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The Major Contribution of Leibniz to Infinitesimal Calculus

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I.

A study of the work of Leibniz is of importance for at least two reasons. In the first place, Leibniz was not alone among great men in presenting in his early works almost all the important mathematical ideas contained in his mature work. In the second place, the main ideas of his philosophy are to be attributed to his mathematical work, not vice versa. He was perhaps, the earliest to realize fully and correctly the important influence of a calculus on discovery. The almost mechanical operations which one goes through when one is using a calculus enables one to discover facts of mathematics or logic without any of that expenditure of the energy of thought which is so necessary when one is dealing with a department of knowledge that has not yet been reduced to the domain of operation of a calculus. These operations were developed and perfected by Gottfried Wilhelm Leibniz and thus places all mathematicians of today in his debt.

II.

Leibniz may be said to have lived not one life but several. As a diplomat, historian, philosopher, and mathematician he did enough in each field to fill one ordinary working life. Younger than Newton by about four years, he was born at Leipzig on July 1, 1646, and living only seventy years against Newton's eighty-five, died in Hanover on November 14, 1716. His father was a professor of moral philosophy and came of a good family which had served the government of Saxony for three generations. Thus young Leibniz' earliest years were passed in an atmosphere of scholarship heavily charged with politics.

At the age of six he lost his father, but not before he had acquired from him a passion for history. Although self-taught by incessant reading in his father's library, he attended a school in Leipzig. At eight he began the study of Latin and by twelve had mastered it sufficiently to compose creditable Latin verse. From Latin he passed on to Greek which he also learned largely by his own efforts.

At the age of fifteen Leibniz entered the University of Leipzig as a student in law. The law, however, did not occupy all his time. In his first two years he read widely in philosophy and for the first time became aware of the new world which the modern or natural philosophers, Kepler, Galileo, and Descartes had discovered. Seeing that this newer philosophy could be understood only by one acquainted with mathematics, Leibniz passed the summer of 1663 at the University of Jena, where he attended the mathematical lectures of Erhard Weigel, a man of considerable local reputation but scarcely a mathematician.

On returning to Leipzig he concentrated on law. By 1666, at the age of twenty, he was thoroughly prepared for his doctor's degree in law. This is the year in which Newton began the rustication at Woolsthorpe that gave him the calculus and his law of universal gravitation. The Leipzig faculty, bilious with jealousy, refused Leigniz his degree, officially on account of his youth, actually because he knew more about law than the whole dull lot of them.

Before this he had taken his bachelor's degree in 1663 at the age of seventeen with a brilliant essay, foreshadowing one of the cardinal doctrines of his mature philosophy. Disgusted at the pettiness of the Leipzig faculty Leibniz left his native town for good and proceeded to Nuremberg where, on November 5, 1966, at the affiliated University to Altdorf, he was not only granted his doctor's degree at once for his essay on a new method of teaching law, but was begged to accept the University professorship of law. Leibniz declined saying he had very different ambitions.[1]

Around 1671 Leibniz invented a more versatile computing machine capable of counting, addition, substraction, multiplication, and division. This was not his main contribution to modern calculating machinery. The main contribution

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[1]E. T. Bell, Men of Mathematics (New York: Simon and Schuster, Inc., 1937), p. 22.

was his recognition of the advantages of the binary scale, or notation, over the dinary.[2] He imagined a steam engine, studied Sanskirt, and tried to promote the unity of Germany.[3]

Up until 1672 Leibniz knew but little of what in his time was modern mathematics. He was then twenty-six when his real mathematical education began at the hands of Huygens, whom he met in Paris in the intervals between one diplomatic plot and another. Huygens presented Leibniz with a copy of his mathematical work on the pendulum. Fascinated by the power of the mathematical method in competent hands, Leibniz begged Huygens to give him lessons, which Huygens, seeing that Leibniz had a first-class mind, gladly did. Leibniz had already drawn up an impressive list of discoveries he had made by means of his own methods. Under Huygens' expert guidance Leibniz quickly found himself. He was a born mathematician.

The lessons were interrupted from January to March, 1673, during Leibniz' absence in London as an attache' for the Elector. While in London, Leibniz met the English mathematicians and showed them some of his work, only to learn that it was already known. His English friends told him of Mercator's quadrature of the hyperbola. This introduced

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[2] E. T. Bell, Mathematics: Queen and Servant of Science (New York: McGraw-Hill Book Company, 1951), p. 248.

[3] Dirk J. Struik, A Concise History of Mathematics (New York: Dover Publications, 1948), p. 156.

Leibniz to the method of infinite series, which he carried on. One of his discoveries may be noted: if π is the ratio of the circumference of a circle to its diameter,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots$$

the series continuing in the same way indefinitely. Thiis is not a practical way of calculating the numerical value of π (3.1415926...), but the simple connection between π and all the odd numbers is striking.

During his stay in London Leibniz attended meetings of the Royal Society, where he exhibited his calculating machine. For this and his other work he was elected a foreign member of the Society before his return to Paris in March, 1673. He and Newton subsequently (1700) became the first foreign members of the French Academy of Sciences.

The remaining forty years of Leibniz' life were spent in the trival service of the Brunswick family. In all he served three masters as librarian, historian, and general brains of the family. It was a matter of great importance to such a family to have an exact history of all its connections with other families as highly favored by heaven as itself. Leibniz was no mere cataloguer of books in his function as family librarian, but an expert genealogist and searcher of mildewed archives as well, whose function it was to confirm the claims of his employers to half the thrones of Europe or, failing confirmation, to manufacture evidence by judicious suppression. His historical researches took him all through Germany and thence to Austria and Italy in 1687-90.

On being called to Berlin in 1700 as tutor to the young Electress, Leibniz found time to organize the Berlin Academy of Sciences. He became its first president. The Academy was still one of the three or four leading learned bodies in the world till the Nazis "purged" it. Similar ventures in Dresden, Vienna, and St. Petersburg came to nothing during Leibniz' lifetime, but after his death the plans for the St. Petersburg Academy of Sciences which he had drawn up for Peter the Great were carried out. The attempt to found a Viennese Academy was frustrated by the Jesuits when Leibniz visited Austria for the last time, in 1714. Their opposition was only to have been expected after what Leibniz had done for Arnauld. That they got the better of the master diplomat in an affair of petty academic politics shows how badly Leibniz had begun to slip at the age of sixty-eight. He was no longer himself, and indeed his last years were but a wasted shadow from his former glory.

Having served princes all his life he now received the usual wages of such service. Ill, fast ageing, and harassed by controversy, he was kicked out.

Leibniz returned to Brunswick in September, 1714, to learn that his employer the Elector George Louis had left for London to become the first German king of England. Nothing would have pleased Leibniz better than to follow George to London, although his enemies at the Royal Society and elsewhere in England were now numerous and vicious enough owing to the controversy with Newton. However, George, now socially a gentleman, had no further use for Liebniz' diplomacy, and curtly ordered the brains that had helped to lift him into civilized society to stick in the Hanover library and get on with their history of the illustrious Brunswick family.

When Leibniz died two years later the diplomatically doctored history was still incomplete.[4] As a diplomat and statesman Leibniz was as good as the cream of the best of them in any time or any place, and far brainer than all of them together.

III.

The seventeenth century was one of activity and advancement in the world of math. Analytic methods had become familiar tools to most of the mathematicians of the period; geometry was being employed to verify and demonstrate analytic conclusions and special attention was focused on problems dealing with the infinite.

The time was indeed ripe, in the second half of the seventeenth century, for someone to organize the views, methods, and discoveries involved in the infinitesimal analysis into a new subject characterized by a distinctive method of procedure.[5]

Unfortunately, not one but two men did just that. The methods of the calculus developed by Sir Isaac Newton in

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[4] Bell, Men of Mathematics, op. cit., p. 130.

[5] Carl B. Boyer, The Concepts of the Calculus (Wakefield, Mass.: 1949), p. 385.

England and Gottfried Wilhelm Leibniz on the continent were essentially the same, yet the dispute over the rights of the two discoveries developed into a controversy which has not yet been settled. Both these men and their followers stooped to tactics which were most unworthy of men of intelligence and honor; as a result, the development of math in England was brought to a standstill for a full century.[6] It must be remembered that these two inventors are not responsible for the beginning of the conflict, only the continuation of the conflict.

The matter was first started in the year 1699 by Fatio de Duillier, a Swiss mathematician who had been living in London since 1691; he was a correspondent of Huygens, and from letters that Fatio sent Huygens, it would appear that the attack had been quietly in preparation for some time. Whether he had Newton's sanction or not cannot be ascertained, yet it seems certain from the correspondence that Newton had given Fatio information with regard to his writing. Fatio then concludes that Newton is the first discoverer and that Leibniz, as second discoverer, has borrowed from Newton. These accusations hurt Leibniz all the more, because he had deposited copies of his correspondence with Newton in the hands of Wallis for publication. As Fatio was a member of

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[6]Dorothy V. Schrader, "The Newton-Leibniz Controversy Concerning the Discovery of the Calculus," Math Teacher, 55 (May, 1962), p. 385.

the Royal Society, Leibniz took it for granted that Fatio's attack was with the approval of that body; he asked therefore that the papers in the hands of Wallis should be published in justice to himself. He received a reply from Sloane, one of the secretaries of the Society, informing him that the assumption with regard to any such participation of the Society in the attack was groundless; and in consequence of this he took no further notice of the matter, and the whole thing lapsed into oblivion.

In the year 1708 the attack against Leibniz was renewed by Keill; and the charge that Leibniz had borrowed from Newton was most directly made. Leibniz had no one in England who was in a position to substantiate his claims, for Wallis had died in 1703; so he appealed directly to the Royal Society. This body in consequence appointed a commission composed of members of the Society to consider the papers concerned in the matter.[7] As the British sporting instinct presently began to assert itself, Newton acquiesced in the disgraceful attack and himself suggested or consented to shady schemes of downright dishonesty designed at any cost to win the international championship--even that of national honor. Leibniz and his backers did likewise. The upshot of it all was that the obstinate British practically rotted mathematically for all of a century after Newton's

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[7]J. M. Child, The Early Mathematical Manuscripts of Leibniz (Chicago: Open Court Publishing Company, 1920), p. 6.

death, while the more progressive Swiss and Franch, follow-ing the lead of Leibniz, and developing his incomparably better way of merely writing the calculus, perfected the subject and made it the simple, easily applied implement of research that Newton's immediate successors should have had the honor of making it.[8]

Authorities now generally agree that Leibniz invented the calculus independently of any knowledge of Newton's fluxions; though Newton had the idea of calculus earlier than Leibniz.[9]

IV.

It is extremely useful to have knowledge of the true origins of memorable discoveries, especially those that have been found not by accident but by dint of mediation. It is not so much that thereby history may attribute to each man his own discoveries and that others should be encouraged to earn like commendation, as that the art of making discoveries should be extended by considering noteworthy examples of it.

Among the most renowned discoveries of all times must be considered that of a new kind of mathematical analysis, known by the name of the differential calculus; and this even if the essentials are at the present time considered to

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[8]Bell, Men of Mathematics, op. cit., pp. 113-114.

[9]R. H. Moorman, "Mathematics and Philosophy," Math Teacher, 51 (January, 1958), p. 35.

be sufficiently demonstrated neverless the origin and the method of the discovery are not yet known to the world at large. Its author invented it nearly forty years ago, and nine years later published it in a concise form; and from that time it has not been only published in his memoirs, but also has been a method of general employment; while many splendid discoveries have been made by its assistance, such have been included in the Acta Eruditorium, Leipsic, and also such have been published in the memoirs of the Royal Academy of Sciences; so that it would seem that a new aspect has been given to mathematical knowledge arising out of its discovery.

Now there never existed any uncertainty as to the name of the true inventor, until recently, in 1712, certain upstarts, either in ignorance of the literature of the times gone by, of through envy, or with some hope of gaining notoriety by the discussion, or lastly from obsequious flattery, have set up a rival to him; and by their praise of the rival to him, the author has suffered no small disparagement in the matter, for the former has been credited with having known far more than is to be found in the subject under discussion. Moreover, they have changed the whole point of the issue, for in their screed, in which under the title of Commercium Epistolicum D. Johannis Collinsii (1712) they have set forth their opinion in such a manner as to give a dubious credit to Leibniz, they have said very little about the calculus; instead, every other page is made up of what they call infinite series. Such things were first given as discoveries by Nicolaus Mercator of Holstein, who obtained them by the process of division, and Newton gave the more general form by extraction of roots. This certainly is a useful discovery, for by its arithmetical approximation are reduced to an analytical reckoning; but it has nothing to do with the differential calculus. Moreover, even this they make use of fallacious reasoning; for whenever this rival works out a quadrature by the addition of the parts by which a figure is gradually increased, at once they hail it as the use of the differential calculus. (The differential calculus was not the employment of an infinitesimal and a summation of such quantities; it was the use of the idea of these infinitesimals being differences and the employment of the notation invented by himself, the rules that governed the notation, and the fact that differentiation was the inverse of a summation and perhaps the greatest point of all was that the work had not to be referred to a diagram.)

Now it certainly never entered the mind of any one else before Leibniz to institute the notation peculiar to the new calculus by which the imagination is freed from a perpetual reference to diagrams, as was made by Vieta and Descartes in their ordinary on Apollonian geometry, and to lines which were called "mechanical" by Descartes, were excluded by the latter in his calculus. But now by the calculus of Leibniz the whole of geometry is subjected to analytical computations, and those transcendent lines that Descartes called mechanical are also reduced to equations chosen to suit them, by considering the differences, as dx, ddx, etc., and the sums that are the inverses of these differences, as functions of the x's; and this, by merely introducing the calculus, whereas before this no other functions were admissable but x, xx, x^3, $\sqrt{x}$, etc., that is to say, powers and roots.

Nevertheless, he did not lack for friends to look after his fair name; and indeed a certain mathematician, one of the first rank of our time well skilled in this branch of learning and perfectly unbiased, whose good-will the opposite party had tried in vain to obtain, plainly stated, giving reasons of his own finding, and let it be known, not altogether with strict justice, that he considered that not only had that rival not invented the calculus, but that in addition he did not understand it to any great extent. Another friend of the inventor published these and other things as well in a short phamplet, in order to check their base contentions. However, it was of greater service to make known the manner and reasoning by which the discoverer arrived at this new kind of calculus; for this indeed has been unknown up till now, even to those perchance, who would like to share in this discovery. Indeed he himself had decided to explain it, and to give an account of the course of his researches in analysis partly from memory and partly from extant writings and remains of old manuscripts, and in this manner to illustrate in due form in a little book the history of this higher learning and the method of its discovery. But since at the time this was found to be impossible owing to the necessities of other business, he allowed this short statement of part of what there was to tell upon the matter to be published in the meantime by a friend who knew all about it, so that in some measure public curiously should be satisfied.

The author of this new analysis, in the first flower of his youth, added to the study of history and jurisprudence other more profound reflections for which he had a natural inclination. Among the latter he took a keen delight in the properties and combinations of number; indeed reprinted without his sanction. Also while still a young boy, when studying logic he perceived that the ultimate analysis of truths that depended on reasoning reduced to two things, definitions and identical truths, and that these alone of the essentials were primitive and undemonstrable. When it was stated in contradictions that identical truths were useless and nugatory, he gave illustrative proofs to the contrary. Among these he gave a demonstration that that mighty axiom, "The whole is greater than its part," could be proved by a syllogism of which the major term was a defination and the minor term an identity. For if one of two things is equal to a part of another the former is called the less, and the later the greater; and this is to be taken as the defination. Now if to this defination there be added the following identical and undemonstrable axiom, "Every thing possessed of magnitude is equal to itself," i.e., A=a, then we have the syllogism:

> Whatever is equal to a part of another, is less than the other: (by the defination)
>
> But the part is equal to a part of the whole: (i.e., to itself, by identity)
>
> Hence the part is less than the whole.

As an immediate consequence of this he observed that from the identity a=a, or at any rate its equivalent, A-A=0, as may be seen at a glance by straightforward and reduction, the following very pretty properly of differences arises, namely:

$$A \quad -A+B \quad -B+C \quad -C+D \quad -D+E \quad -E \quad = 0$$

$$+ L + M + N + P$$

If now A, B, C, D, E are supposed to be quantities that continually increase in magnitude, and the differences between successive terms are denoted by L, M, N, P, it will then follow that successive terms are:

$$A+L+M+N+P-E = 0,$$

i.e. $L+M+N+P = E-A$;

that is, the sums of the differences between successive terms, no matter how great their number, will be equal to the difference between the terms at the beginning and the end of the series. For example, in place of A, B, C, D, E, let us take the squares, 0, 1, 4, 9, 16, 25, and instead of the differences given above, the odd number, 1, 3, 5, 7, 9, will to disclosed; thus

0 1 4 9 15 25

1 3 5 7 9

From which is evident that

$$1+3+5+7+9 = 25 - 0 = 25$$

and $$3+5+7+9 = 25 - 1 = 24;$$

and the same will hold good whatever the number of terms or the differences may be, or whatever numbers are taken as the first and last terms. Delighted by this easy, elegant theorem, our young friend considered a large number of numerical series, and also proceeded to the second differences or differences of the differences, the third differences or the differences between the differences of the differences, and so on. He also observed that for the natural numbers, i.e., the numbers in order proceeding from 0, the second differences vanished, as also did the third differences for the squares, the fourth differences for the cubes, and the fifth for the biquadrates, the sixth for the surdesolids, and so on; also that the differences for the natural numbers were constant and equal to 1; the second differences for the square, 1.2, or 2; the third forthe cubes, 1.2.3, or 6; the fourth for the biquadrates, 1.2.3.4, or 24; the fifth for the surdesolids, 1.2.3.4.5, or 120, and so on. These things it is admitted had been previously noted by others, but they were new to him, and by their easiness and elegance were in in themselves an inducement to further advances. But especially he considered what he called "combinatory numbers," such as are usually tabulated as in the margin.

1	1	1	1	1	1
1	2	3	4	5	6
1	3	6	10	15	21
1	4	10	20	35	56
1	5	15	35	70	126
1	6	21	56	126	252
1	7	28	84	210	462

Here, a preceeding series, either horizontal or vertical, always contains the first differences of the series immediately following it, the second differences of the one next after that, the third differences of the third, and so on. Also each series, either horizontal or vertical contains the sums of the series immediately preceding it, the sums of the sums or the second sums of the series next before that, the third sums of the third, and so on. But, to give something not yet common knowledge, he also brought to light certain general theorems on differences and sums, such as the following. In the series, a, b, c, d, e, etc., where the terms continually decrease without limit we have

Terms	a b c d e etc.
1st diff.	f g h i k etc.
2nd diff.	l m n o p etc.
3rd diff.	q r s t u etc.
4th diff.	β γ δ ϵ θ etc.
etc.	γ μ ν ρ ν etc.

Taking a as the first term, and w as the last, he found

a-w = 1f + 1g + 1h + 1i + 1k + etc.
a-w = 1l + 2m + 3r + 4o + 5p + etc.
a-w = 1q + 3r + 6s + 10t + 15u + etc.
a-w = 1β + 4γ + 10δ + 20ϵ + 35θ + etc.
etc.

Again we have

$$a - w \quad \begin{array}{l} + 1f \\ + 1f - 1l \\ + 1f - 2l + 1q \\ + 1f - 3l + 3q - 1 \\ + 1f - 4l + 6q - 4 + 1 \\ \quad \text{etc. etc. etc.} \end{array}$$

This theorem is one of the fundamental theorems in the theory of the summation of series by finite differences, namely,

$$\Delta m u_n = u_{n+m} - {}_mC_1 \cdot u_{n+m+1} + {}_mC_2 \cdot u_{n+m-2} - \text{etc.}$$

which is usually called the direct fundamental theorem; for although Leibniz could not have expressed his results in this form since he did not know the sums of the figurate numbers as generalized formulas, and apparently his only a special case, yet it must be remembered that any term of the first series can be chosen as the first term. Hence adopting a notation invented by him at a later date, and denoting any term of the series generally by y, we may call the first difference dy, the second ddy, the third d^3y, the fourth d^4y; and calling any term of another of the series x, we may denote the sum of its terms by $\int x$, the sum of their sums on their second sums by $\int\int x$, the third sum by $\int^3 x$, and the fourth sum by $\int^4 x$, hence supposing that

$$1 + 1 + 1 + 1 + 1 + \text{etc.} = x,$$

or that x represents the natural numbers, for which dx = 1, then

$$1 + 3 + 6 + 10 + \text{etc.} = \int x,$$
$$1 + 4 + 10 + 20 + \text{etc.} = \int\int x,$$
$$1 + 5 + 15 + 35 + \text{etc.} = \int^3 x,$$

and so on. Finally it follows that

$$y - w = dy.x - ddy.\int x + d^3y.\int\int x - d^4y.\int^3 x + \text{etc.}$$

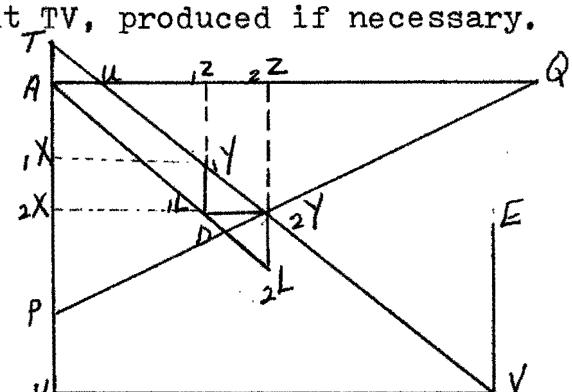
and this is equal to y, if we suppose that the series is continued to infinity, or that w becomes zero. Hence also follows the sum of the series itself, and we have

$$\int y = yx - dy.\int x + ddy.\int\int x - d^3y.\int^3 x + \text{etc.}$$

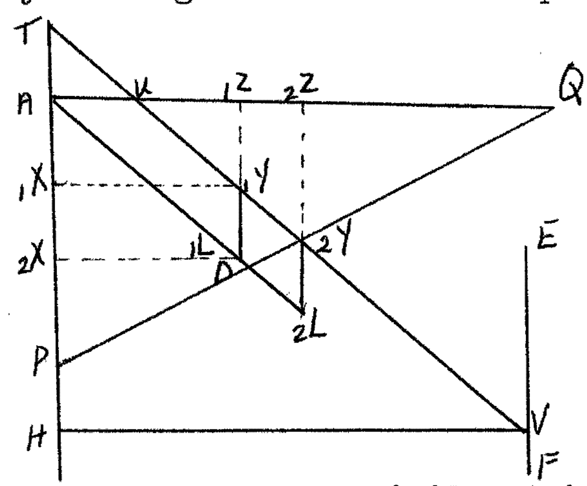
On his return from England to France in the year 1673, having meanwhile satisfactorily performed his work for the Most Nobel Elector of Mainz, he still by his favor remained in the service of Mainz; but his time being left more free, at the instigation of Huygens he began to work at Cartesian analysis, and in order to obtain an insight into the geometry of quadratures he consulted the Synopsis Geometriae of Honoratus Fabri, Gregory St. Vincent, and a little book by Dettonville. Later on from one example given by Dettonville, a light burst upon him, which strange to say Pascal himself had not perceived in it. For when he proves the theorem of Archimedes for measuring the surface of a sphere or parts of it, he used a method in which the whole surface of the solid formed by a notation round any axis can be reduced to an equivalent plane figure. From it he made the following general theorem.

Portions of a straight line normal to a curve, intercepted between the curve and an axis, when taken in order and applied at right angles to the axis give rise to a figure equivalent to the moment of the curve about the axis.

When he showed this to Huygens the latter praised him highly and confessed to him that by the help of this very theorem he had found the surface of parabolic canoids and others of the same sort, stated without proof many years before in his work on the pendulum clock. Leibniz stimulated by this and pondering on the fertility of this point of veiw, since previously he had considered infinitely small things such as intervals between the ordinates in the method of Cavalieri and such only, studied the triangle $_1YD_2Y$, which he called the Characteristic Triangle, whose sides D_1Y, D_2Y are respectively equal to $_1X_2X$, $_1Z_2Z$, parts of the coordinates or coabscissae AX, AZ, and its third side $_1Y_2Y$ a part of the tangent TV, produced if necessary.



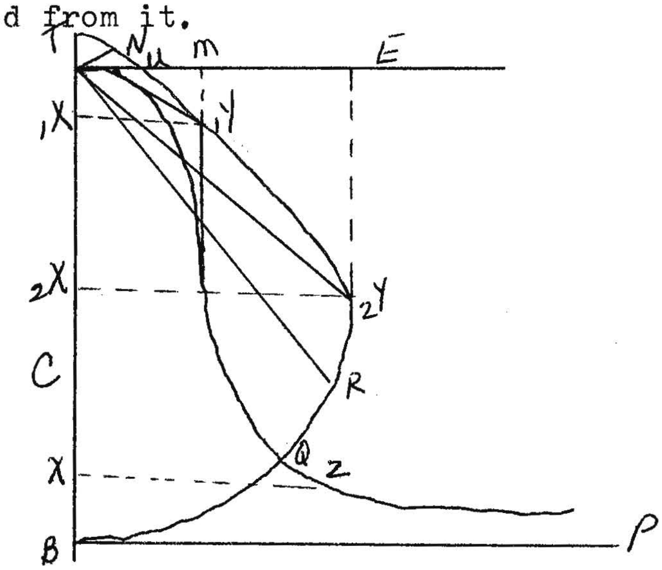
Even though this triangle is indefinite (being infinitely small), yet he perceived that it was always possible to find definite triangles similar to it. For suppose that AXX, AZZ are two straight lines at right angles, and AX, AZ the coabscissae, YX, YZ the coordinates, TUV the tangent, PYQ the perpendicular, XT, ZU the subtangents, XP, ZQ the subnormals; and lastly let EF be drawn parallel to the axis AX; let the tangent TY meet EF in V, and from D draw VH perpendicular to the axis. Then the triangles ${}_1YD_2Y$, TXY, YZU, TAU, YXP, QZY, QAP, THV, and as many more of the sort as you like, are all similar. For example, from the similar triangles ${}_1YD_2Y$, ${}_2Y_2XP$, we have $P_2Y.{}_1YD = {}_2Y_2X.{}_2Y_1Y$; that is, the rectangle contained by the perpendicular P_2Y and ${}_1YD$ (or the element of the axis ${}_1X_2X$) is equal to the rectangle contained by the ordinate ${}_2Y_2X$ and the element of the curve about the axis. Hence the whole moment of the curve is obtained by forming the sum of these perpendiculars to the axis.



Also, on account of the similar triangles ${}_1YD_2Y$, THV, we have ${}_1Y_2Y:{}_2YD = TV:VH$, or $VH.{}_1Y_2Y = TV.{}_2YD$; that is, the rectangle contained by the constant length VH and the element of the coabscissae, ${}_1Z_2Z$. Hence the plane figure produced by applying the lines TV in order at right angles to AX is equal to the rectangle contained by the curve when straightened out and the constant length HV.

Again, from the similar triangles ${}_1YD_2Y$, ${}_2Y_2XP$, we have ${}_1YD:D_2Y = {}_2Y_2X:{}_2XP$, and thus ${}_2XP.{}_1YD = {}_2Y_2X.D_2Y$, or the sum of the subnormals ${}_2XP$, taken in order and applied to the axis, either to ${}_1YD$ or to ${}_1X_2X$, will be equal to the sum of the products of the ordinates ${}_2Y_2X$ and their elements, ${}_2YD$, taken in order. But straight lines that continually increase from zero, when each is multiplied by its elements of increase, form altogether a triangle. Let then AZ Always be equal to ZL, then we get the right-angled triangle AZL, which is half the square of AZ; and thus the figure that is produced by taking the subnormals in order and applying them perpendicular to the axis will always equal to half the square on the ordinate. Thus to find the area of a given figure another figure is sought such that its subnormals are respectively equal to the ordinates of the given figure, and then this second figure is the quadratrix of the given one; and thus from this extremely elegant consideration we obtain the reduction of the areas of surfaces described by notation to plane quadratures, as well as the rectification of curves; at the same time we can reduce these quadratures of figures to an inverse problem of tangents. From this Leibniz wrote down a large collection of theorems of two kinds. For in some of them only definite magnitudes were dealt with and others truly depended on infinitely small magnitudes and advanced to a much greater extent.

Leibniz worked these things out at Paris in the year 1673 and part of 1674. But in the same year 1674 Leibniz came upon the well-known arithmetical tetragonism; and it will be worth while to explain how this was accomplished. He once happened to have occasion to break up an area into triangles formed by a number of straight lines meeting in a point, and he perceived that something new could be readily obtained from it.



In this figure, let any number of straight lines, AY, be drawn to the curve AYR, and let any axis AC be drawn, and AE a normal or coaxis to it; and let the tangent at Y to the curve cut them in T and U. From A draw AN perpendicular to the tangent; then it is plain that the elementary triangle $A{}_1Y_2Y$ is equal to half the rectangle contained by the element of the curve ${}_1Y_2Y$ and AN. Now draw the characteristic triangle mentioned above, ${}_1YD_2Y$, of which the hypotenuse is a portion of the tangent or the element of the arc, and the sides are parallel to the axis and the coaxis. It is then plain from similar triangles ANU, ${}_1YD_2Y$, that ${}_1Y_2Y : {}_1YD = AU : A$, or AU. ${}_1X_2X$ is equal to AN.${}_1Y_2Y$, and this is equal to double the triangle $A{}_1Y_2Y$. Thus if every AU is supposed to be transferred to XY, and taken in it as AZ, then the trilinear space AXZA so formed will be equal to twice the segment AY⌣A, (the symbol ⌣ is here to be read as "and then along the arc to"), included between the straight line AY and the arc AY. In this way are obtained what he called the figures of segments or the proportionals of a segment. A similar method holds good for the case in which the point is not taken on the curve, and in this manner he obtained the proportional trilinear figures for sectors cut off by lines meeting in a point; and even when the straight lines had their extremities not in a line but in a curve, none the less on that account were useful theorems made out. It is sufficient for our purpose to consider the figures of segments and that too only for the circle. In this case, if the point A is taken at the beginning of the quadrant AYQ, the curve AZQZ will cut the circle at Q, the other end of the quadrant, and thence descending will be asympotic to the base BP (drawn at right angles to the diameter at its other end B); and although extending to infinity, the whole figure included between the diameter AB, the base BP..., and the curve AZQZ...asympotic to it, will be equal to the circle on AB as diameter.

Take the radius as unity, put AX or UZ = x, and AU or AZ = z, then we have x = 2zz:, 1 + zz and the sum of all the x's applied to AU, which at the present time we call $\int x\,dz$, is the trilinear figure AUZA, which is the complement of the trilinear figure AXZA, and this has been shown to be doublt the circular segment.

The author obtained the same results by the method of transmutations, of which he sent an account to England. It is reguired to form the sum of all the ordinates (1 - xx) = y; suppose y = $\pm$ 1 $\mp$ xz, from which x = 2z:, 1 + zz, and y = $\pm$ zz $\mp$ 1,:, zz + 1; and thus again all that remains to be done is the summation of rationals.

From the above it was at once apparent that, using the method by which Mercator had given the arithmetical tetragonism of the hyperbola by means of an infinite series, that of the circle might also be given, though not so symmetrically, by dividing by 1 + zz, as in the same way that the former had divided by 1 + z. The author, however, soon found a general theorem for the area of any central conic. Namely, the sector include by the arc of a conic section, starting from the vertex, and two straight lines joining its ends to the center, is equal to the rectangle contained by the semi-transverse axis and a straignt line of length

$$t \pm 1/3p + 1/sp \pm 1/7p + \ldots,$$

where t is the portion of the tangent at the vertex intercepted between the vertex and the tangent at the other extremity of the arc, and unity is the square on the semi-conjugate axis or the rectangle contained by the halves of the latus-rectum and the transverse axis, and $\pm$ is to be taken to mean + for the hyperbola and - for the circle or the ellipse. Hence, if the square of the diameter is taken to be unity, then the area of the circle is

$$\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots$$

In the year 1672, while conversing with Huygens on the properties of number, the latter propounded to him this problem: To find the sum of a decreasing series of fractions of which the numerators are all unity and the denominators are the triangular number; of which he said that he found the sum among the contributions of Hudde on the estimation of probability. Leibniz found the sum to be 2, which agreed with that given by Huygens. While doing this he found the sums of a number of arithmetical series of the same kind in which the numbers are any combinatory numbers whatever, and communicated the results to Oldenburg in February 1673, as his opponents have stated. When later he saw the Arithmetical Triangle of Pascal, he formed on the same plan his own Harmonic Triangle.

ARITHMETICAL TRIANGLE

in which the fundamental series is an arithmetical progression

1, 2, 3, 4, 5, 6, 7, ...

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

HARMONIC TRIANGLE

in which the fundamental series is a harmonic progression;

1/1
1/2 1/2
1/3 1/6 1/3
1/4 1/12 1/12 1/4
1/5 1/20 1/30 1/20 1/5
1/6 1/30 1/60 1/60 1/30 1/6
1/7 1/42 1/105 1/140 1/105 1/42 1/7

where if the denominator of any series descending obliquely to infinity or of any parallel finite series, are each divided by the term that corresponds in the first series, the combinatory numbers are produced, namely those that are contained in the arithmetrical triangle. Moreover this property is common to either triangle, namely, that the oblique series are the sum and difference series of one another. In the Arithmetrical Triangle any given series is the sum series of the series that immediately precedes it, and the difference series of the one that follows it; in the Harmonic Triangle, on the other hand, each series is the sum series of the series that follows it and the difference series of the row preceding it. From which it follows that

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\dots=\frac{1}{0}$$

$$\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}+\frac{1}{28}+\dots=\frac{2}{1}$$

$$\frac{1}{1}+\frac{1}{4}+\frac{1}{10}+\frac{1}{20}+\frac{1}{35}+\frac{1}{56}+\frac{1}{84}+\dots=\frac{3}{2}$$

$$\frac{1}{1}+\frac{1}{5}+\frac{1}{15}+\frac{1}{35}+\frac{1}{70}+\frac{1}{126}+\frac{1}{210}+\dots=\frac{4}{3}$$

Now he had found out these things before he had turned to Cartesian analysis; but when he had had his thoughts directed to this, he considered that any term of a series could in most cases be denoted by some general notation, by which it might be referred to some simple series. For instance, if the general term of the series of natural numbers is denoted by x, then general term of the series of the squares would be X^2 that of the cubes would be X^3 and so on. Any triangular number, such as 0, 1, 3, 6, 10, would be

$$\frac{X.X+1}{1.2} \text{ or } \frac{XX + X}{2}$$

any pyramidal number, such as 0, 1, 4, 10, 20, etc. would be

$$\frac{X.X+1.X+2}{1.2.3} \text{ or } \frac{X^3+3XX+2X}{6}$$

and so on.

From this it was possible to obtain the difference series of a given series, and in some cases its sum as well, when it was expressed numerically. For instance, the square is xx, the next greater square is xx+2x+1, and the difference of these are 2x+1; i.e. the series of odd numbers is the difference series for the series of squares. For if x is 0, 1, 2, 3, 4, etc., then 2x+1 os 1, 3, 5, 7, 9. In the same way the difference between x^3 and $x^3+3xx+3x+1$ is 3xx+3x+1, and thus the latter is the general term of the difference series for the series of the cubes. Further, if the value of the general term can thus be expressed by means of a variable x so that the variable does not enter into a a denominator or an exponent, he perceived that he could always find the sum series of the given series. For instance, to find the sum of the squares, since it is plain that the variable cannot be raised to a higher degree than the cube, he supposed its general term z to be

$z = lx^3 + mxx + nx$, where dz has to be xx;
we have $dz = l\,d(x^3) + m\,d(xx) + n$, (where $dx = 1$); now
$d(x^3) = 3xx + 3x + 1$, and $d(xx) = 2x + 1$, as already found;
hence $dz = 3lxx + 3lx + l + 2mx + m + n \simeq xx$;
$\therefore\ l = 1/3,\ m = -1/2,\ 1/3 - 1/2 + n = 0$, or $n = 1/6$;
and the general term of the sum-series for the squares is $1/3x^3 - 1/2xx + 1/6x$ or $2x^3 - 3xx + x, : 6$.

As an example, if it is desired to find the sum of the first nine or ten squares, i.e., from 1 to 81 or from 1 to 100, take for x the value 10 or 11, the number next greater than the root of the last square, and $2x^3-3xx+x, :6$ will be 2000-300+10,:6 = 285, or 2.1331-3.121+11,:6 = 385. Nor is it much more difficult with this formula to sum the first 100 or 1000 squares. The same method holds good for any power of the natural numbers or for expressions which are made up of such powers, so that it is possible to sum as many terms as we please of such series by a formula. But our friend saw that it was not always easy to rpoceed in the same way when the variable entered into the denominator, as it was not always possible to find the sum of a numerical series; however on the following up of this ame analytical method, he found ingeneral, and published the results, that a sum series could always be found, or the matter reduced to finding the sum of a number of fractional terms such as $1/x$, $1/xx$, $1/x^3$, etc., which at any rate, if the number of terms taken is finite, can be summed, though hardly in a short way; but if it is a question of an infinite number of terms, then terms such as $1/x$ cannot be summed at all, because the total of an infinite number of terms such as $1/xx$, $1/x^3$, etc., be summed except by taking quadratures. So, in the year 1682, in the month of February, he noted that if the numbers 1.3, 3.5, 5.7, 7.9, 9.11, etc. or 3, 15, 35, 63, 99, etc., are taken and from them is formed the series of fractions

$$\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \frac{1}{63} + \frac{1}{99} + \ldots$$

then the sum of this series continued to infinity is nothing else but 1/2; while, if every other fraction is left out, 1/3+1/35+1/99+etc. expresses the magnitude of a semicircle of which the square on the diameter is represented by 1. Thus, suppose x = 1, 2, 3, etc. Then the general term of 1/3+1/15+1/35+1/63+... is $\frac{1}{4xx+8x+3}$; it is required to find the general term of the sum series.

Let us try whether it can have the form $e/(bx+c)$, the reasoning being simple; then we shall have

$$\frac{e}{bx+c} - \frac{e}{bx+b+c} = \frac{eb}{bbxx+bbx+bc+2bcx+cc} \simeq \frac{1}{4xx+8x+3}$$

hence, equating coefficients in the two formulas, we have

$$b=2,\ eb=1,\ \text{or}\ e=1/2$$

$$bb+2bc=8\ \text{or}\ 4+4c=8,\ \text{or}\ c=1;$$

and finally we should have also $bc + cc = 3$ which is the case. Hence the general term of the sum series if $(1 \cdot 2)/(2x+1)$ or $1/(4x+2)$, and these numbers of the form $4x+2$ are the doubles of the odd numbers. Finally he gave a method for applying the differential calculus to numerical series when the variable entered into the exponent as in a geometrical progression where, taking any radix b the term is b^x where x stands for a natural number. The terms of the differential series will be
and from this it is plain that the differential series of the given geometrical series is also a geometrical series proportional to the given series. Thus the sum of a geometrical series may be obtained.[10]

V.

A (real) alternating series of which the absolute values of the terms form a monotonic null sequence, is invariably convergent. If $\sum_{\nu=0}^{\infty} (-1)^{\nu} b_{\nu}$, with $b_{\nu} \searrow 0$, is such a series, then its value lies between b_0 and $b_0 - b_1$, more generally, between any two successive partial sums.

Proof: For arbitrary matural ν and ρ,

$$|(-1)^{\nu+1} b_{\nu+1} + \dots + (-1)^{\nu+\rho} b_{\nu+\rho}| = |b_{\nu+1} - b_{\nu+2} + \dots + (-1)^{\rho-1} b_{\nu+\rho}|$$

The sum between the absolute value sign aan be written in

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[10]This whole section is reprinted from J. M. Child, The Early Mathematical Manuscripts of Leibniz already footnoted. Since it is essential to present Leibniz' own thoughts, I have included this section. Please remember that these words are his own, not mine. The passages come from pages 22-27, 28-34, 37-45, 49-52.

the form

$$b_{\nu+1} - (b_{\nu+2} - b_{\nu+3}) - \dots - \begin{cases} b_{\nu+p}, \text{ if } p \text{ is even} \\ b_{\nu+p-1} - b_{\nu+p}, \text{ if } p \text{ is odd} \end{cases}$$

Since $\{b_\nu\}$ is decreasing, this shows that this sum $\geqq 0$ and therefore the absolute value sign on the right may be removed. If this sum is then written in the form

$$b_{\nu+1} - (b_{\nu+2} - b_{\nu+3}) - \dots - \begin{cases} b_{\nu+p-1} - b_{\nu+p}, \text{ if } p \text{ is even} \\ b_{\nu+p}, \text{ if } p \text{ is odd} \end{cases}$$

then this shows further that the sum is $\leqq b_{\nu+1}$. Since $b_\nu \searrow 0$, this is $< \varepsilon$ for all $\nu > \mu$, if μ is chosen so that $b_\mu < \varepsilon$.[11]

<u>VI</u>

Let any curve AEC be referred to a rt. $\angle BAD$; let $AB \sqcap DC \sqcap x$ and let the last $x \sqcap b$; also let $BC \sqcap AD \sqcap y$ and the last $y \sqcap c$. Then it is plain that

$$\text{omn. } \overline{yx \text{ to } x} = \frac{b^2c}{2} - \text{omn. } \overline{\frac{x^2}{2} \text{ to } y} \quad \dots \quad (1)$$

For the moment of the space ABCEA about AD is made up of rectangle contained by $BC (= y)$ and $AB (= x)$. Also the moment about AD of the space ADCEA, the compliment of the former is made up of the sum of the squares on DC halved $\left(= \frac{x^2}{2}\right)$; and if this moment is taken away from the whole moment of the rectangle ABCD about AD, i.e., from c into omn. x, or from $\frac{b^2c}{2}$, there will remain the moment of the space ABCEA. Hence the equation that is obtained is

$$\text{omn } yx \text{ to } x + \text{omn. } \frac{x^2}{2} \text{ to } y = \frac{b^2c}{2} \quad \dots \quad (2)$$

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[11] Konrad Knopp, <u>Infinite Sequences and Series</u> (New York: Dover Publications, 1956), p. 68.

In this way be obtain the quadrature of the two joined in one in every case; and this is the fundamental theorem in the center of gravity method.

Let the equation expressing the nature of the curve be

$$ay^2+bx^2+cxy+dx+ey+f=0 \qquad (3)$$

and suppose that $xy=z$, then $y=z/x$.

Substituting this value in (3), we have

$$\frac{az^2}{x^2}+bx^2+cz+dx+\frac{ez}{x}+f=0 \qquad (4)$$

and on removing the fraction,

$$az^2+bx^4+cx^2z+dx^3+exz+fx^2=0 \qquad (5)$$

Again, let $x^2=2w$; then substitution in (3), we have

$$ay^2+2bw+cxy+dx+ey+f=0 \qquad (6)$$

and:

$$x=\frac{-ay^2-2bw-ey-f}{cy+d} \qquad (7)$$

$$x=\sqrt{2w} \qquad (8)$$

and squaring each side, we have

$$a^2y^2+4aby^2w+2aey^3+2afy^2+4b^2w^2+4bewy+4bfw+e^2y^2+2fey+f^2-2c^2y^2w-4cdyw-2d^2w=0 \qquad (9)$$

If a curve is described according to equation (5), and one according to (9), the quadrature of the figure of the one will depend on the quadrature of the figure of the other, and vice versa.[12]

VII.

$$\overline{dx}=1,\ \overline{dx^2}=2x,\ \overline{dx^3}=3x^2, \text{ etc.}$$

$$\overline{d\tfrac{1}{x}}=-\frac{1}{x^2},\ \overline{d\tfrac{1}{x^2}}=-\frac{2}{x^2},\ \overline{d\tfrac{1}{x^3}}=\frac{3}{x^2}, \text{ etc.}$$

$$\overline{d\sqrt{x}}=\frac{1}{\sqrt{x}}, \text{ etc.}$$

[12]Child, op. cit., p. 65-66.

From these the following general rules may be derived for the differences and sums of the simple powers:

$\overline{dx^e} = e, x^{e-1}$, and conversely $\int \overline{x^e} = \dfrac{x^{e+1}}{e+1}$,

Hence, $\overline{d\frac{1}{x^2}} = \overline{dx^{-2}}$ will be $-2x^{-3}$ or $-2/x^3$,

and $\overline{d\sqrt{x}}$ or $dx^{1/2}$ will be $-\frac{1}{2}x^{-1/2}$ or $-\frac{1}{2}\sqrt{1/x}$.

Let $y = x^2$, then $\overline{dy} = 2x\overline{dx}$ or $\dfrac{\overline{dy}}{dx} = 2x$.

By the same method, the general rules is established as:

$$\frac{\overline{dx^e}}{dx} = ex^{e-1}, \text{ and } \int \overline{x^e \overline{dx}} = \frac{x^{e+1}}{e+1}.$$

Suppose that the equation

$$ay^2 + byx + cz^2 + fx^2 + gy^2 + h^2 = 0$$

is utilized and $y+dy = y$ and $x+dx = x$, by omitting these things that should be omitted, another equation is formed:

$$\left.\begin{array}{c} \overline{ay^2 + byx + cx^2 + f^2x + g^2y + h^2 = 0} \\ a^2\overline{dy}\,y + by\,\overline{dx} + 2cx\,\overline{dx} + f^2\overline{dx} + g^2\overline{dy} \\ \underline{bx\overline{dy}} \\ a\overline{dy}^2 + b\,\overline{dx}\,\overline{dy} + c\overline{dx}^2 = 0 \end{array}\right\} = 0.$$

It can be extended indefinitely: Let there be any number of letters, and any formulas composed from them; for example, let there be the formula made up of three letters,

$$ay^2 \quad bx^2 \quad cz^2 \quad fyx \quad gyx \quad hxz \quad ly \quad mx \quad nz \quad p = 0,$$

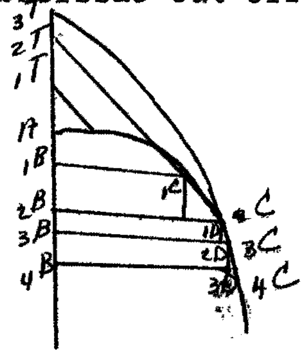
From this comes another equation

$$\begin{array}{c} \underline{ay^2 \quad bx^2 \quad cz^2 \quad fyx \text{ simi} - \quad ly \quad mx \quad \text{simi} - \quad p} \\ 2a\,\overline{dy}\,y \quad 2b\,\overline{dx}\,x \quad 2c\,\overline{dz}\,z \quad fy\,\overline{dx} \quad \text{larly} \quad l\overline{dy} \quad m\overline{dx} \quad \text{larly} \\ \underline{fx\,\overline{dy}} \\ a\overline{dy}^2 \quad b\,\overline{dx}^2 \quad c\overline{dz}^2 \quad f\overline{dx}\,\overline{dy}\ldots \end{array}$$

It is plain from this that by the same method tangent planes to surfaces may be obtained, and in every case that it does not matter whether or not the letters x, y, z have any known relation, for this can be substituted afterward.[13]

VIII.

Let CC be a line, of which the axis is AB, and let BC be ordinates perpendicular to this axis, these being called y, and let AB be the abscissae cut off along the axis, these being called x.



Then CD, the differences of the abscissae, will be called dx; such are ${}_1C_1D$, ${}_2C_2D$, ${}_2C_2D$, ${}_3C_3D$, etc. Also the straight lines ${}_1D_1C$, ${}_2D_3C$, ${}_3D_4C$, the differences of the ordinates, will be called dy. If now these dx and dy are taken to be infinitely small, or the two points on the curve are understood to be a distance apart that is less that any given length, i.e., if ${}_1D_2C$, ${}_2D_3C$, etc. are considered as the momentaneous increments of the line BC, increasing continuously as it descends along AB, then it is plain that the straight line joining these two joints, ${}_2C\,{}_1C$ say, when produced to meet the axis in ${}_1T$, will be the tangent to the curve, and ${}_1T\,{}_1B$ will be to the ordinate ${}_1B_1C$ as ${}_1C_1D$ is to ${}_1D_2C$; or, if, ${}_1T_1B$ or ${}_2T_2B$, etc. are in general called t, then t:y::dx:dy. Thus to find the

d

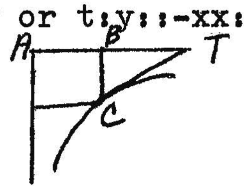
[13]Child, <u>op. cit</u>., p. 124-126.

differences of series is to find tangents:

For example, it is required to find the tangent to the hyperbola. Here, since $y = \frac{aa}{x}$, supposing that in the diagram, x stands for AB the abscissa along on asymptote, and a for the side of the power, or of the area of the rectangle AB.BC; then

$$dy = -\frac{aa}{xx}dx,$$

as will be soon seen when this method of calculus is set forth; hence dx:dy or t:y::-xx:aa::-x: $\frac{aa}{x}$::-x:y; therefore t = -y,



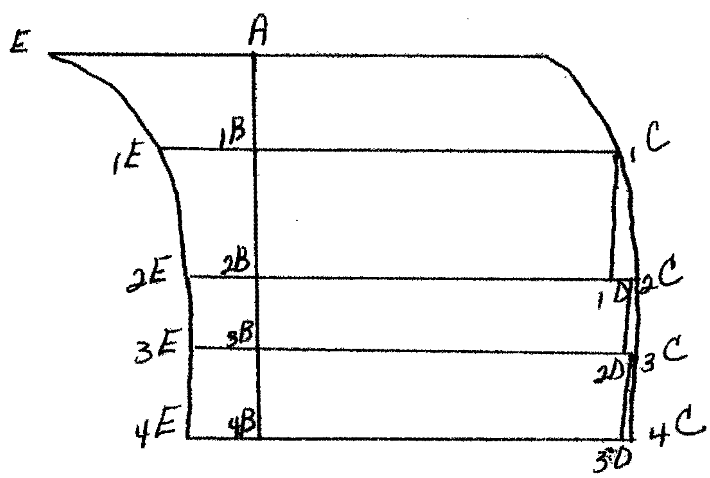
that is, in the hyperbola BT will be equal to AB, but on account of the sign -x, BT must be taken not toward A but in the opposite direction.

Moreover, differences are the opposite to sums; thus ${}_4B_4C$ is the sum of all the differences such as ${}_3D_4C$, ${}_2D_3C$, etc. as far as A, even if they are infinite in number. This fact is represented thus, $\int dy = y$. Also representing the area of a figure by the sum of all the rectangles contained by the ordinates and the differences of the abscissae, i.e., by the sum ${}_1B_1D + {}_2B_2D + {}_3B_3D +$ etc. For the narrow triangle ${}_1C_1D_2C$, ${}_2C_2D_xC$, since they are infinitely small compared with the said rectangles, may be omitted without risk, and thus represented the area of the figure by $\int y\,dx$, or the sum of the rectangle contained by each y and the dx that corresponds to it. Here, if the dx's are taken equal to one another, the method of Cavalieri is obtained.

Obtain the area of a figure by finding the figure of its summatrix or quadratrix, and of this the ordinates are to the ordinates of the given figure in the ratio of sums to differences; for instance, let the curve of the figure required to be squared be EE, and let the ordinates to it, EB, which is called e, be proportional to the differences of the ordinates BC, or to dy; that is let ${}_1B_1E : {}_2B_2E :: {}_1D_2C : {}_2D_2C$, and so on; or again, let $A_1B : {}_1B_1C, {}_1C_1D : {}_1D_2C$, etc. or dx:dy by in the ratio of a constant or never-varying straight line a to ${}_1B_1E$ or e; then

$$dx : dy :: a : e, \text{ or } edx = ady;$$
$$\therefore \int edx = \int ady.$$

Since edx is the same as e multiplied by its corresponding dx, such as the rectangle ${}_3B_4E$, which is formed from ${}_3B_3E$ and ${}^3B^4B$; thence $\int edx$ is the sum of all such rectangles, ${}_3B_4E + {}_2B_1E + {}_3B_2E$+etc., and this sum is the figure A_4B_4EA, if it is supposed that the dx's, or the intervals between the ordinates e, or BC, are infinitely small. Again, ady is the rectangle contained by a and dy, such as is contained by ${}_3D_4C$ and the constant length a, and the sum of



these rectangles, namely $\int ady$, or ${}_3D_4C.a+{}_2D_3C.a+{}_1D_2C.a+$etc. is the same as ${}_3D_4C+{}_2D_3C+{}_1D_2C+$etc. into a, that is, the same as ${}_4B_4C.a$; therefore $\int ady = a\int dy = ay$. Therefore $\int edx = ay$, that is, the area A_4B_4EA will be equal to the rectangle contained by ${}_4B_4C$ and the constant line a, and generally ABEA is equal to the rectangle contained by BC and a (See Proof I).

Thus, for quadratures it is only necessary, being given the line EE, to find the summatrix line CC, and this can always be found by calculus, whether such a line is treated in ordinary geometry or whether it is transcendent and cannot be expressed by algebraical calculation; of this matter in another place.

Now the triangle for the line I called the characteristic of the line, because of its most powerful aid there can be found theorems about the line which are seen to be admirable, and its center of gravity; for ${}_1C_2C$ is equal to $\sqrt{dx.dx+dy.dy}$. From this comes a method for finding the length of a curve by means of some quadrature; i.e., in the case of the parabola, if $y = \frac{xx}{2a}$, then $dy = \frac{xdx}{a}$, and hence ${}_1C_2C = \frac{dx}{a}\sqrt{aa+xx}$; hence, ${}_1C_2C:dx$ as the ordinate of the hyperbola, depends on the quadrature of the hyperbola, as has already been found by others.

If $t:y::dx:dy$; thence $tdy = ydx$, and therefore $\int tdy = \int ydx$. This equation, enunciated geometrically, gives an elegant theorem due to Gregory, namely that if

BAF is a right angle, and AF = BG, and FG is parallel to AB and equal to BT, that is, ${}_1F_1G = {}_1B_1T$, then tdy, or the sum of the rectangles contained by t (${}_4F_4G$ or ${}_4B_4T$) and dy (${}_3F_4F$ or ${}_3D_4C$) is equal to the rectangles ${}_4F_3G+{}_3F_2G+{}_2F_1G$+etc., or the area of the figure A_4F_4GA is equal to ydx, that is, to the figure A_4B_4CA; or generally, the figure AFGA is equal to the figure ABCA.[14]

IX.

Differences and sums are the inverses of one another, that is to say, the sum of the differences of a series is a term of the series, and the difference of the sums of a series is a term of the series and thus $\int dx = x$ for the former and d x = x for the latter.

Thus, let the differences of a series, the series itself, and the sums of the series, be

Differences		1		2		3		4		5...dx	
Series	0		1		3		6		10	15...x	
Sums		0		1		4		10		20	25... x

Then the terms of the series are the sums of the differences, or x = dx; thus 3 = 1+2, 6 = 1+2+3, etc.; on the other hand, the differences of the sums of the series are terms of the series, or $d\int x = x$; thus 3 is the difference between 1 and 4, 6 between 4 and 10.

Also, da = o, if it is given that a is a constant quantity, since a-a = o.

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[14]Child, op. cit., p. 137-141.

ADDITION AND SUBTRACTION:

The difference or sum of a series, of which the general term is made up of the general terms of other series by addition or subtraction, is made up in exactly the same manner from the differences or sums of these series; or

$$x+y-v = \int \overline{dx+dy-dv}, \int \overline{x+y-v} = \int x + \int y - \int v$$

SIMPLE MULTIPLICATION:

Here $dxy = xdx+ydy$, or $xy = \int xdx + \int ydy$ this is what is said about figures taken together with their complements being equal to the circumscribed rectangle. It is demonstrated by the calculus as follows:

dxy is the same thing as the difference between two successive xy's; let one of these be xy, and the other x+dx into y+dy; then

$$dxy = \overline{\overline{x+dx}.\overline{y+dy}} = xdy+ydx+dxdy;$$

the omission of the quantity dxdy, which is infinitely small in comparison with the rest, for it is supposed that dx and dy are infinitely small, will leave xdy+ydx; the signs vary according as y and x increase together, or one increases as the other decreases.

SIMPLE DIVISION:

$$d\frac{y}{x} = \frac{xdy-ydx}{xx}$$

For, $d\frac{y}{x} = \frac{y+dy}{x+dx} - \frac{y}{x} = \frac{xdy-ydx}{xx+xdx}$, which becomes equal to $\frac{xdy-ydx}{xx}$ (if xx is equal to xx+xdx, since xdx can be omitted as being infinitely small in comparison with xx); also, if y = aa, then dy = o, and the result becomes $\frac{-aadx}{xx}$, which is the value used before.

From this any one can deduce by calculus the rules for Compound Multiplication and Division; thus,

$$dxvy = xydv+xvdy+yvdx,$$

$$d\frac{y}{vz} = \frac{vzdy-yvdz-yzdv}{vv.zz};$$

as can be proved from what has gone before; then

$$d\frac{y}{x} = \frac{xdy-ydx}{xx};$$

hence, by putting zv for x, and zdv+vdz for dx or dzv in the above, the proof is complete. Powers follow: $dx^2 = 2xdx$, $dx^3 = 3x^2dx$, and so on. For, putting y = x, and v = x, then write dx^2 for dxy, and this is equal to xdy+ydx, or equal to 2xdx. Similarly, for dx^3 write dxyv, that is xydv+xvdy+yvdx, or equal to $3x^2dx$. By the same method, $dx^e = e.x^{e-1}dx$.

Hence also, $d\frac{1}{x^h} = \frac{-h\,dx}{x^{h+1}}$.

For, if $\frac{1}{x^h} = x^e$, then $e = -h$ and $x^{e-1} = \frac{1}{x^{h+1}}$, as is well known. The same will do for fractions. The procedure is the same for irrationals or roots. $d\sqrt[r]{x^h} = dx^{h:r}$, or $e \cdot x^{e-1}dx$ (or substituting once more $h:r$ for e, and $\overline{h-r}:r$ for $e-1$) $\frac{h}{r} \cdot x^{\overline{h-r:r}}dx$ thus comes $d\sqrt[r]{x^h}$.

Moreover, conversely,

$$\int x^e \cdot dx = \frac{x^{e+1}}{e+1}, \int \frac{1}{x^e}dx = -\frac{1}{\overline{e-1} \cdot x^{e-1}}, \int \sqrt[r]{x^h} \cdot dx = \frac{r}{r+h}\sqrt[r]{x^{h+r:r}}.$$

These are the elementary principles of the differential and summatory calculus, by means of which highly complicated formulas can be dealt with, not only for a fraction or an irrational quantity, or anything else; but also an indifinite quantity, such as x or y, or any other thing expressing generally the terms of any series, may enter into it.[15]

X.

"Jack of all trades, master of none" has its spectacular exceptions like any other folk proverb, and Gottfried Wilhelm Leibniz is one of them. Mathematics was but one of the many fields in which Leibniz showed conspicuous genius. The mechanical operations which mathematician perform to discover new facets of logic and fact all date back to Leibniz. All the great ideas of importance of a calculus in assisting deduction are found in his writings. It was Leibniz' main wish that some day some would delve deeply into mathematical ideas and join the beauty of their minds to the labor of his. This is exactly what every mathematician has done. He accomplished his wish and with no reserve is truly proclaimed the master of infinitesimal calculus.

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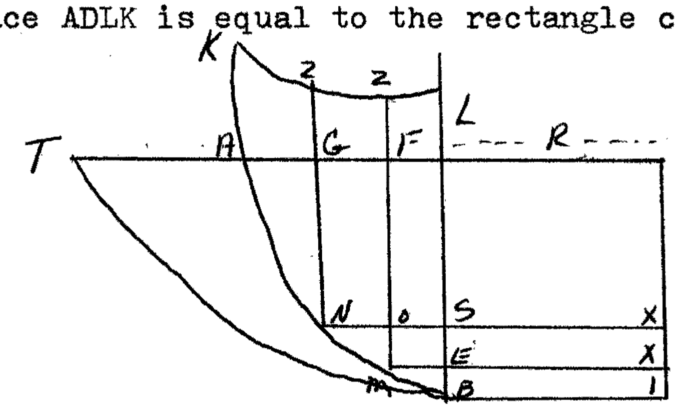
[15]Child, op. cit., p. 142-144.

PROOF I

Let AMB be a curve of which the axis is AD and let BD be perpendicular to AD; also let KZL be another line such that, when any point M is taken in the curve AB, and through it are drawn MT a tangent to the curve AB, and MFZ parallel to DB, cutting KZ in Z and AD in F, and R is the space ADLK is equal to the rectangle contained by R and DB.



For, if DH = R and the rectangle BDHI is completed, and MN is taken to be an indifinitely small arc of the curve AB, and MEX, NOS are drawn parallel to AD; then NO:MD = TF:FM = R:FZ;

NO:FZ = MO.R and FG.FZ = ES.EX.

Hence, since the sum of such rectangles as FG.FZ differs only in the least degree from the space ADLK, and the rectangles ES.EX form the rectangles DHIB, the theorem is obvious.

I.

NOVA METHODUS PRO MAXIMIS ET MINIMIS, ITEMQUE TANGENTIBUS, QUAE NEC FRACTAS NEC IRRATIONALES QUANTITATES MORATUR, ET SINGULARE PRO ILLIS CALCULI GENUS *).

Sit (fig. 111) axis AX, et curvae plures, ut VV, WW, YY, ZZ, quarum ordinatae ad axem normales, VX, WX, YX, ZX, quae vocentur respective v, w, y, z, et ipsa AX, abscissa ab axe, vocetur x. Tangentes sint VB, WC, YD, ZE, axi occurrentes respective in punctis B, C, D, E. Jam recta aliqua pro arbitrio assumta vocetur dx, et recta, quae sit ad dx, ut v (vel w, vel y, vel z) est ad XB (vel XC, vel XD, vel XE) vocetur dv (vel dw, vel dy, vel dz) sive differentia ipsarum v (vel ipsarum w, vel y, vel z). His positis, calculi regulae erunt tales.

Sit a quantitas data constans, erit da aequalis 0, et $d\overline{ax}$ erit aequalis adx. Si sit y aequ. v (seu ordinata quaevis curvae YY aequalis cuivis ordinatae respondenti curvae VV) erit dy aequ. dv. Jam *Additio et Subtractio*: si sit z — y + w + x aequ. v, erit $d\overline{z - y + w + x}$ seu dv aequ. dz — dy + dw + dx. *Multiplicatio*: $d\overline{xv}$ aequ. xdv + vdx. seu posito y aequ. xv, fiet dy aequ. xdv + vdx. In arbitrio enim est vel formulam, ut xv, vel compendio pro ea literam, ut y, adhibere. Notandum, et x et dx eodem modo in hoc calculo tractari, ut y et dy, vel aliam literam indeterminatam cum sua differentiali. Notandum etiam, non dari semper regressum a differentiali Aequatione, nisi cum quadam cautione, de quo alibi. Porro *Divisio*: $d\frac{v}{y}$ vel (posito z aequ. $\frac{v}{y}$) dz aequ. $\frac{\pm vdy \mp ydv}{yy}$.

Quoad *Signa* hoc probe notandum, cum in calculo pro litera substituitur simpliciter ejus differentialis, servari quidem eadem signa, et pro + z scribi + dz, pro — z scribi — dz, ut ex addi-

*) Act. Erud. Lips. an. 1684.

LEIBNIZ'S FIRST PAPER ON THE CALCULUS (As reprinted by C. I. Gerhardt, 1858)

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