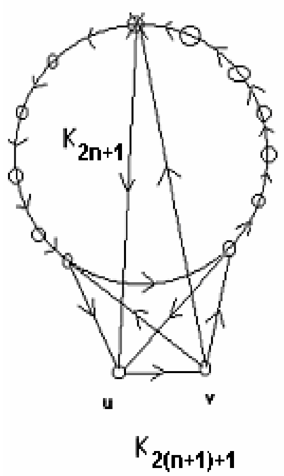
Consider K_5 with five pebbles on its vertices, and v_0 as target. If $p(v_1) \geq 2$, we can move one pebble to v_0. So $p(v_1) < 2$. Consider $A = \{v1, v2\}$ and $B = \{v3, v4\}$. Without loss of generality A has at least 3 pebbles on its vertices.

If the number of pebbles on A is at least four, then we can move one pebble to x using the path v_2-v_1- v_0.

If $p(v_1)+ p(v_2) = 3$, and $p(v_1)=1$, then we use the path v_2-v_1-v_0 to move one pebble to v_0.

If, however, $p(v_1)= 0$ and $p(v_2)= 3$, then B has two pebbles on its vertices. If $p(v_3)=2$, move one pebble from v_3 to v_2, and A now has four pebbles. If $p(v_4)=2$, move one pebble to v_1, and A now has four pebbles. If $p(v_3)=p(v_4)=1$, move one pebble from v_2 to v_4, so there are two pebbles at v_4. Move one pebble to v_3, giving us two pebbles at v_3, and then move one pebble to v_0.

For the induction step, assume that the result holds for K_{2n+1} and show that it holds for $K_{2(n+1)+1}$. Consider a distribution of $(2n+1)+1$ pebbles on $K_{2(n+1)+1}$ with v_o as the target. Let $u = v_i$ and $v = v_{i-1}$ such that v_o is adjacent to $u = v_i$ as shown below.



If u, v and the edges associated with them are removed, then it can easily be checked that the resulting graph is the alternating complete graph K_{2n+1}, which we will denote M. (Note that two edges are removed at each vertex.

Assume that there are $2(n+1)+1$ pebbles on $K_{2(n+1)+1}$. If M has $2n+1$ pebbles on it, then by the induction hypothesis, one pebble can be moved to v_0. Also if $p(u)+p(v) \geq 4$, then we can move one pebble to v_0 using the path u-v-v_0.

So suppose that M has fewer than $2n+1$ pebbles on it. Then $p(u)+p(v) > 2$, and $p(u)+p(v) < 4$, or $p(u)+p(v)=3$. If $p(v) \geq 1$, then using the path u-v-v_0, we can move one pebble to v_0. So, assume $p(u)= 3$ and $p(v)= 0$. Then M has $2n$ pebbles on it. Move one pebble from u to any vertex v_r in M such that u is adjacent to v_r. (We are guaranteed that such a v_r exists since $id(u) = od(u)$ and u is adjacent to or from at least two vertices in M.) Now M has $2n+1$ pebbles on it, and one pebble can be moved to v_0 by the induction hypothesis.

Hence alternating complete graphs are demonic. It is known that undirected complete graphs on any number of vertices are demonic[2].

The results obtained in this paper allow us to conclude that the bounds of the pebbling number of a directed graph on n vertices are the same as that of an undirected graph: $n \leq f(G_D) \leq 2^{(n-1)}$ where n is the order of a graph G_D.

Open Questions

We conclude this paper with some open questions on pebbling on directed graphs.

- What are other classes of directed graphs which are demonic?
- Does Graham's conjecture [$f(G_D \times H_D) \leq f(G_D) \times f(H_D)$],where $G_D \times H_D$ is the Cartesian product of two graphs, hold?
- What are the optimal pebbling numbers of directed graphs? The optimal pebbling number of a graph G, $f_{opt}(G)$, is the least number such that there exists a solvable distribution of $f_{opt}(G)$ pebbles on G.
- If G_{D1} and G_{D2} are strongly connected directed graphs with the same set of vertices and edges with different orientations, then under what conditions is $f(G_{D1}) = f(G_{D2})$?

Acknowledgments

We thank the University Berry Scholars Program for their continuing support. We are grateful for the helpful comments made by the referee(s).

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