1-1967

Vectors: Problems Worked for Math

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Problems Worked For
Math H 491
Special Studies
Honors Program

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Pages : 1—28

Honors Paper #31

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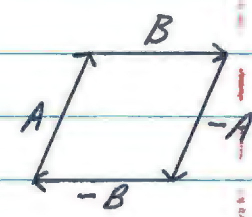
1. A, B are vectors forming consecutive sides of a parallelogram. Find the vectors forming the other two sides.

$-A$

$-B$



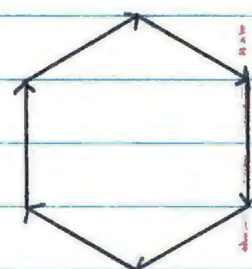
2. A, B are vectors forming consecutive sides of a regular hexagon. Find the vectors forming the other four sides.

$B-A$

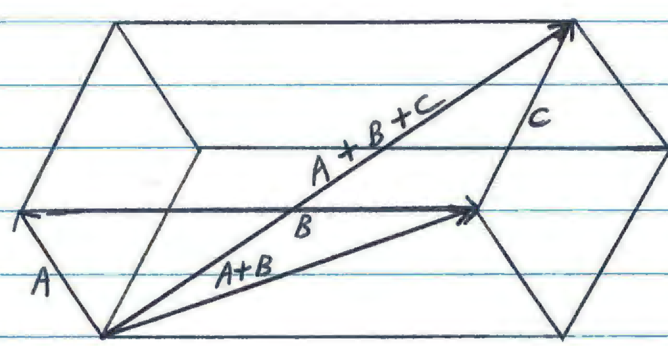
$-A$

$-B$

$A-B$



5. The vectors A, B, C form coterminous edges of a parallelepiped. Show that $A+B+C$ is equal to a vector diagonal



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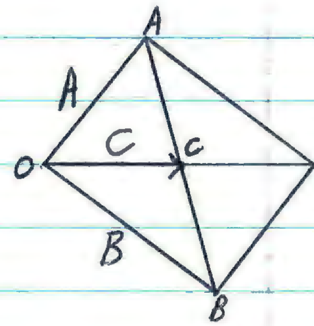
6. If A, B are vectors from an origin O to the points A, B, find the vector, C, from O to the middle point of AB.



$$\frac{BC}{BA} = \frac{1}{2}$$

$$\frac{C-B}{A-B} = \frac{1}{2}$$

$$C - B = \frac{1}{2}(A-B)$$

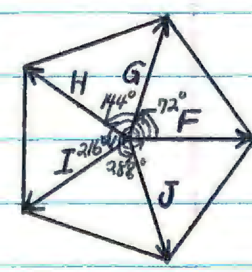
$$C = \frac{1}{2}(A-B) + B$$

$$C = \frac{1}{2}A - \frac{1}{2}B + B$$

$$C = \frac{1}{2}A + \frac{1}{2}B$$

$$C = \frac{1}{2}(A+B)$$

8. From the center of a regular pentagon vectors are drawn to its vertices. Show that their sum is zero.



$$\vec{F} = 1\,F_x + 0\,F_y$$

$$\vec{G} = .30902\,G_x + .95106\,G_y$$

$$\vec{H} = -.80902\,H_x + .58779\,H_y$$

$$\vec{I} = -.80902\,I_x - .58779\,I_y$$

$$\vec{J} = .30902\,J_x - .95106\,J_y$$

$$F_x + G_x + H_x + I_x + J_x = 0 \qquad |F| = |G| = |H| = |I| = |J| = 1$$

$$F_y + G_y + H_y + I_y + J_y = 0$$

$$\therefore \vec{F} + \vec{G} + \vec{H} + \vec{I} + \vec{J} = 0$$

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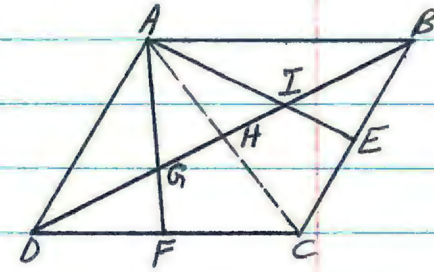

9. $\frac{DF}{DC} = \frac{1}{2}$

$\frac{CE}{CB} = \frac{1}{2}$

AF, DH Medians $\triangle ACD$

$\frac{HG}{GD} = \frac{1}{2}$

$HG = \frac{1}{3} HD$

$3HG = HD$

BH, AE Medians $\triangle ABC$

$\frac{HI}{IB} = \frac{1}{2}$

$HI = \frac{1}{3} HB$

$HD = HB$

$HI = \frac{1}{3} HD$

$BD = BH + HD$

$BD = HD + HD$

$BD = 2HD$

$BD = 2(3HG)$

$BD = 6HG$

$\frac{1}{6} BD = HG$

$\frac{1}{6} BD = HI$

$HG = HI$

$HG + HI = GI$

$GI = \frac{1}{3} BD$

$HD = HB$

$GD = \frac{2}{3} HD$

$IB = \frac{2}{3} HB$

$GD = IB \qquad GD + IB = \frac{2}{3} BD$

$\therefore GD = \frac{1}{3} BD$

$IB = \frac{1}{3} BD$

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13. The vectors $A = i + j - k$
$B = 3i - 2j - k$
extend from the origin of coördinates. Show that the line joining their ends is parallel to the x-y plane and find its length.

$$B - A = 2i - 3j$$
$$|B-A| = \sqrt{2^2 + (-3)^2}$$
$$|B-A| = \sqrt{13}$$

The line joining their ends is parallel to the x-y plane and one unit below it. Its length is $\sqrt{13}$.

14. The vectors from the origin to the points A, B, C, D are
$A = i + j + k$,
$B = 2i + 3j$,
$C = 3i + 5j - 2k$,
$D = k - j$.
Show that the lines AB and CD are parallel and find the ratio of their lengths.

$B - A = i + 2j - k$ $\quad AB = |B-A|$ $\quad CD = |D-C|$

$D - C = -3i - 6j + 3k$ $\quad AB = \sqrt{1^2 + 2^2 + (-1)^2}$ $\quad CD = \sqrt{(-3)^2 + (-6)^2 + 3^2}$

$D - C = -3(B-A)$ $\quad AB = \sqrt{6}$ $\quad CD = \sqrt{54}$

$\therefore CD \parallel AB$ $\quad \frac{CD}{AB} = \frac{3\sqrt{6}}{\sqrt{6}} = 3$ $\quad CD = 3\sqrt{6}$

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15. Show that the vectors $i - j,\ j - k,\ k - i$ are parallel to a plane.

$$A = i - j$$
$$B = j - k$$
$$C = k - i$$

$$A \times B = [(-1)(-1) - (0)(1)]\,i + [(0)(0) - (1)(-1)]\,j + [(1)(1) - (-1)(0)]\,k$$

$$A \times B = i + j + k$$

$$C \cdot (A \times B) = -1 + 1 = 0$$

$$\therefore \cos\theta = 0$$

$$\theta = 90^\circ$$

$$\therefore C \perp A \times B$$

$\therefore$ C $\parallel$ the plane of A and B

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18. Show that the vectors

$$A = 2i - j + k$$
$$B = i - 3j - 5k$$
$$C = 3i - 4j - 4k$$

form the sides of a rt. triangle.

$$AB = \sqrt{(1-2)^2 + (-3+1)^2 + (-5-1)^2}$$
$$AB = \sqrt{1 + 4 + 36}$$
$$AB = \sqrt{41}$$

$$BC = \sqrt{(3-1)^2 + (-4+3)^2 + (-4+5)^2}$$
$$BC = \sqrt{4 + 1 + 1}$$
$$BC = \sqrt{6}$$

$$AC = \sqrt{(3-2)^2 + (-4+1)^2 + (-4-1)^2}$$
$$AC = \sqrt{1 + 9 + 25}$$
$$AC = \sqrt{35}$$

$$(AB)^2 = (AC)^2 + (BC)^2$$
$$41 = 35 + 6$$
$$41 = 41$$

$\therefore \triangle ABC$ is a right triangle

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19. Find the cosine of the angle between the two vectors

$$A = i - 2j - 2k$$
$$B = 2i + j - 2k$$

$$A \cdot B = |A|\,|B| \cos\theta$$
$$2 - 2 + 4 = \sqrt{1+4+4} \cdot \sqrt{4+1+4} \cdot \cos\theta$$
$$4 = \sqrt{9} \cdot \sqrt{9} \cdot \cos\theta$$
$$4 = 9 \cos\theta$$
$$\frac{4}{9} = \cos\theta$$
$$.4444 = \cos\theta$$

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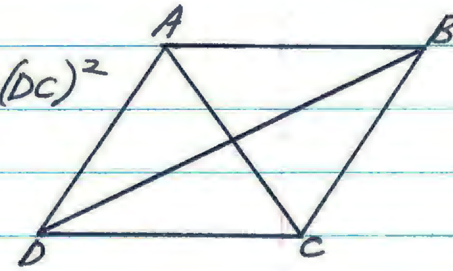
21. Prove that the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its four sides.

$$(AC)^2 + (DB)^2 \overset{?}{=} (DA)^2 + (AB)^2 + (CB)^2 + (DC)^2$$



$$(DB)^2 = (DC)^2 + (CB)^2 - 2(DC)(CB)\cos\angle DCB$$
$$(AC)^2 = (AB)^2 + (CB)^2 - 2(AB)(CB)\cos\angle ABC$$

$$\angle ABC = 180^\circ - \angle DCB$$
$$\cos\angle ABC = -\cos\angle DCB$$

$$AB = DC$$
$$CB = DA$$

$$(AC)^2 = (DC)^2 + (CB)^2 - 2(DC)(CB)(-\cos\angle DCB)$$
$$(AC)^2 = (DC)^2 + (CB)^2 + 2(DC)(CB)\cos\angle DCB$$

$$(AC)^2 + (DB)^2 = (DC)^2 + (DC)^2 + (CB)^2 + (CB)^2$$

$$(AC)^2 + (DB)^2 = (AB)^2 + (DC)^2 + (CB)^2 + (DA)^2$$

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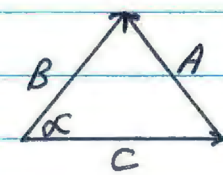
22. By squaring both sides of the equation

$$A = B - C$$

and interpreting the result geometrically, prove the formula $a^2 = b^2 + c^2 - 2bc \cos\alpha$.



$$A = B - C$$

$$|A|^2 = |B|^2 - 2B \cdot C + |C|^2$$

$$B \cdot C = |B|\,|C| \cos\alpha$$

$$A^2 = B^2 - 2BC\cos\alpha + C^2$$

$$A^2 = B^2 + C^2 - 2BC\cos\alpha$$

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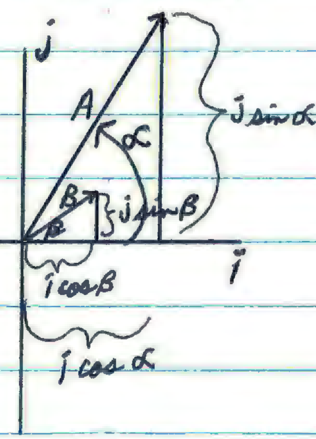
23. Show that $A = \mathbf{i}\cos\alpha + \mathbf{j}\sin\alpha$, $B = \mathbf{i}\cos\beta + \mathbf{j}\sin\beta$ are unit vectors in the xy-plane making angles α, β with the x-axis. By means of the scalar product obtain the formula for $\cos(\alpha-\beta)$.



$$\frac{\mathbf{i}\cos\alpha}{A} = \cos\alpha \qquad \frac{\mathbf{i}\cos\beta}{B} = \cos\beta$$

$$\mathbf{i}\cos\alpha = A\cos\alpha \qquad \mathbf{i}\cos\beta = B\cos\beta$$

$$\mathbf{i} = A \qquad \mathbf{i} = B$$

$$\frac{\mathbf{j}\sin\alpha}{A} = \sin\alpha \qquad \frac{\mathbf{j}\sin\beta}{B} = \sin\beta$$

$$\mathbf{j}\sin\alpha = A\sin\alpha \qquad \mathbf{j}\sin\beta = B\sin\beta$$

$$\mathbf{j} = A \qquad \mathbf{j} = B$$

$$\theta = \alpha - \beta$$

$$A \cdot B = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$|A|\,|B|\cos\theta = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\left(\sqrt{\cos^2\alpha + \sin^2\alpha}\right)\left(\sqrt{\cos^2\beta + \sin^2\beta}\right)\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\cos^2 + \sin^2 = 1$$

$$\therefore \cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

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27. Given

$$A = i - 2j, \quad B = j + k$$

find the component of A ~~a~~along B.

$a = |A| \cos\theta$ is component of A along B.

$$A \cdot B = |A|\,|B| \cos\theta \qquad |B| = \sqrt{1^2 + 1^2}$$

$$\frac{A \cdot B}{|B|} = |A| \cos\theta \qquad |B| = \sqrt{2}$$

$$\frac{-2}{\sqrt{2}} = |A| \cos\theta \qquad A \cdot B = (1)(0) + (-2)(1) + (0)(1)$$

$$A \cdot B = -2$$

$$-\sqrt{2} = |A| \cos\theta = a$$

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29. Given
$$A = \tfrac{1}{7}\,(2i + 3j + 6k),$$
$$B = \tfrac{1}{7}\,(3i - 6j + 2k),$$
$$C = \tfrac{1}{7}\,(6i + 2j - 3k),$$

show that A, B, C are of unit length, mutually perpendicular, and that $A \times B = C$.

$$|A| = \sqrt{\left(\tfrac{2}{7}\right)^2 + \left(\tfrac{3}{7}\right)^2 + \left(\tfrac{6}{7}\right)^2}$$
$$|A| = \sqrt{\tfrac{4}{49} + \tfrac{9}{49} + \tfrac{36}{49}}$$
$$|A| = \sqrt{\tfrac{49}{49}}$$
$$|A| = 1$$

$$|B| = \sqrt{\left(\tfrac{3}{7}\right)^2 + \left(-\tfrac{6}{7}\right)^2 + \left(\tfrac{2}{7}\right)^2}$$
$$|B| = \sqrt{\tfrac{9}{49} + \tfrac{36}{49} + \tfrac{4}{49}}$$
$$|B| = \sqrt{\tfrac{49}{49}}$$
$$|B| = 1$$

$$|C| = \sqrt{\left(\tfrac{6}{7}\right)^2 + \left(\tfrac{2}{7}\right)^2 + \left(-\tfrac{3}{7}\right)^2}$$
$$|C| = \sqrt{\tfrac{36}{49} + \tfrac{4}{49} + \tfrac{9}{49}}$$
$$|C| = \sqrt{\tfrac{49}{49}}$$
$$|C| = 1$$

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29. (Continued)

$$A \cdot B = |A|\,|B| \cos\theta$$

$$\left(\tfrac{2}{7}\right)\left(\tfrac{3}{7}\right) + \left(\tfrac{3}{7}\right)\left(-\tfrac{6}{7}\right) + \left(\tfrac{6}{7}\right)\left(\tfrac{2}{7}\right) = (1)(1)\cos\theta$$

$$\frac{6}{49} - \frac{18}{49} + \frac{12}{49} = (1)(1)\cos\theta$$

$$0 = \cos\theta$$

$$90^\circ = \theta$$

$$\therefore A \perp B$$

$$B \cdot C = |B|\,|C| \cos\theta$$

$$\left(\tfrac{3}{7}\right)\left(\tfrac{6}{7}\right) + \left(-\tfrac{6}{7}\right)\left(\tfrac{2}{7}\right) + \left(\tfrac{2}{7}\right)\left(-\tfrac{3}{7}\right) = (1)(1)\cos\theta$$

$$\frac{18}{49} - \frac{12}{49} - \frac{6}{49} = \cos\theta$$

$$0 = \cos\theta$$

$$90^\circ = \theta$$

$$\therefore B \perp C$$

$$A \cdot C = |A|\,|C| \cos\theta$$

$$\left(\tfrac{2}{7}\right)\left(\tfrac{6}{7}\right) + \left(\tfrac{3}{7}\right)\left(\tfrac{2}{7}\right) + \left(\tfrac{6}{7}\right)\left(-\tfrac{3}{7}\right) = (1)(1)\cos\theta$$

$$\frac{12}{49} + \frac{6}{49} - \frac{18}{49} = \cos\theta$$

$$0 = \cos\theta$$

$$90^\circ = \theta$$

$$\therefore A \perp C$$

$$A \times B \stackrel{?}{=} C \qquad |C| = 1$$

$$1 = 1$$

$$\therefore A \times B = C$$

$A \times B$ is a vector $\perp$ both A and B, with magnitude of $|A||B| \sin\theta = 1$

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30. Find the area of the parallelogram formed by the two vectors

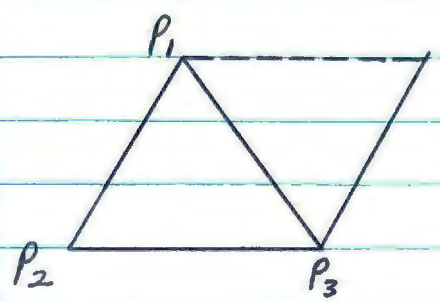
$$A = 3i + 2j \,, \quad B = 2j - 4k$$

$$\text{Area} = |A \times B|$$
$$\text{Area} = |-8i + 12j + 6k|$$
$$\text{Area} = \sqrt{64 + 144 + 36}$$
$$\text{Area} = \sqrt{244}$$
$$\text{Area} \approx 15.62$$

31. Find the area of the triangle formed by the points

$$P_1(1,1,1), \quad P_2(1,2,3), \quad P_3(2,3,1).$$

$$\overrightarrow{P_1P_2}\ (0,1,2)$$
$$\overrightarrow{P_2P_3}\ (1,1,-2)$$



$$\text{Area} = \frac{|\overrightarrow{P_2P_3} \times \overrightarrow{P_1P_2}|}{2}$$
$$\text{Area} = \frac{|4, -2, 1|}{2}$$
$$\text{Area} = \frac{\sqrt{16+4+1}}{2}$$
$$\text{Area} = \frac{\sqrt{21}}{2}$$

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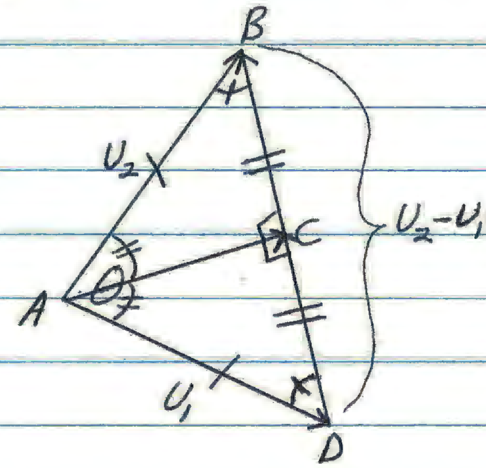
32. If U_1 and U_2 are vectors of unit length and θ is the angle between them, show that

$$\sin \tfrac{1}{2}\theta = \tfrac{1}{2}\,|U_2 - U_1|.$$



$$\frac{DC}{DB} = \frac{1}{2}$$

$$\frac{AC - U_1}{U_2 - U_1} = \frac{1}{2}$$

$$AC - U_1 = \tfrac{1}{2}(U_2 - U_1)$$

$$DC = \tfrac{1}{2}(U_2 - U_1)$$

$$DC = CB$$

$$CB = \tfrac{1}{2}(U_2 - U_1)$$

$$\frac{CB}{U_2} = \sin \tfrac{1}{2}\theta$$

$$CB = U_2 \sin \tfrac{1}{2}\theta \qquad U_2 = 1$$

$$\tfrac{1}{2}|U_2 - U_1| = \sin \tfrac{1}{2}\theta$$

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33. The vectors from the origin to the points A, B, C are

$$A = i + j - 2k,$$
$$B = 2i - j + k,$$
$$C = i + 3j - k.$$

Find a vector N perpendicular to the plane ABC. By projecting A upon N find the distance from the origin to the plane.

$$A - B = -i + 2j - 3k$$
$$A - C = 0 - 2j - k$$

$$(A-B) \times (A-C) = -8i - j + 2k = N$$

$$A \cdot N = -8 - 1 - 4 = |A|\,|N| \cos\theta$$
$$A \cdot N = -13 = |A|\,|N| \cos\theta$$

$$|N| = \sqrt{64+1+4}$$
$$|N| = \sqrt{69}$$

$$\frac{A \cdot N}{|N|} = |A| \cos\theta$$

$$\frac{-13}{\sqrt{69}} = |A| \cos\theta$$

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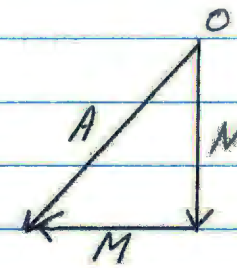
34. In problem 33 find the vector which is the projection of A upon the plane ABC

Use vector $N' = -N = 8i + j - 2k$

$$A = i + j - 2k$$
$$B = 2i - j + k$$
$$C = i + 3j - k$$



The distance from the origin to the plane is the length of N'' and is $\frac{13}{\sqrt{69}}$.

N'' is proportional to N'

$\therefore N'' = 8Ki + Kj - 2Kk$ K - a constant

$$N'' = K(8i + j - 2k)$$

$$|N''| = \frac{13}{\sqrt{69}} = \sqrt{64K^2 + K^2 + 4K^2}$$

$$\frac{13}{\sqrt{69}} = \sqrt{69K^2}$$

$$\frac{13}{\sqrt{69}} = K\sqrt{69}$$

$D\sqrt{69}$

$$\frac{13}{69} = K$$

$$N'' = \frac{13}{69}(8i + j - 2k)$$

$$N'' = \frac{104}{69} i + \frac{13}{69} j - \frac{26}{69} k$$

$$M = A - N'' = \left(\frac{69}{69} - \frac{104}{69}\right)i + \left(\frac{69}{69} - \frac{13}{69}\right)j + \left(-\frac{138}{69} + \frac{26}{69}\right)k$$

$$M = -\frac{35}{69} i + \frac{56}{69} j - \frac{112}{69} k$$

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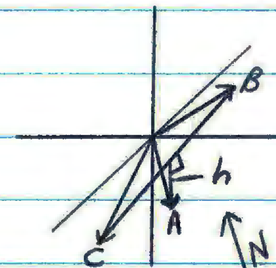
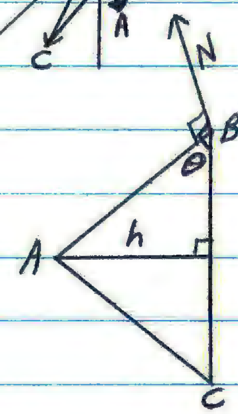
35. Using the values in problem 33, find the vector from A perpendicular to the line BC and terminating on that line.



$A = i + j - 2k$
$B = 2i - j + k$
$C = i + 3j - k$



$AB = B - A = i - 2j + 3k$
$BC = C - B = -i + 4j - 2k$

$N = AB \times BC$
$N = -8i - j + 2k$

$|N \cdot h| = |N|\,|h| \cos 90°$
$|N \cdot h| = 0$
$|(C - B) \cdot h| = 0$

$|h|^2 = a^2 + b^2 + c^2$
$|h| = \sqrt{a^2 + b^2 + c^2}$
$h = ai + bj + ck$

$$N \cdot h = -8ai - bj + 2ck = 0$$
$$(C-B) \cdot h = -ai + 4bj - 2ck = 0$$
$$-9a + 3b = 0$$
$$3b = 9a$$
$$b = 3a$$
$$-8a - 3a + 2c = 0$$
$$2c = 11a$$
$$c = \tfrac{11}{2} a$$

$$|AB \times BC| = |AB|\,|BC| \sin\theta$$
$$|-8i - j + 2k| = |AB|\,|BC| \sin\theta$$
$$\frac{|-8i - j + 2k|}{|BC|} = |AB| \sin\theta = |h|$$
$$\frac{\sqrt{69}}{\sqrt{21}} = |h|$$
$$\frac{69}{21} = |h|^2$$

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35. (Continued)

$$|h|^2 = a^2 + b^2 + c^2$$

$$\frac{69}{21} = a^2 + 9a^2 + \frac{121}{4}a^2$$

$$\frac{69}{21} = \frac{161}{4}a^2$$

$$\frac{276}{3381} = a^2$$

$$\sqrt{\frac{276}{3381}} = a$$

$$\vec{h} = ai + bj + ck$$

$$\vec{h} = ai + 3aj + \frac{11}{2}a\,k$$

$$\vec{h} = \sqrt{\frac{276}{3381}}\ i + 3\sqrt{\frac{276}{3381}}\ j + \frac{11}{2}\sqrt{\frac{276}{3381}}\ k$$

38. Find the value of $A \cdot B \times C$ if

$A = i - j - 6k,$

$B = i - 3j + 4k,$

$C = 2i - 5j + 3k.$

$$B \times C = (-9+20)i \quad (8-3)j \quad (-5+6)k$$

$$B \times C = 11i + 5j + k$$

$$A \cdot B \times C = 11 - 5 - 6$$

$$A \cdot B \times C = 0$$

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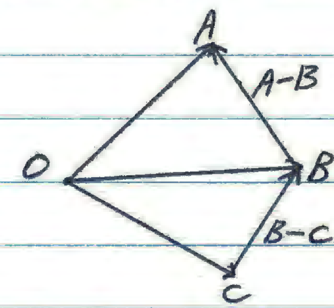
40. If A, B, C are vectors from the origin to the points A, B, C show that

$$A \times B + B \times C + C \times A$$

is perpendicular to the plane ABC.



If $A \times B + B \times C + C \times A \perp$ plane ABC then

$$(A\times B + B\times C + C\times A)\times[(A-B)\times(B-C)] = 0$$
$$(A\times B + B\times C + C\times A)\times[A\times(B-C) - B\times(B-C)] = 0$$
$$(A\times B + B\times C + C\times A)\times[A\times B - A\times C - \cancel{B\times B} + B\times C] = 0$$
$$(A\times B + B\times C + C\times A)\times[A\times B + B\times C - A\times C] = 0$$
$$-A\times C = C\times A$$
$$(A\times B + B\times C + C\times A)\times(A\times B + B\times C + C\times A) = 0$$
$$0 = 0$$

$\therefore A\times B + B\times C + C\times A \perp$ plane ABC.

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41. Prove the formula

$$(A \times B) \cdot (B \times C) \times (C \times A) = (ABC)^2.$$

$$(A \times B) \cdot (B \times C) \times (C \times A) \stackrel{?}{=} (ABC)^2$$

$$(A \times B) \cdot [(BCA)C - (BCC)A]$$

$$(A \times B) \cdot [(B \cdot C \times A) \cdot C - \cancel{(B \cdot C \times C) \cdot A}]$$

$$(A \times B) \cdot (B \cdot C \times A) \cdot C$$

$$C \cdot (A \times B) \cdot B \cdot (C \times A)$$

$$A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B)$$

$$A \cdot (B \times C) \cdot A \cdot (B \times C)$$

$$(ABC)^2 = (ABC)^2$$

42. Show that

$$A \times (B \times C) + B \times (C \times A) + C \times (A \times B) = 0.$$

$$A \times (B \times C) + B \times (C \times A) + C \times (A \times B) \stackrel{?}{=} 0$$

$$\cancel{(A \cdot C)B} - \cancel{(A \cdot B)C} + \cancel{(B \cdot A)C} - \cancel{(B \cdot C)A} + \cancel{(C \cdot B)A} - \cancel{(C \cdot A)B} \stackrel{?}{=} 0$$

$$0 = 0$$

43. Show that

$$A \times B \cdot C \times D + B \times C \cdot A \times D + C \times A \cdot B \times D = 0.$$

$$A \times B \cdot C \times D + B \times C \cdot A \times D + C \times A \cdot B \times D \stackrel{?}{=} 0$$

$$\cancel{(A \cdot C)(B \cdot D)} - \cancel{(A \cdot D)(B \cdot C)} + \cancel{(B \cdot A)(C \cdot D)} - \cancel{(B \cdot D)(C \cdot A)}$$

$$+ \cancel{(C \cdot B)(A \cdot D)} - \cancel{(C \cdot D)(A \cdot B)} \stackrel{?}{=} 0$$

$$0 = 0$$