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Partha P. Banerjee
University of Dayton, pbanerjee1@udayton.edu

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Signal Flow Analysis

Partha P. Banerjee
University of Dayton

Signal flow graphs are a viable alternative to block diagrammatic representation of a system. What makes signal flow graphs attractive is that certain features from graph theory can be applied to the simplification and the synthesis of complex systems.

100.1 Introduction

The relationship between the input and output of a certain system can be represented in terms of a *block diagram.* The block diagram represents the operator that operates on the input to produce the output, and can be represented either in the time domain or in the Laplace domain for a time-dependent input and output. The relationship between the input and the output in the Laplace domain is called the *transfer function* of the system. In this case, the input is the independent variable and the output is the dependent variable. Sometimes, when there are intermediate dependent variables, the relationships between each other as well as the input(s) and output(s) can also be represented by block diagrams. Alternatively, instead of block diagrams, the dependent variables and the inputs can be denoted as *nodes,* and connections or *paths* between the nodes can denote the mathematical operator linking the two variables or nodes. This is used to draw what is called a *signal flow graph.* A simple diagram representing the similarities between a block diagram and a signal flow graph is shown in Figure 100.1.

100.2 Signal Flow Graphs for Feedback Systems

In many systems, there is feedback (positive or negative) from the output to the input. Negative feedback makes a system more stable, while positive feedback causes a system to become unstable and is the principle behind the operation of oscillators. Feedback is depicted in a block diagram through a feedback transfer function $G(s)$ between the output and the input, as shown in Figure 100.2. Note that in this case, $U(s)$ is the input, and the output $X_2(s)$ is fed back to the input through $G(s)$. The input $X_1(s)$ to $H(s)$ can be expressed as

$$X_1(s) = U(s) - X_2(s)G(s) \tag{100.1}$$

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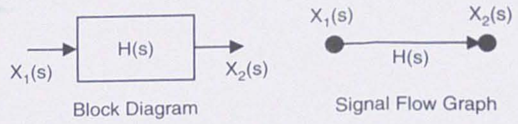

FIGURE 100.1 Connection between a block diagram and a signal flow graph.

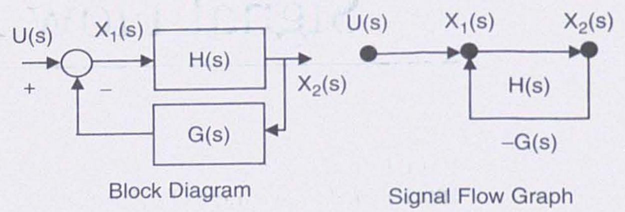

FIGURE 100.2 Connection between a block diagram and a signal flow graph for a system with negative feedback.

where

$$X_2(s) = X_1(s)H(s) \tag{100.2}$$

The same is depicted in the signal flow graph drawn on the right. Note that upon manipulating Equation (100.1) and Equation (100.2), a direct relationship can be found between the input $U(s)$ and the output $X_2(s)$ as

$$X_2(s) = \frac{H(s)}{1+G(s)H(s)}U(s) = H_{eq}U(s) \tag{100.3}$$

In other words, the feedback system represented by the block diagram in Figure 100.2 can be reduced to a block diagram similar to Figure 100.1, where the input is now $U(s)$ and the transfer function relating the output $X_2(s)$ to the input is now $H_{eq}(s)$, defined in Equation (100.3). This is shown in Figure 100.3. The equivalent signal flow graph also reduces to a form similar to Figure 100.1, with $X_1(s)$ replaced by $U(s)$ and $H(s)$ replaced by $H_{eq}(s)$, as shown in Figure 100.3. This also suggests that in a signal flow graph, it may be possible to reduce the number of nodes through a systematic node elimination procedure. This is facilitated through using Mason's theorem for reduction of systems, to be described below.

In general, signal flow graphs may be more complicated and comprise nodes, paths, and *loops*. An example of a *feedback loop* appears in the signal flow graph of Figure 100.2; however, *self-loops* are possible as well. An example of a more complicated signal flow graph involving many loops is shown in Figure 100.4. This signal flow graph corresponds to the set of equations

$$\begin{aligned} X_1(s) &= U(s) - X_2(s)G_1(s)\,, \\ X_2(s) &= X_1(s)H_1(s) + X_2(s)G_2(s)\,, \\ Y(s) &= X_2(s)H_2(s)\,. \end{aligned} \tag{100.4}$$

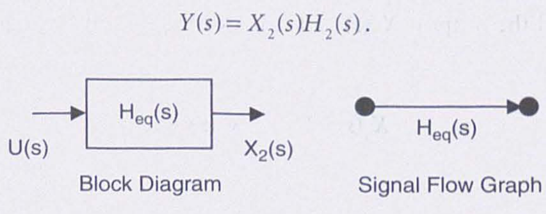

FIGURE 100.3 Reduced block diagram and corresponding signal flow graph from Figure 100.2.

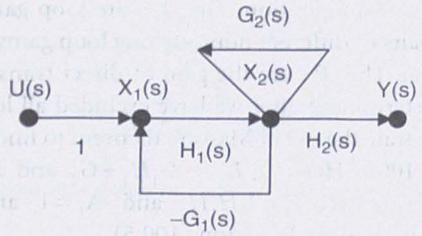

FIGURE 100.4 A more complicated signal flow graph corresponding to the system of equations showing feedback loops including self-loops.

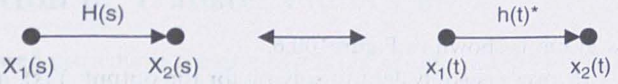

FIGURE 100.5 Equivalence between signal flow graphs in Laplace and time domains.

One can readily go from the set of equations to the signal flow graph and vice versa.

In passing, we would like to point out that the second of the relations in Equation (100.4) above can be rewritten in the form

$$X_2(s) = X_1(s)\frac{H_1(s)}{1-G_2(s)}$$

This implies that self-loops can be eliminated by dividing all incoming path gains by $1-G_i(s)$, where $G_i(s)$ is the self-loop gain for the node X_i. Other types of simplification of signal flow graphs, such as *node elimination,* are discussed later. We would like to remind readers that signal flow graphs can be drawn for signals depicted in either the Laplace domain or the time domain. The time domain equivalent of the signal flow graph in Figure 100.4 would involve the same nodes and loops, except that the nodes would be depicted as $u(t), x_1(t), x_2(t), y(t)$, which are the inverse Laplace transforms of $U(s), X_1(s), X_2(s), Y(s)$, respectively, and the loops would correspond to operators in the time domain such as $h_1(t), h_2(t), g_1(t), g_2(t)$, which are the inverse Laplace transforms of $H_1(s), H_2(s), G_1(s), G_2(s)$, respectively. Figure 100.5 shows the equivalence between the signal flow graphs in the Laplace and time domains. It should be noted that multiplication in the Laplace domain corresponds to convolution in time, denoted as a * in Figure 100.5.

For instance, if $H_1(s) = s$, then $h_1(t) = d[\delta(t)]/dt$, and it can be shown from the properties of convolution that $h_1(t) * x_1(t) = [d[\delta(t)]/dt] * x_1(t) = dx_1(t)/dt$, so that the operator $h_1(t)* = d/dt$. If $H_1(s) = c$, a constant, then $h_1(t)* = c$, which is the same multiplicative constant.

100.3 Reduction of Signal Flow Graphs

We now enunciate *Mason's theorem* for reduction of a signal flow graph. It states that the equivalent transfer function from input $U(s)$ to output $Y(s)$ can be written as

$$H_{eq}(s) = Y(s)/U(s) = \sum_i P_i \Delta_i / \Delta \qquad (100.5)$$

where

$$\Delta = 1 - \sum L_j + \sum{}' L_k L_l - \sum{}' L_m L_n L_o + \ldots \qquad (100.6)$$

is the determinant of the feedback configuration. The L_i s are loop gains; ΣL_j is the sum of all loop gains; $\Sigma' L_k L_l$ is the sum of all pairs of different nontouching loop gains, etc. Two loops are nontouching if they have no nodes in common. The P_i s are the gain of direct transmittances from the input to the output. Also Δ_i is the system determinant after we have excluded all loops that touch the P_i path.

As an example, we will demonstrate the use of Mason's theorem to find the equivalent transfer function for the system shown in Figure 100.4. Here, $\Sigma L_j = -G_1 H_1 + G_2$, and all higher order sums in (100.6) are zero, so that $\Delta = 1 + G_1 H_1 - G_2$. Also, $P_1 = 1.H_1 H_2$ and $\Delta_1 = 1$, and there is only one direct path from the input to the output. Hence, using Equation (100.5),

$$H_{eq}(s) = Y(s)/U(s) = \frac{H_1 H_2}{1 - G_2 + G_1 H_1} \tag{100.7}$$

The reduced signal flow graph is shown in Figure100.6.

In retrospect, Mason's theorem is equivalent to solving for the output $Y(s)$ in terms of the (known) input $U(s)$ from a set of linear algebraic equations of the form $\underline{A}\underline{X} = \underline{B}$. According to Cramer's rule, the solution for the j-th component of the vector $\underline{X}$ is $Xj = |\underline{Dj}| / |\underline{A}|$, where the matrix D_j has $\underline{B}$ as its j-th column, and the corresponding columns of $Y(s)$ as its other columns. Upon applying this to the example depicted in Figure 100.4, we see that the dependent variables X_1, X_2, Y can be solved by first rewriting (100.4) in the form of a vector-matrix equation of the type $\underline{A}\underline{X} = \underline{B}$ as

$$\begin{bmatrix} 1 & G_1 & 0 \\ H_1 & 1-G_2 & 0 \\ 0 & H_2 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ Y \end{bmatrix} = \begin{bmatrix} U \\ 0 \\ 0 \end{bmatrix} \tag{100.8}$$

the solution for Y, using Cramer's rule is

$$Y = \begin{vmatrix} 1 & G_1 & U \\ H_1 & 1-G_2 & 0 \\ 0 & H_2 & 0 \end{vmatrix} / \begin{vmatrix} 1 & G_1 & 0 \\ H_1 & 1-G_2 & 0 \\ 0 & H_2 & 1 \end{vmatrix} = \frac{H_1 H_2}{(1-G_2) + H_1 G_1} U, \tag{100.9}$$

which yields the result for the equivalent transfer function identical to Equation (100.7) above.

The reduction of the signal flow graph shown above using Mason's theorem can also be achieved through a repeated *node elimination* process. The rules of elimination are as follows. Assume that we would like to eliminate the node X_2. First, we need to eliminate the self-loop around X_2. The self-loop of gain G_2 is eliminated by dividing all incoming transmittances by $1 - G_2$. This makes the transmittance from X_1 to X_2 equal to $H_1/(1-G_2)$. Now, in the reduced signal flow diagram, all nodes except X_2 are drawn, and all original branches not entering or leaving X_2 are inserted. Finally, we add branches representing every possible path (in the signal flow diagram without self-loops) through X_3. For instance, we can go from X_1 to X_1 through $-H_1 G_1/(1-G_2)$, and X_1 to Y through $H_2 G_1/(1-G_2)$. The reduced signal flow graph is shown in Figure 100.7.

A similar procedure can be used to eliminate the node X_1. As before, it entails first removing the self-node at X_1, which makes the transmittance from U to X_1 equal to $(1-G_2)/(1-G_2+H_1 G_1)$. Upon now eliminating X_1, the transmittance from U to Y is $[(1 - G_2)/(1 - G_2 + H_1 G_1)] \times [H_2 G_1/(1 - G_2)]$

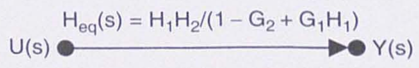

FIGURE 100.6 Equivalent reduced signal flow graph derived from Figure 100.4.

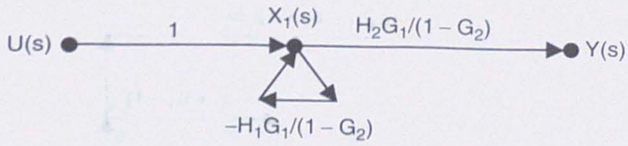

FIGURE 100.7 Reduction of the signal flow graph in Figure 100.4 through elimination of node X_2.

$= H_2G_1/(1 - G_2 + H_1G_1)$, which is identical to the equivalent transfer function $H_{eq}(s)$ in Equation (100.7) derived using Mason's theorem. The resulting signal flow graph is as shown in Figure 100.6.

100.4 Realization of Transfer Functions

Thus far, we have discussed procedures for simplifying signal flow graphs to derive the transfer function of the system. We will now turn our attention to synthesizing signal flow graphs given a particular transfer function. Using the $H_{eq}(s)$ in Equation (100.7) as an illustration, assume that

$$H_1(s) = s + a_1,\ H_2(s) = a_2,\ G_1(s) = s + b_1, G_2(s) = s + b_2\ . \tag{100.10}$$

Then

$$H_{eq}(s) = \frac{Y(s)}{U(s)} = \frac{a_2 s + a_1 a_2}{s^2 + (a_1 + b_1 - 1)s + (a_1 b_1 - b_2 + 1)}\ . \tag{100.11}$$

As is often the case, the degree of the polynomial in the denominator is equal to or greater than the degree of the polynomial in the numerator. Then the degree of the polynomial in the denominator is defined as the *order* of the system, and is equal to the number of *states* of the system. We can therefore define two state variables $X_1(s), X_2(s)$ for the system, related through $X_1(s) = X_2(s)/s$, or equivalently, $X_2(s) = sX_1(s)$. In the time domain this implies that $\chi_2(t) = d\chi_1(t)/dt$, where $\chi_1(t), \chi_2(t)$ are the inverse Laplace transforms of $X_1(s), X_2(s)$, respectively. Conversely, $\chi_1(t)$ is the integral of $\chi_1(t)$.

For convenience, Equation (100.11) is reexpressed in the form

$$H_{eq}(s) = \frac{Y(s)}{U(s)} = \frac{a_2/s + a_1 a_2/s^2}{1 + (a_1 + b_1 - 1)/s + (a_1 b_1 - b_2 + 1)/s^2}\ . \tag{100.12}$$

Since this is a second-order system, one needs two integrators. The integrator outputs are called $X_1(s), X_2(s)$, and the integrator inputs are called $X_1'(s), X_2'(s)$, respectively, as shown in Figure 100.8.

The second step is the realization of the denominator in Equation (100.12). Since Mason's theorem states that all loops that touch have a $\Delta = 1 - \sum L_j$, it is convenient to construct loops incorporating feedback which have a node in common, viz., $X_1(s)$, and with feedback loop gains equal to $-(a_1 + b_1 - 1)/s$ and $-(a_1 b_1 - b_2 + 1)/s^2$, as illustrated by the dashed lines in Figure 100.9. Finally, to construct the numerator, we ensure that all direct paths also pass through one node, viz., $X_1(s)$. The path gains are a_2/s and $a_1 a_2/s^2$, as illustrated by the dotted lines in Figure 100.9.

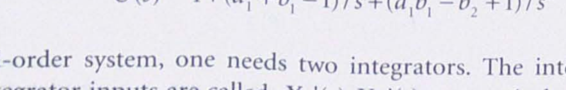

FIGURE 100.8 First step in the realization of the transfer function in Equation (100.12).

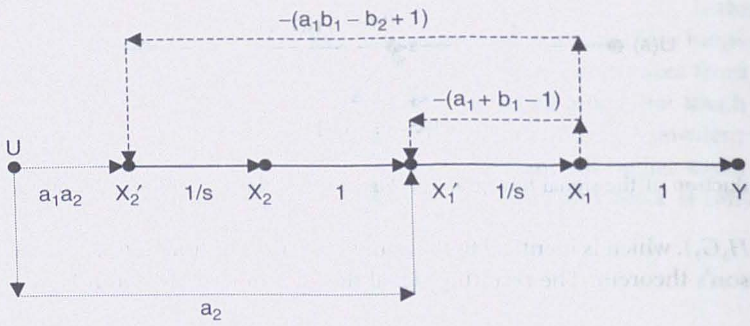

FIGURE 100.9 Signal flow graph for the transfer function in Equation (100.12).

The signal flow graph shown in Figure 100.9, also called *Type I* realization, is not unique. *Type II* is an alternate realization that assumes that all feedback loops and parallel paths go through $X_2'(s)$ rather than $X_1(s)$. Finally, *Type III* involves a realization that is based on first decomposing Equation (100.12) into partial fractions in the form

$$H_{eq}(s) = \sum a_i/(s+b_i),\ b_i \in \Re \tag{100.13}$$

by first factorizing the denominator. This yields a realization of the transfer function in terms of parallel loops. In cases where the denominator has complex roots, it can be decomposed into partial fractions involving sums of terms as in Equation (100.11) and Equation (100.13). Details can be found in Truxal (1972).

100.5 Boundary Conditions and Signal Flow Graphs

Signal flow graphs can be suitably adapted to incorporate initial conditions imposed on a certain state. For instance, assume that in the time domain, states $x_1(t), x_2(t)$ are related through the set of coupled differential equations as

$$\begin{aligned} dx_1(t)/dt &= x_2, \\ dx_2(t)/dt &= -a_2x_1 - a_1x_2 \end{aligned} \tag{100.14}$$

where a_1, a_2 are constants. Equation (100.14) is the state variable formulation of a second order ODE of the form $d^2x/dt^2 + a_1dx/dt + a_2x = 0$. With the definitions $x_1 = x, x_2 = dx_1/dt$, this ODE can be rewritten as Equation (100.14). Note also that Equation (100.14) can be recast in the form

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{100.15}$$

which is a special case of the vector ODE

$$d\underline{x}/dt = \underline{A}\underline{x} + \underline{B}\underline{u} \tag{100.16a}$$

Together with the output equation

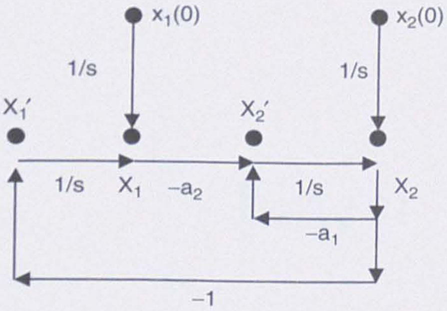

FIGURE 100.10 Signal flow diagram for realization of the system modeled by Equation (100.15).

$$\underline{y} = \underline{C}\,\underline{x} + \underline{D}\,\underline{u} \tag{100.16b}$$

one can describe the behavior of the entire linear system.

Upon Laplace transforming Equation (100.14), we get

$$\begin{aligned} sX_1(s) + X_2(s) &= x_1(0), \\ a_2X_1(s) + (s + a_1)X_2(s) &= x_2(0). \end{aligned} \tag{100.17}$$

Similar to the way the signal flow graph from the transfer function in Equation (100.12) was realized, we can draw the signal flow diagram for Equation (100.17), as shown in Figure 100.10.

100.6 Conclusion

We have summarized the salient points of signal flow graphs, their reduction, and their synthesis. As seen from the discussions above, they are an analogue to block diagrams in the analysis of linear systems. In some cases, signal flow graphs can give valuable information about the *controllability* and *observability* of linear systems as well. Loosely speaking, a state is controllable if it can be changed by an appropriate set of inputs. A state is observable if the output(s) depend on the particular state. However, formal tests for controllability and observability can be made on the basis of the matrices $\underline{A}, \underline{B}, \underline{C}, \underline{D}$ defined in Equation (100.16) above. This is outside the scope of this chapter.

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