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# Dissertation: The Least Fixed Point Property for $\omega$ -chain Continuous Functions

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UNIVERSITY OF CALIFORNIA  
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The Least Fixed Point Property for  
 $\omega$ -Chain Continuous Functions

A Dissertation submitted in partial satisfaction  
of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

by

Joe Don Mashburn

June, 1981

Dissertation Committee:

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ABSTRACT OF THE DISSERTATION

The Least Fixed Point Property  
for  $\omega$ -Chain Continuous Functions

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Joe Don Mashburn

Doctor of Philosophy, Graduate Program in Mathematics

University of California, Riverside, June 1981

Professor Albert R. Stralka, Chairman

The basic definitions are given in the first section, including those for  $\omega$ -chain continuity,  $\omega$ -chain completeness, and the least fixed point property for  $\omega$ -chain continuous functions. Some of the relations between completeness and fixed point properties in partially ordered sets are stated and it is briefly shown how the question basic to the dissertation arises.

In the second section, two examples are given showing that a partially ordered set need not be  $\omega$ -chain complete to have the least fixed point property for  $\omega$ -chain continuous functions. The first example shows that the least fixed point of an  $\omega$ -chain continuous function is not in gen-

eral equal to  $\sup\{f^n(o):n \in \mathbb{N}\}$ . The second example is a partially ordered set which has the least fixed point property for  $\omega$ -chain continuous functions even though it contains an unbounded countable chain. In this case, the least fixed point is always equal to  $\sup\{f^n(o):n \in \mathbb{N}\}$ . The free meet-semilattice generated by this second example is also obtained and shown to have the least fixed point property for  $\omega$ -chain continuous functions.

Retracts are discussed in section 3, where it is seen that they are not sufficient to characterize those partially ordered sets having the least fixed point property for  $\omega$ -chain continuous functions. A lemma which will be useful later is proved characterizing partially ordered sets which have unbounded countable chains as retracts.

In section 4 the relation between finite width and the least fixed point property for  $\omega$ -chain continuous functions is explored. Two useful lemmas are proved which will allow us to restrict the cases that need to be checked when proving theorems later. A third lemma is obtained which says that in a partially ordered set of finite width, if A and B are disjoint subsets and B is an upper end, then there are only a finite number of elements of B which are the suprema of countable chains of A. From this lemma, the first theorem is derived. This states that if C is a nonempty unbounded countable chain in a partially ordered set, P, of finite width, then any cofinal subchain of C of order type  $\omega$  is a

retract of  $P$ . This leads to the second theorem: a characterization of  $\omega$ -chain complete partially ordered sets of finite width. This says that a partially ordered set of finite width is  $\omega$ -chain complete only if it has the least fixed point property for  $\omega$ -chain continuous functions. It is known that the implication can be reversed.

Section 5 introduces the notion of a layered partially ordered set and discusses some of its problems. The first theorem in this section says that if  $P$  is a layered partially ordered set in which every antichain generates a closed lower end, then any chain in  $P$  of order type  $\omega$  is a retract of  $P$ . The second theorem says that if  $P$  is a partially ordered set in which every antichain generates a closed lower end and every lower end of  $P$  which contains a countable chain but none of its upper bounds is layered, then  $P$  is  $\omega$ -chain complete if it has the least fixed point property for  $\omega$ -chain continuous functions. It is a corollary of a theorem by Edwin Miller that every partially ordered set in which every antichain is finite and every chain contains a countable cofinal subchain is layered. From this and the previous theorem it is easy to see that a partially ordered set in which every antichain is finite and every chain contains a countable cofinal subchain is  $\omega$ -chain complete if and only if it has the least fixed point property for  $\omega$ -chain continuous functions.

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## I. INTRODUCTION

A partially ordered set is chain complete if each of its chains has a least upper bound. It is  $\omega$ -chain complete if each of its countable chains has a least upper bound. Since the empty set is considered to be a countable chain, any partially ordered set which is chain or  $\omega$ -chain complete has a least element, denoted  $o$ .

A function,  $f$ , mapping a partially ordered set,  $P$ , to a partially ordered set,  $Q$ , is  $\omega$ -chain continuous if and only if for every nonempty countable chain,  $C$ , of  $P$  which has a supremum in  $P$ ,  $f(\sup_P(C))$  equals  $\sup_Q(f(C))$ . An  $\omega$ -chain continuous function, then, must preserve order.

A partially ordered set,  $P$ , has the (least) fixed point property if and only if every order preserving function from  $P$  to itself has a (least) fixed point. It has the (least) fixed point property for  $\omega$ -chain continuous functions if and only if every  $\omega$ -chain continuous function from  $P$  to itself has a (least) fixed point.

In 1955 A. Tarski ([T], Thm. 1) and A. C. Davis ([Da], Thm. 2) characterized complete lattices as those lattices having the fixed point property. In 1976 G. Markowsky ([M], Thm. 11) characterized chain complete partially ordered sets as those having the least fixed point property. About the same time partially ordered sets and categories were being developed as models for theoretical computer

science ([Sc1],[Sc2],[Sm]). The existence of fixed points is very important for these models, so it is necessary to be sure that the partially ordered set or category in question has the appropriate fixed point property. It was in this context that G. Plotkin asked in 1978 if partially ordered sets with the least fixed point property for  $\omega$ -chain continuous functions have a characterization similar to that for partially ordered sets with the least fixed point property. That is, is a partially ordered set which has the least fixed point property for  $\omega$ -chain continuous functions necessarily  $\omega$ -chain complete? It is known that if a partially ordered set,  $P$ , is  $\omega$ -chain complete, then it has the least fixed point property for  $\omega$ -chain continuous functions. If  $f:P \rightarrow P$  is  $\omega$ -chain continuous, then

$$f(\sup\{f^n(o) : n \in \mathbb{N}\}) = \sup(f(\{f^n(o) : n \in \mathbb{N}\})) = \sup\{f^n(o) : n \in \mathbb{N}\}.$$

A partially ordered set which does not have a least element cannot have the least fixed point property for  $\omega$ -chain continuous functions, since the identity map is an  $\omega$ -chain continuous function which, in that case, would not have a least fixed point. Therefore, partially ordered sets will always be assumed to have a least element. The symbol  $\mathbb{N}$  will be used to denote the positive integers;  $\mathbb{N}_0$  will denote the nonnegative integers; and  $\mathbb{N}_\omega$  will denote the positive integers with infinity.

## II. SOME EXAMPLES

The following example is a partially ordered set which is not  $\omega$ -chain complete, yet has the least fixed point property for  $\omega$ -chain continuous functions.

EXAMPLE 1. Let  $P_1$  be the set  $\mathbb{N}_0 \times \{0,1,2\}$ . Order  $P_1$  by setting  $(n,m) \leq (n',m')$  if and only if one of the following conditions holds:

- 1)  $n=m=0$
- 2)  $n \leq n'$  and  $m=m' \in \{0,1\}$
- 3)  $m=n'=0$
- 4)  $m=0$  and  $m'=2$
- 5)  $m=m'=1$  and  $n'=0$
- 6)  $0 < n \leq n'$ ;  $m=1$ ; and  $m'=2$
- 7)  $n=0$  and  $m=m'=2$

$P_1$  can be represented by the diagram below.

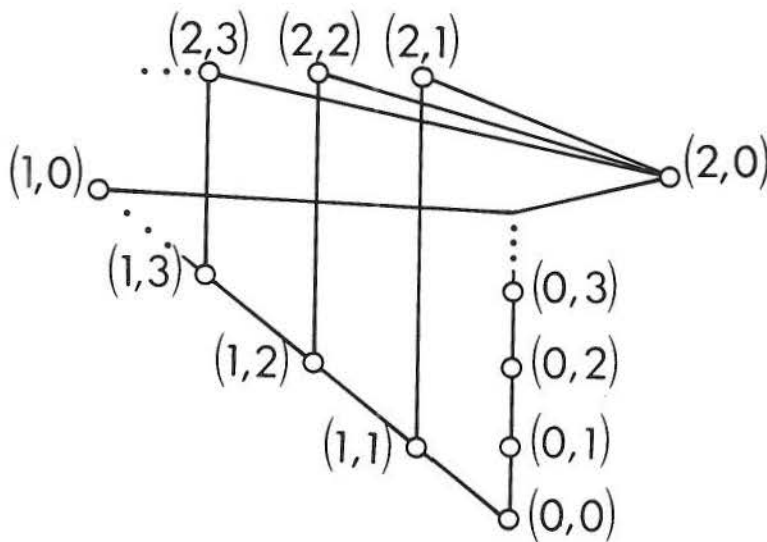


FIG. 1

Let  $f$  be an  $\omega$ -chain continuous function from  $P_1$  to itself such that  $f((0,0))$  is not  $(0,0)$ . If  $f((0,0))$  is an element of  $P_1 - (\mathbb{N} \times \{0\})$ , then  $f(P_1)$  is contained in  $P_1 - (\mathbb{N} \times \{0\})$ . In that case,  $f$  must have a least fixed point since  $P_1 - (\mathbb{N} \times \{0\})$  is  $\omega$ -chain complete. Therefore, assume that  $f((0,0))$  is an element of  $\mathbb{N} \times \{0\}$ . Let  $U$  denote the set of upper bounds of  $\mathbb{N} \times \{0\}$ . Then  $f(P_1)$  is contained in  $(\mathbb{N} \times \{0\}) \cup U$ . If  $f$  has a fixed point in  $\mathbb{N} \times \{0\}$ , then  $f$  has a least fixed point in  $\mathbb{N} \times \{0\}$ . Assume that  $f$  has no fixed points in  $\mathbb{N} \times \{0\}$ . This forces  $f(U)$  to be a subset of  $U$ .

Assume that  $f((0,2))$  is  $(0,1)$ . Then for every  $n$  in  $\mathbb{N}$ ,  $f((n,2))$  equals  $(0,1)$ , and  $f((n,1)) \leq (0,1)$ . If  $f(\mathbb{N} \times \{1\})$  is contained in  $\mathbb{N} \times \{0\}$ , then  $(0,1)$  must also be in  $\mathbb{N} \times \{0\}$ , contradicting the assumption that  $f(U)$  is a subset of  $U$ . Thus, there is an  $N$  in  $\mathbb{N}$  such that, for every  $n \geq N$ ,  $f((n,1))$  equals  $(0,1)$ . Hence  $f((0,1))$  equals  $(0,1)$ .

If  $f((0,2))$  equals  $(0,2)$  then, for every  $n$  in  $\mathbb{N}$ ,  $f((n,2))$  must be above  $(0,2)$ . Since there is no element of  $P_1$  greater than  $(0,2)$  and  $(0,1)$ ,  $f((n,1))$  cannot equal  $(0,1)$  for any  $n$  in  $\mathbb{N}$ . Thus,  $f((0,1))$  cannot equal  $(0,1)$ , and  $(0,2)$  is the least fixed point of  $f$ .

If  $f((0,2))$  equals  $(m,2)$  for some  $m$  in  $\mathbb{N}$ , then, for every  $n$  in  $\mathbb{N}$ ,  $f((n,2))$  equals  $(m,2)$ . It again follows that  $f((0,1))$  cannot equal  $(0,1)$ ; and  $P_1$  has the least fixed point property for  $\omega$ -chain continuous functions.

As noted earlier, if a partially ordered set is  $\omega$ -chain

complete, then the least fixed point of an  $\omega$ -chain continuous function,  $f$ , from the set to itself is just  $\sup\{f^n(o):n \in \mathbb{N}\}$ . In  $P_1$ , however, this is not always true. Define a function,  $f$ , from  $P_1$  to itself by

$$f(x) = \begin{cases} (0,1) & \text{if } x \notin \mathbb{N} \times \{0\} \\ (n+1,0) & \text{if } x = (n,0) \end{cases}$$

This function is clearly  $\omega$ -chain continuous and has  $(0,1)$  as its only fixed point. But  $\sup\{f^n((0,0)):n \in \mathbb{N}\}$  does not even exist.

The partially ordered set  $P_1$  has the least fixed point property for  $\omega$ -chain continuous functions because of the countable chain which sneaks around the antichain of upper bounds of  $\mathbb{N} \times \{0\}$ . The next example shows that a partially ordered set can have an unbounded countable chain and still have the least fixed point property for  $\omega$ -chain continuous functions. Also, the least fixed point will be  $\sup\{f^n(o):n \in \mathbb{N}\}$ .

EXAMPLE 2. Let  $Q$  be the set  $(\mathbb{N} \times \mathbb{N}_\infty) \cup \{o\}$ . Order  $Q$  by setting  $x \leq y$  if and only if one of the following conditions holds.

- 1)  $x = o$
- 2)  $x = (j,k)$ ;  $y = (n,m)$ ; and either  $m = \infty = k$  and  $j \leq n$ , or  $j = n$  and  $k \leq m$

Then  $Q$  can be represented by the diagram below.

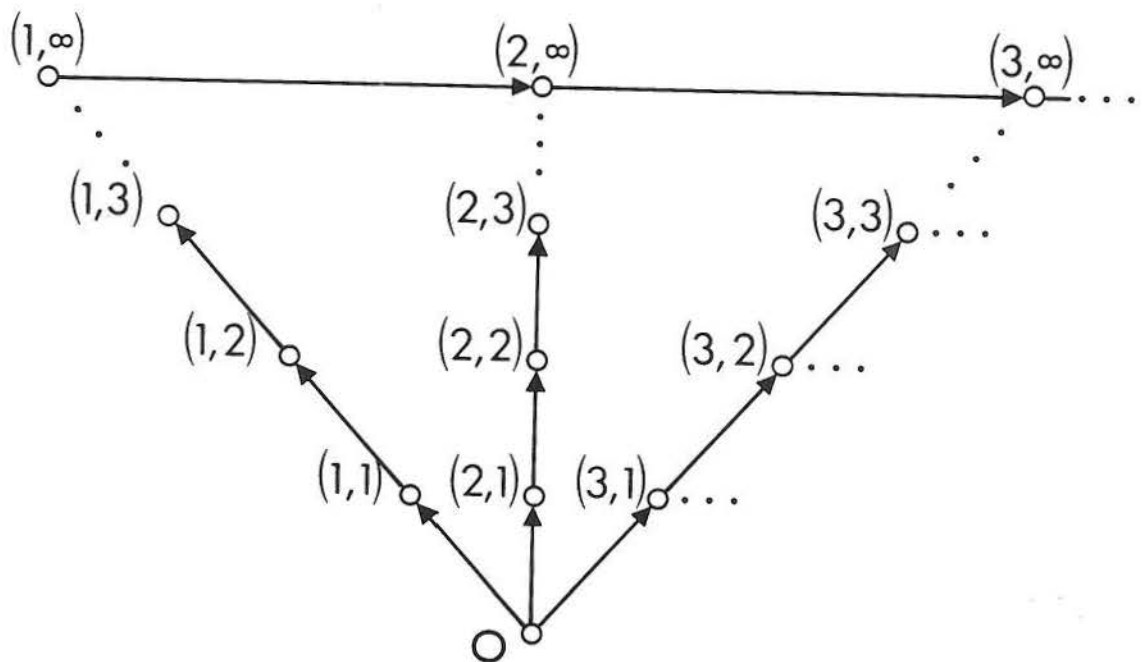


FIG. 2

Let  $X$  be the set  $\mathbb{N} \times \{\infty\}$ . For every  $n$  in  $\mathbb{N}$ , let  $X_n$  be the set  $\{n\} \times \mathbb{N}$ . Let  $P_2$  be the set  $Q \cup \mathbb{N}^{\mathbb{N}}$  and order  $P_2$  by setting  $x \leq y$  if and only if  $x$  and  $y$  are elements of  $Q$  and  $x \leq y$  in  $Q$ ; or  $y = (n_1, n_2, \dots)$  is an element of  $\mathbb{N}^{\mathbb{N}}$ ,  $x$  is an element of  $Q$ , and  $x \leq (m, n_m)$  for some  $m$  in  $\mathbb{N}$ . Then  $X$  is an unbounded countable chain in  $P_2$  and  $\mathbb{N}^{\mathbb{N}}$  is an antichain.

Let  $f$  be an  $\omega$ -chain continuous function from  $P_2$  to itself such that  $f(o)$  is not  $o$ . If  $\sup\{f^n(o) : n \in \mathbb{N}\}$  exists, then it is the least fixed point of  $f$ . It may therefore be assumed that there is no  $m$  in  $\mathbb{N}$  such that  $f^m(o)$  is an element of  $\mathbb{N}^{\mathbb{N}}$ . If  $\{f^n(o) : n \in \mathbb{N}\}$  is contained in  $X_m$  for some  $m$  in  $\mathbb{N}$ , then  $\sup\{f^n(o) : n \in \mathbb{N}\}$  exists. Therefore, assume that for some  $N$  in  $\mathbb{N}$ , for every  $n \geq N$ ,  $f^n(o)$  is an element of  $X$ .

Let  $M$  be the smallest positive integer such that  $f(o) \leq (M, \infty)$ . Then  $f(\{(n, \infty) : n \geq M\})$  is contained in  $X$ ; and so, for every  $n \geq M$ , there is a  $j_n$  in  $\mathbb{N}$  such that, if  $k \geq j_n$ ,  $f((k, n))$  equals  $f((n, \infty))$ . Let  $x$  be the element of  $\mathbb{N}^{\mathbb{N}}$  whose  $n^{\text{th}}$  component is  $j_n$  for every  $n$  in  $\mathbb{N}$ . Then for every  $n$  in  $\mathbb{N}$ ,  $f(x) \geq f((j_n, n)) = f((n, \infty))$ . It follows that  $f(x)$  is an upper bound of  $\{f^n(o) : n \in \mathbb{N}\}$ ; that  $\{f^n(o) : n \in \mathbb{N}\}$  is finite; and that  $\sup\{f^n(o) : n \in \mathbb{N}\}$  exists.

This example can be used to show another difference between fixed point properties for  $\omega$ -chain continuous functions and those for order preserving functions. R. E. Smithson ([S2], Thm 1.2) proved the following theorem, which will not be true for the fixed point property for  $\omega$ -chain continuous functions and  $\omega$ -chain completeness.

**THEOREM 1.** Let  $P$  be a partially ordered set with a least element such that for every chain,  $C$ , of  $P$ , the set of upper bounds of  $C$  is down directed. If  $P$  has the fixed point property, then  $P$  is chain complete.

This theorem has the following obvious corollary.

**COROLLARY 1.** A meet-semilattice has the fixed point property if and only if it is chain complete.

These theorems do not carry over to  $\omega$ -chain continuous functions and  $\omega$ -chain completeness, however, since  $\mathfrak{F}(P_2)$ , the free meet-semilattice generated by  $P_2$ , contains an unbounded countable chain, but has the least fixed point property for  $\omega$ -chain continuous functions. The pairs of elements of  $P_2$  which do not already have infima are those pairs in which both elements are from the antichain  $\mathbb{N}^{\mathbb{N}}$ , and those in which

one element is from  $\mathbb{N}^{\mathbb{N}}$  and the other is from  $X$ .

EXAMPLE 3. Let  $P_3$  be the set  $P_2 \cup (\text{FS}'(\mathbb{N}^{\mathbb{N}})) \cup (X \times \text{FS}(\mathbb{N}^{\mathbb{N}}))$ , where  $\text{FS}(\mathbb{N}^{\mathbb{N}})$  is the set of all nonempty finite subsets of  $\mathbb{N}^{\mathbb{N}}$  and  $\text{FS}'(\mathbb{N}^{\mathbb{N}})$  is the set of all finite subsets of  $\mathbb{N}^{\mathbb{N}}$  which have at least two elements. Define an order on  $P_3$  by setting  $x \leq y$  if and only if one of the following conditions holds:

- 1)  $x = A \in \text{FS}'(\mathbb{N}^{\mathbb{N}})$ ;  $y \in P_2$ ; and  $y \in A$
- 2)  $x \in P_2$ ;  $y = A \in \text{FS}(\mathbb{N}^{\mathbb{N}})$ ; and  $x \leq a$  for every  $a$  in  $A$
- 3)  $x = ((n, \infty), A) \in X \times \text{FS}(\mathbb{N}^{\mathbb{N}})$ ;  $y \in P_2$ ; and  $(n, \infty) \leq y$  or  $A \leq y$
- 4)  $x \in P_2$ ;  $y = ((n, \infty), A) \in X \times \text{FS}(\mathbb{N}^{\mathbb{N}})$ ; and  $x \leq (n, \infty)$  and  $x \leq A$
- 5)  $x = A \in \text{FS}'(\mathbb{N}^{\mathbb{N}})$ ;  $y = B \in \text{FS}'(\mathbb{N}^{\mathbb{N}})$ ; and  $B \subset A$
- 6)  $x = ((n, \infty), A) \in X \times \text{FS}(\mathbb{N}^{\mathbb{N}})$ ;  $y = B \in \text{FS}'(\mathbb{N}^{\mathbb{N}})$ ; and  $B \subset A$
- 7)  $x = ((m, \infty), A) \in X \times \text{FS}(\mathbb{N}^{\mathbb{N}})$ ;  $y = ((n, \infty), B) \in X \times \text{FS}(\mathbb{N}^{\mathbb{N}})$ ; and  $m \leq n$  and  $B \subset A$

Then  $P_3$  is the free meet-semilattice generated by  $P_2$ . Notice that  $X$  is still unbounded in  $P_3$ . In fact, it is still the only chain that does not have a supremum.

Let  $C = \{c_n : n \in \mathbb{N}\}$  be an infinite chain in  $P_3 - X$ . Clearly, if  $C$  is a subset of  $P_2$  then  $C$  has a supremum. Therefore, assume that  $C$  is a subset of  $\text{FS}'(\mathbb{N}^{\mathbb{N}}) \cup (X \times \text{FS}(\mathbb{N}^{\mathbb{N}}))$ . For every  $k$  in  $\mathbb{N}$ , let  $A_k$  be an element of  $\text{FS}(\mathbb{N}^{\mathbb{N}})$  such that either  $c_k = ((n_k, \infty), A_k)$  or  $c_k = A_k$ . Then, since every  $A_k$  is finite,  $A = \bigcup_{k \in \mathbb{N}} A_k$  is in  $\text{FS}(\mathbb{N}^{\mathbb{N}})$  and  $c_k \leq A_k \leq A$  for every  $k$ . If there were an  $m$  in  $\mathbb{N}$  such that, for every  $k$  in  $\mathbb{N}$ ,  $n_k \leq m$ , then  $C$



would be finite, since the sequence  $A_1, A_2, \dots$  must be finite. If  $B$  is an element of  $FS'(\mathbb{N}^{\mathbb{N}})$  and  $B$  is an upper bound of  $C$ , then  $B$  is a subset of  $A_k$  for all  $k$  and therefore  $A \leq B$ . If  $x$  is an element of  $\mathbb{N}^{\mathbb{N}}$  and  $x$  is an upper bound of  $C$ , then  $x$  is an element of  $A_k$  for all  $k$  and so  $A \leq x$ . Thus  $A$  is the supremum of  $C$ . The proof that  $P_3$  has the least fixed point property for  $\omega$ -chain continuous functions is now the same as that for  $P_2$ .

### III. RETRACTS AND AN EXAMPLE

A function,  $f$ , from a partially ordered set,  $P$ , to itself is a retraction if and only if  $f$  is  $\omega$ -chain continuous and is the identity on  $f(P)$ . The image of a retraction on  $P$  is called a retract of  $P$ . Obviously, if  $P$  has the least fixed point property for  $\omega$ -chain continuous functions, then so does every retract of  $P$ . Is it possible to make the implication go the other way? If  $C$  is a non-empty countable chain in  $P$  which does not have a supremum in  $P$ ;  $U$  is a set of upper bounds of  $C$ ; and  $CUU$  is a retract of  $P$ , then if  $P$  is to have the least fixed point property,  $U$  must have a least element. This is the case in the examples given so far. In [M], Markowsky used such a property for his characterization of chain complete partially ordered sets.

Lemma 1 gives a better idea of when a partially ordered set can have an unbounded countable chain as a retract. Recall that any nonempty countable chain contains a cofinal subchain of order type  $\omega$ .

DEFINITION. A subset,  $D$ , of a partially ordered set,  $P$ , is closed if and only if for every nonempty countable chain,  $C$ , in  $D$ , if  $C$  has a supremum in  $P$  then  $\sup(C)$  is in  $D$ . A subset of  $P$  is open if and only if its complement is closed.

LEMMA 1. An unbounded countable chain  $C = \{c_n : n \in \mathbb{N}\}$  of a

partially ordered set,  $P$ , is a retract of  $P$  if and only if there is a collection of disjoint closed subsets of  $P$ ,

$\{D_n : n \in \mathbb{N}\}$ , such that

$$1) P = \bigcup_{n \in \mathbb{N}} D_n$$

$$2) \forall n \in \mathbb{N}, c_n \in D_n$$

$$3) \text{ if } p \in D_m; q \in D_n; \text{ and } p \leq q, \text{ then } m \leq n.$$

PROOF. Assume that  $C$  is a retract of  $P$  and let  $f: P \rightarrow C$  be a retraction. For every  $n$  in  $\mathbb{N}$ , let  $D_n$  be the set  $f^{-1}(c_n)$ . Clearly the  $D_n$ 's form a collection of disjoint subsets of  $P$ ;  $P$  equals  $\bigcup_{n \in \mathbb{N}} D_n$ ; and, since  $f$  preserves order, if  $p$  is an element of  $D_m$ ,  $q$  is an element of  $D_n$ , and  $p \leq q$ , then  $m \leq n$ . Let  $X = \{x_n : n \in \mathbb{N}\}$  be a chain in  $D_m$  which has a supremum in  $P$ . Then  $f(\sup(X)) = \sup(f(X)) = c_m$ , so that  $\sup(X)$  is an element of  $D_m$ .

Now let  $\{D_n : n \in \mathbb{N}\}$  be a collection of disjoint closed subsets of  $P$  having the properties listed in the lemma. Define a function,  $f$ , from  $P$  to itself by  $f(x) = c_n$  if and only if  $x$  is an element of  $D_n$ . This function is obviously order preserving, so it remains to show that it preserves the supremums of nonempty countable chains. Let  $X = \{x_n : n \in \mathbb{N}\}$  be an infinite chain which has a supremum in  $P$ . Assume that  $\sup(X)$  is an element of  $D_m$ . Then  $X$  is contained in  $\bigcup_{n \leq m} D_n$ .

Since  $X$  is infinite, one of these  $D_n$ 's must contain an infinite number of elements of  $X$ . Then it will contain a cofinal

subchain of  $X$  and hence  $\sup(X)$ . Thus only  $D_m$  contains an infinite number of elements of  $X$ . So  $f(\sup(X))$  equals  $\sup(f(X))$ . Example 4 shows that it is not enough to consider retracts as Markowsky was able to do for order preserving functions.

EXAMPLE 4. Let  $P_4$  be the subset of  $\mathbb{N}_\infty^{\mathbb{N}}$  consisting of those elements  $\bar{x}=(x_1, x_2, \dots)$  such that  $x_n$  equals  $\infty$  for only a finite number of  $n$ 's, and if  $x_{n+1}$  equals  $\infty$  then so does  $x_n$ . Order  $P_4$  with the usual product ordering. Then  $P_4$  is a lattice and every nonempty subset of  $P_4$  has an infimum. Thus, every bounded chain in  $P_4$  has a supremum.

If  $C$  is a nonempty bounded countable chain in  $P_4$ ;  $U$  is a set of upper bounds of  $C$ ; and  $f:P_4 \rightarrow CUU$  is a retraction, then  $\sup_{CUU}(C)$  is an element of  $U$  since  $f(\sup_{P_4}(C)) = \sup_{CUU}(f(C)) = \sup_{CUU}(C)$ . Therefore  $U$  has a least element.

For every  $n$  in  $\mathbb{N}$  let  $\bar{c}_n$  be the element of  $P_4$  having  $\infty$  as its first  $n$  components and 1 everywhere else. Then  $C = \{\bar{c}_n : n \in \mathbb{N}\}$  is an unbounded chain. Let  $\{D_n : n \in \mathbb{N}\}$  be a collection of disjoint closed subsets of  $P_4$  such that, for each  $n$ ,  $\bar{c}_n$  is an element of  $D_n$  and if  $p$  and  $q$  are elements of  $D_m$  and  $D_n$  respectively, then if  $p \leq q$ ,  $m \leq n$ . For every  $n$ , let  $E_n$  be the set  $\bigcup_{m \leq n} D_m$ .

Set  $x_1$  equal to 1. Assume that  $x_k$  has been defined for every  $k \leq n$ . Let  $X_n$  be the set of all positive integers,  $x$ ,

such that  $(x_1, \dots, x_n, x, 1, 1, 1, \dots)$  is an element of  $E_n$ . If  $X_n$  is empty, set  $x_{n+1}$  equal to 1. If not, then it must be finite since  $E_n$  is closed and does not contain  $\bar{c}_{n+1}$ . In this case, set  $x_{n+1}$  equal to  $(\max(X_n))+1$ . Each  $x_n$  defined by this process is a positive integer, so  $\bar{x}=(x_1, x_2, \dots)$  is in  $P_4$ . But  $\bar{x}$  is not contained in any  $D_n$ . If  $\bar{x}$  were in  $D_n$  for some  $n$ , then  $\bar{y}=(x_1, \dots, x_{n+1}, 1, 1, 1, \dots)$  would be in  $D_m$  for some  $m \leq n$ , since  $\bar{y} \leq \bar{x}$ . Thus,  $\bar{y}$  would be in  $E_n$ , which contradicts the definition of  $x_{n+1}$ .

Let  $C'=\{\bar{c}'_n; n \in \mathbb{N}\}$  be an unbounded chain in  $P_4$  and let  $f$  be an  $\omega$ -chain continuous map from  $P_4$  to  $C'$ . For every  $n$  in  $\mathbb{N}$ , define a sequence whose supremum is  $\bar{c}'_n$  as follows. For every  $j, k \in \mathbb{N}$ , let

$$x_{nj k} = \begin{cases} j & \text{if } j \leq c'_{nj} \text{ and } k \leq j \\ c'_{nj} & \text{if } c'_{nj} < j \text{ and } k \leq j \\ 1 & \text{if } j < k \end{cases}$$

where  $\bar{c}'_n=(c'_{n1}, c'_{n2}, \dots)$ , and set  $\bar{x}_{nj}=(x_{nj1}, x_{nj2}, \dots)$ . Let  $X_n$  be the set of all the  $\bar{x}_{nj}$ 's. Notice that for every  $n$  in  $\mathbb{N}$ , every element of  $X_n$  is less than an element of  $C$ . Therefore, if  $C'$  is the image of  $P_4$  under  $f$ ,  $f(C)$  must be a cofinal subchain of  $C'$ . Assume that this is so and define a function,  $g$ , taking  $C'$  onto a cofinal subchain of  $C$  by  $g(x)=\min\{\bar{c}_n : x \leq f(\bar{c}_n)\}$ . Then  $g \circ f$  is a retraction from  $P_4$  onto a cofinal subchain of  $C$ . But this is impossible, so  $C'$  cannot be the image of  $P_4$  under  $f$ . Thus, no unbounded chain is a retract of  $P_4$ . In fact, the same argument will show that

there is no  $\omega$ -chain continuous function from  $P_4$  onto an unbounded countable chain.

Even though  $P_4$  satisfies this nice property for retracts, it does not have the least fixed point property for  $\omega$ -chain continuous functions. Denote the point  $(1,1,\dots)$  by  $\bar{1}$ , and, for every  $\bar{x}=(x_1,x_2,\dots)$  in  $P_4$ , let  $\bar{x}+\bar{1}$  be the point  $(x_1+1,x_2+1,\dots)$ , where  $\infty+1$  equals  $\infty$ . Define a function,  $f$ , taking  $P_4$  into itself by  $f(\bar{x})=\bar{x}+\bar{1}$ . This function is clearly order preserving. Let  $C=\{\bar{c}_n:n\in\mathbb{N}\}$  be a chain which has a supremum in  $P_4$ . Then  $f(\sup(C))=(\sup(C))+\bar{1}=\sup\{\bar{c}_n+\bar{1}:n\in\mathbb{N}\}=\sup(f(C))$ . Thus,  $f$  is an  $\omega$ -chain continuous functions which has no fixed points.

#### IV. FINITE WIDTH

The partially ordered sets in the first three examples all contained infinite antichains. It is also clear that any chain has the least fixed point property for  $\omega$ -chain continuous functions if and only if it is  $\omega$ -chain complete. So it seems reasonable that any partially ordered set in which the cardinalities of all its antichains are finite and bounded would have the least fixed point property for  $\omega$ -chain continuous functions if and only if it is  $\omega$ -chain complete. Before this is shown to be true, some notation and a definition are in order.

DEFINITION. A partially ordered set,  $P$ , has finite width if there is a positive integer,  $N$ , such that, for every antichain,  $A$ , of  $P$ , the cardinality of  $A$  is less than or equal to  $N$ .

For every element,  $x$ , of  $P$ , the lower end generated by  $x$ , denoted  $\downarrow x$ , is the set of all elements of  $P$  less than or equal to  $x$ . For a subset,  $X$ , of  $P$ , the lower end generated by  $X$ , denoted  $\downarrow X$ , is  $\bigcup_{x \in X} \downarrow x$ . The sets  $\uparrow x$  and  $\uparrow X$  are defined dually.

LEMMA 2. If a partially ordered set,  $P$ , contains a countable chain,  $C$ , whose set of upper bound,  $U$ , contains a chain,  $D$ , which is not bounded below by any element of  $U$  and if  $C$  is a retract of  $\bigcap_{d \in D} \downarrow d$ , then  $P$  does not

have the fixed point property for  $\omega$ -chain continuous functions.

PROOF. Let  $D'$  be a subchain of  $D$  such that every subset of  $D'$  has a largest element and every element of  $D$  has an element of  $D'$  below it. Denote  $D'$  by  $\{d_\alpha : \alpha < \sigma\}$  for some ordinal  $\sigma$ . Set  $E_0$  equal to  $P - \downarrow d_0$  and, for every  $\alpha > 0$ , let  $E_\alpha$  be the set  $(\downarrow d_\alpha) - (\downarrow d_{\alpha+1})$ . Let  $f$  be a retraction of  $\bigcap_{d \in D} \downarrow d$  on-

to  $C$ . Define a function,  $g: P \rightarrow D' \cup C$ , by

$$g(x) = \begin{cases} d_\alpha & \text{if } x \in E_\alpha \\ f(x) & \text{if } x \in \bigcap_{d \in D} \downarrow d \end{cases}$$

Let  $x$  and  $y$  be elements of  $P$  such that  $x \leq y$ . If  $x$  and  $y$  are elements of  $\bigcap_{d \in D} \downarrow d$ , then  $g(x) = f(x) \leq f(y) = g(y)$ . If  $x$  is an element of  $\bigcap_{d \in D} \downarrow d$  and  $y$  is an element of  $P - \bigcap_{d \in D} \downarrow d$ , then  $g(x)$  is in  $C$ ;  $g(y)$  is in  $D'$ ; and  $g(x) \leq g(y)$ . If  $x$  and  $y$  are elements of  $P - \bigcap_{d \in D} \downarrow d$ , then let  $\alpha < \sigma$  such that  $y$  is an element of  $E_\alpha$ . Then if  $x$  is in  $E_\beta$ ,  $\beta$  must be at least as large as  $\alpha$ . Thus  $g(x) = d_\beta \leq d_\alpha = g(y)$ . So  $g$  preserves order.

Let  $X$  be a nonempty countable chain in  $P$  which has a supremum in  $P$ . If  $\sup(X)$  is an element of  $\bigcap_{d \in D} \downarrow d$ , then  $X$  is a subset of  $\bigcap_{d \in D} \downarrow d$  and  $g(\sup(X)) = f(\sup(X)) = \sup(f(X)) = \sup(g(X))$ . If  $\sup(X)$  is an element of  $P - \bigcap_{d \in D} \downarrow d$ , then, because  $g$  preserves order and  $\bigcap_{d \in D} \downarrow d$  is closed, it may be assumed that  $X$  is con-



tained in  $P - \bigcup_{d \in D} \downarrow d$ . Let  $\alpha < \sigma$  such that  $\sup(X)$  is in  $E_\alpha$ . Since  $\downarrow E_{\alpha+1}$  equals  $\downarrow d_{\alpha+1}$  and is closed,  $X$  cannot be contained in  $\downarrow E_{\alpha+1}$ . Thus, from some point on, every element of  $X$  is in  $E_\alpha$ . Hence,  $g(\sup(X))$  equals  $\sup(g(X))$ . Therefore,  $CUD'$  is a retract of  $P$ .

Because  $CUD'$  does not have the fixed point property for  $\omega$ -chain continuous functions,  $P$  does not either.

LEMMA 3. If a partially ordered set,  $P$ , contains a countable chain,  $C$ , whose set of upper bounds,  $U$ , contains two minimal elements,  $x$  and  $y$ , with a common upper bound,  $z$ , and if  $C$  is a retract of  $(\downarrow x) \cap (\downarrow y)$ , then  $P$  does not have the least fixed point property for  $\omega$ -chain continuous functions.

PROOF. Let  $f$  be a retraction from  $(\downarrow x) \cap (\downarrow y)$  onto  $C$ , and define a function,  $g: P \rightarrow CU\{x, y, z\}$ , by

$$g(p) = \begin{cases} f(p) & \text{if } p \in (\downarrow x) \cap (\downarrow y) \\ x & \text{if } p \in (\downarrow x) - (\downarrow y) \\ y & \text{if } p \in (\downarrow y) - (\downarrow x) \\ z & \text{if } p \in P - [(\downarrow x) \cup (\downarrow y)] \end{cases}$$

Let  $p$  and  $q$  be elements of  $P$  such that  $p \leq q$ . The function,  $g$ , obviously preserves order in each of the subsets of  $P$  given in the definition of  $g$ . If  $p$  is an element of  $(\downarrow x) \cap (\downarrow y)$  and  $q$  is not, then  $g(p)$  is an element of  $C$  and  $g(q)$  is in  $\{x, y, z\}$ . Therefore,  $g(p)$  is less than  $g(q)$ . Assume that  $p$  is an element of  $(\downarrow x) - (\downarrow y)$ . Then  $q$  is an element

either of  $(\downarrow x) - (\downarrow y)$  or of  $P - [(\downarrow x) \cup (\downarrow y)]$ . In either case,  $g(p)$  is less than or equal to  $g(q)$ . The other case being analogous,  $g$  preserves order.

Since each of the subsets given in the definition of  $g$  is closed and  $g$  is  $\omega$ -chain continuous on all of them,  $g$  is  $\omega$ -chain continuous on  $P$ . Because  $CU\{x, y, z\}$  does not have the least fixed point property for  $\omega$ -chain continuous functions,  $P$  does not either.

A subset,  $X$ , of a partially ordered set,  $P$  is an upper end if and only if  $X$  equals  $\uparrow X$ . It is a lower end if and only if  $X$  equals  $\downarrow X$ .

LEMMA 4. If a partially ordered set,  $P$ , has finite width and  $A$  and  $B$  are nonempty disjoint subsets of  $P$  such that  $B$  is an upper end, then there is only a finite number of points in  $B$  which are suprema of countable chains in  $A$ .

PROOF. Let  $L$  be the set of elements of  $B$  which are suprema of countable chains in  $A$ . Assume that  $L$  is infinite and let  $M$  be a maximal antichain of  $L$ . Since  $M$  is finite, there is some element of  $M$ , say  $p_1$ , such that  $p_1$  is comparable to an infinite number of elements of  $L$ . Either  $(\uparrow p_1) \cap L$  is infinite or  $(\downarrow p_1) \cap L$  is. If  $(\uparrow p_1) \cap L$  is infinite, let  $K_1$  be the set  $(\uparrow p_1) \cap L$ . Otherwise, set  $K_1$  equal to  $(\downarrow p_1) \cap L$ .

Let  $n \in \mathbb{N}$  and assume that elements  $p_1, p_2, \dots, p_n$  and subsets  $K_{n-1}$  and  $K_n$  of  $L$  have been defined and satisfy the following properties.

- 1)  $\{p_1, \dots, p_n\}$  is a chain in  $L$
- 2)  $K_n$  is infinite
- 3)  $K_n = (\uparrow p_n) \cap K_{n-1}$  or  $K_n = (\downarrow p_n) \cap K_{n-1}$

Let  $M$  be a maximal antichain of  $K_n - \{p_1, \dots, p_n\}$ . Since  $M$  is finite, one of its elements,  $p_{n+1}$ , is comparable to an infinite number of elements of  $K_n$ . Define  $K_{n+1}$  to be  $(\uparrow p_{n+1}) \cap K_n$  if that set is infinite, and  $(\downarrow p_{n+1}) \cap K_n$  otherwise.

The chain  $\{p_n : n \in \mathbb{N}\}$  thus obtained is infinite and so contains either an increasing or a decreasing chain. Assume that  $\{p_n : n \in \mathbb{N}\}$  is increasing, that is, that  $p_1 < p_2 < p_3 < \dots$ .

For each  $n$  in  $\mathbb{N}$ , let  $X_n = \{x_{nm} : m \in \mathbb{N}\}$  be a chain in  $A$  which has  $p_n$  as a supremum. Because  $\{p_n : n \in \mathbb{N}\}$  is increasing, there is, for every  $n$  in  $\mathbb{N}$ , an  $N_{nm}$  in  $\mathbb{N}$ , where  $m < n$ , such that if  $j \geq N_{nm}$ , then for any  $x_{mk}$  in  $X_m$ ,  $x_{nj} \not\leq x_{mk}$ . If  $x_{mk}$  were less than  $x_{nj}$  for every  $k$  in  $\mathbb{N}$ , then  $p_m$  would be less than or equal to  $x_{nj}$ . This is not possible since  $B$  is an upper end and  $A$  and  $B$  are disjoint. Thus, for every  $j \geq N_{nm}$ , there is an  $M_{nmj}$  in  $\mathbb{N}$  such that if  $k \geq M_{nmj}$ , then  $x_{mk}$  and  $x_{nj}$  are incomparable. Let  $n \in \mathbb{N}$  and, for each  $0 \leq j \leq n-1$ , set  $k_j$  equal to  $\max(\{N_{n-j,i} : 1 \leq i < n-j\} \cup \{M_{n-i,n-j,k_i} : 0 \leq i \leq j-1\})$ . Then  $\{x_{n-j,k_j} : 0 \leq j \leq n-1\}$  is an antichain. Therefore,  $P$  contains arbitrarily large finite antichains, contradicting the assumption that  $P$  has finite width.

By reversing the order of  $P$ , the same argument works when  $\{p_n : n \in \mathbb{N}\}$  is a decreasing chain. Thus,  $L$  is finite.

THEOREM 2. If  $P$  is a partially ordered set of finite width, then any unbounded chain in  $P$  of order type  $\omega$  is a retract of  $P$ .

PROOF. Let  $C = \{c_n : n \in \mathbb{N}\}$  be an unbounded chain of  $P$ . Set  $E_1$  equal to  $P - (\uparrow c_2)$  and, for every  $n \geq 2$ , let  $E_n$  be the set  $(\uparrow c_n) - (\uparrow c_{n+1})$ . For every  $n$  in  $\mathbb{N}$ , let  $L'_n$  be the set of all the elements of  $P - E_n$  which are suprema of nonempty countable chains in  $E_n$ . Let  $L_n$  equal  $L'_n - \bigcup_{k < n} L'_k$ . By Lemma 4 each  $L_n$  is finite. For every  $m, n \in \mathbb{N}$ , let  $K_{nm}$  be the set  $L_n \cap E_m$ . Notice that  $K_{nm}$  is empty if  $n \geq m$ .

Let  $p$  be an element of  $K_{nm}$  and let  $V''_p$  be an open upper end containing  $p$ . Because it is an upper end,  $V''_p$  is also closed. Let  $V'_p$  be the intersection of  $V''_p$  with  $\uparrow c_n$ . There may be elements of  $V'_p$  which are the suprema of nonempty countable chains in  $P - (\uparrow c_n)$ . Let  $A$  be the collection of all such elements of  $V'_p$ . By Lemma 4,  $A$  is finite, so

$V_p = V'_p - \bigcup_{a \in A} (\downarrow a)$  is an open upper end. Now let  $U_p$  be the set

$(V_p - \bigcup_{n \leq k \leq m} (\downarrow c_k)) \cap (\bigcup_{n \leq k \leq m} E_k)$ . This is an open set contained in

$\bigcup_{n \leq k \leq m} E_k$  and is an upper end in that set. Hence it is open

and closed with respect to  $\bigcup_{n \leq k \leq m} E_k$ . Set  $D_n$  equal to

$$[E_n \cup (\bigcup_{k=1}^n \bigcup_{p \in K_{kn}} U_p)] - \bigcup_{k=1}^n \bigcup_{n < j} \bigcup_{q \in K_{kj}} U_q.$$

The following properties will be shown to be true.

$$1) P = \bigcup_{n \in \mathbb{N}} D_n$$

- 2) if  $n \neq m$ , then  $D_n \cap D_m = \emptyset$
- 3) for all  $n$ ,  $c_n \in D_n$
- 4) if  $x, y \in P$ ;  $x \leq y$ ;  $x \in D_m$ ; and  $y \in D_n$ , then  $m \leq n$
- 5) each  $D_n$  is closed

1) Let  $x$  be an element of  $P$ . Since  $P$  equals  $\bigcup_{n \in \mathbb{N}} E_n$ ,  $x$  must

be in  $E_n$  for some  $n$  in  $\mathbb{N}$ . If  $x$  is not in  $D_n$ , then there is some  $1 \leq a \leq n$  and some  $b > n$  and some  $p \in K_{ab}$  such that  $x$  is an element of  $U_p$ . Let  $M$  be the set  $\{j > n : \exists 1 \leq k \leq n, \exists p \in K_{kj} \exists x \in U_p\}$  and set  $m$  equal to  $\max(M)$ . Then  $x$  is an element of

$E_m \cup \left( \bigcup_{k=1}^n \bigcup_{p \in K_{km}} U_p \right)$ . Let  $1 \leq a \leq m$ ; let  $b > m$ ; and let  $p$  be an element

of  $K_{kj}$ . Then  $U_q$  is a subset of  $\bigcup_{a \leq i < b} E_i$ . So if  $x$  is an ele-

ment of  $U_q$ , then  $a \leq n$ . But if  $k \leq n$ , then  $b$  is an element of  $M$ , contradicting the choice of  $m$ . Thus  $x$  is not an element of

$\bigcup_{k=1}^m \bigcup_{j > m} \bigcup_{q \in K_{km}} U_q$  and  $x$  is in  $D_m$ .

2) Let  $m$  and  $n$  be positive integers such that  $m < n$ . Then the intersection of  $E_m$  with  $E_n$  is empty. Let  $1 \leq a \leq m$  and let  $p$  be an element of  $K_{am}$ . Then  $U_p$  is a subset of  $\bigcup_{a \leq i < m} E_i$  which

has an empty intersection with  $E_n$ . Therefore the intersec-

tion of  $\bigcup_{k=1}^m \bigcup_{p \in K_{km}} U_p$  with  $E_n$  is empty. Assume that

$E_m \cap \left( \bigcup_{k=1}^m \bigcup_{p \in K_{kn}} U_p \right)$  is not empty and let  $1 \leq a \leq n$  be an element of

$K_{an}$  such that  $E_m \cap U_x$  is not empty. Now  $U_x$  is a subset of

$\bigcup_{a \leq i < n} E_i$ , so  $a \leq m$ . But then  $U_x$  is a subset of  $\bigcup_{k=1}^m \bigcup_{j > m} \bigcup_{q \in K_{kj}} U_q$ ;

so  $(E_m - (\bigcup_{k=1}^m \bigcup_{j>m} \bigcup_{q \in K_{kj}} U_q)) \cap (\bigcup_{k=1}^n \bigcup_{p \in K_{kn}} U_p)$  is empty. Now assume

that  $(\bigcup_{k=1}^m \bigcup_{p \in K_{km}} U_p) \cap (\bigcup_{j=1}^n \bigcup_{q \in K_{jn}} U_q)$  is not empty. Then there exist

$1 \leq a \leq m$ ,  $1 \leq b \leq n$ ,  $x \in K_{am}$ , and  $y \in K_{bn}$  such that  $U_x \cap U_y$  is not empty.

If  $b \leq m$ , then  $U_y$  is a subset of  $\bigcup_{k=1}^m \bigcup_{j>m} \bigcup_{q \in K_{kj}} U_q$  and  $D_m \cap U_p$  is

empty. Because  $U_y$  is a subset of  $\bigcup_{b \leq i \leq n} E_i$  and  $U_x$  is a subset

of  $\bigcup_{a \leq i \leq m} E_i$ ,  $b$  is less than or equal to  $m$ . Therefore,  $D_m \cap U_p$

is empty. Thus,  $D_m \cap D_n$  is empty.

3) For every  $n$  in  $\mathbb{N}$ ,  $c_n$  is an element of  $E_n$ . By the way they were defined, no  $U_p$  contains a  $c_n$ . So  $c_n$  is an element of  $D_n$ .

4) Let  $x$  and  $y$  be elements of  $P$  such that  $x \leq y$ ,  $x$  is an element of  $D_m$  and  $y$  is an element of  $D_n$ . Let  $x$  be an element of  $E_a$  and  $y$  an element of  $E_b$ . Since  $x$  is less than  $y$ ,  $a$  must be less than or equal to  $b$ . Obviously  $a$  is less than or equal to  $m$  and  $b$  is less than or equal to  $n$ . If  $m$  is equal to  $a$ , then  $m \leq b \leq n$ . If  $m$  is greater than  $a$ , then there exists  $1 \leq k \leq a$  and  $j > a$  and an element,  $p$ , of  $K_{kj}$  such that  $x$  is an element of  $U_p$ . If there are no  $j$ 's greater than  $b$ , then  $m \leq b \leq n$ . If  $j > b$ , then  $U_p$  is an upper end of  $\bigcup_{k \leq i < j} E_i$ , so  $y$  is an element of  $U_p$ . Therefore,  $m$  is less than or equal to  $n$ .

5) Let  $n$  be an element of  $\mathbb{N}$  and let  $X$  be a nonempty countable chain in  $D_n$ . If  $X$  is a subset of  $E_n$ , then either  $\sup(X)$

is an element of  $E_n$  or  $\text{sup}(X)$  is an element of  $K_{kj}$  for some  $1 \leq k \leq n$  and  $j > n$ . If  $\text{sup}(X)$  is an element of  $K_{kj}$ , then the intersection of  $X$  with  $U_{\text{sup}(X)}$  is not empty and  $X$  is not a subset of  $D_n$ . For the same reason,  $\text{sup}(X)$  cannot belong to  $U_p$  for any element,  $p$ , of  $K_{kj}$ , where  $1 \leq k \leq n$  and  $j > n$ . Thus,  $\text{sup}(X)$  is an element of  $D_n$ . If  $X$  is a subset of  $\bigcup_{k=1}^n \bigcup_{p \in K_{kn}^p} U_p$ , then it

may be assumed that  $X$  is a subset of  $U_p$  for some element,  $p$ , of  $K_{kn}$  where  $1 \leq k \leq n$ . Assume that  $\text{sup}(X)$  is not in  $U_p$  and that  $\text{sup}(X)$  is not in  $E_n$ . Let  $\text{sup}(X)$  be an element of  $E_m$ . Since  $U_p$  is a subset of  $\bigcup_{k \leq i < n} E_i$  and  $U_p$  is closed in this set,  $m$  must

be greater than  $n$ . Again it may be assumed that  $X$  is a subset of  $E_j$  for some  $k \leq j \leq n$ . But then  $\text{sup}(X)$  is an element of  $K_{im}$  for some  $1 \leq i \leq n$  and the intersection of  $X$  with

$\bigcup_{k=1}^n \bigcup_{j > n} \bigcup_{q \in K_{kj}^q} U_q$  is not empty, contradicting the assumption

that  $X$  is a subset of  $D_n$ . Therefore,  $\text{sup}(X)$  is an element of  $D_n$ .

It follows from Lemma 1 that  $C$  is a retract of  $P$ .

Therefore,  $P$  does not have the fixed point property for  $\omega$ -chain continuous functions.

**THEOREM 3.** If a partially ordered set with finite width has the least fixed point property for  $\omega$ -chain continuous functions, then it is  $\omega$ -chain complete.

**PROOF.** By Theorem 2, it may be assumed that every countable chain in a partially ordered set,  $P$ , is bounded. Let

$C$  be a nonempty countable chain in  $P$  which does not have a supremum. Let  $U$  be the set of upper bounds of  $C$ .

By Lemmas 2 and 3 it may be assumed that  $U$  contains a set of minimal elements,  $M$ , that every element of  $U$  is above some element of  $M$ , and that no two elements of  $M$  have a common upper bound. Denote the elements of  $M$  by  $m_1, m_2, \dots, m_r$ . Let  $D$  be the set of all elements,  $x$ , of  $P$  such that  $\uparrow x$  is not contained in  $\downarrow(\uparrow m_i) \cup \bigcup_{j \neq i} \downarrow(\uparrow m_j)$  for any  $1 \leq i \leq r$ .

Let  $X = \{x_n : n \in \mathbb{N}\}$  be a chain in  $D$  which has a supremum in  $P$ . Assume that for every  $n$  in  $\mathbb{N}$ , there are  $i \geq n$ ,  $1 \leq j_i \neq k_i \leq r$ , and  $y_{j_i} \geq m_{j_i}$ , and  $y_{k_i} \geq m_{k_i}$  such that  $x_i$  is less than both  $y_{j_i}$  and  $y_{k_i}$ . Then there are an infinite number of pairs,  $(j_i, k_i)$ , and one of the pairs,  $(j, k)$ , must be repeated an infinite number of times. Let  $Y_j$  be the set  $\{y_{j_i} : j_i = j\}$  and  $Y_k$  the set  $\{y_{k_i} : k_i = k\}$ . If  $Y_j$  is finite, then it contains an element which is an upper bound of  $X$ . If it is infinite, it must contain an infinite chain which, by assumption, is bounded. Any upper bound of this chain is, of course, an upper bound of  $X$ . In either case there is a  $y_j \geq m_j$  which is an upper bound of  $X$ . Similarly, there is a  $y_k \geq m_k$  which is an upper bound of  $X$ . Thus,  $\sup(X)$  is less than both  $y_j$  and  $y_k$  and  $\sup(X)$  is an element of  $D$ .

Assume that for every  $n$  in  $\mathbb{N}$ , there is a  $y_n$  in  $P \cap \bigcup_{i=1}^r \downarrow(\uparrow m_i)$  such that  $x_n \leq y_n$ . Let  $Y$  be the set of all these



$y_n$ 's. If  $Y$  is finite, one of its elements must be an upper bound of  $X$ . If  $Y$  is infinite, it must contain an infinite chain which, by assumption, is bounded. Any upper bound of this chain will be an upper bound of  $X$ . In either case there is an element,  $y$ , of  $P - \bigcup_{i=1}^r \downarrow(\uparrow m_i)$  which is an upper bound of  $X$ .

Then  $\sup(X)$  is less than or equal to  $y$  and  $\sup(X)$  is an element of  $D$ . Therefore  $D$  is closed.

Since  $C$  is unbounded in  $D$ , there is, by Theorem 2, a retraction,  $f$ , of  $D$  onto a cofinal subchain,  $C'$  of  $C$ . Define a function,  $g: P \rightarrow C' \cup M$  by

$$g(x) = \begin{cases} m_i & \text{if } x \in \downarrow(\uparrow m_i) - D \text{ } 1 \leq i \leq r \\ f(x) & \text{if } x \in D \end{cases}$$

Let  $x$  and  $y$  be elements of  $P$  with  $x \leq y$ . If  $x$  is in  $D$ , then  $g(x) \leq g(y)$ . If  $x$  is not in  $D$ , then there is some  $1 \leq i \leq r$  such that  $\uparrow x$  is a subset of  $\downarrow(\uparrow m_i) - \bigcup_{j \neq i} \downarrow(\uparrow m_j)$ . But then  $y$  is an element of  $\downarrow(\uparrow m_i) - D$ , so  $g(x) = m_i = g(y)$ . Therefore,  $g$  preserves order.

Clearly,  $g$  is  $\omega$ -chain continuous on  $D$  and on  $\downarrow(\uparrow m_i) - D$  for each  $1 \leq i \leq r$ . Thus  $g$  is  $\omega$ -chain continuous on  $P$ ;  $C' \cup M$  is a retract of  $P$ ; and  $P$  cannot have the least fixed point property for  $\omega$ -chain continuous function.

**COROLLARY 2.** A partially ordered set,  $P$ , having finite width is  $\omega$ -chain complete if and only if it has the least fixed point property for  $\omega$ -chain continuous functions.

## V. LAYERED PARTIALLY ORDERED SETS

The proof of Theorem 2 relied heavily on Lemma 4, which is no longer true if partially ordered sets not of finite width are considered. Another way must be found if Theorem 3 is to be generalized.

Recall that in the first three examples, the partially ordered sets all contained a countable chain which sneaked around an antichain, that is, they all contained an antichain which generates a nonclosed lower end. Notice also that the lower end generated by a finite antichain will always be closed. Thus no partially ordered set of finite width could have an antichain which generates a lower end that is not closed. If a partially ordered set contains an unbounded chain  $C = \{c_n : n \in \mathbb{N}\}$  and there is a collection,  $\{A_n : n \in \mathbb{N}\}$ , of antichains such that  $c_n$  is an element of  $\downarrow A_n$ ;  $P$ , the partially ordered set, equals  $\bigcup_{n \in \mathbb{N}} \downarrow A_n$ ; and for each  $n$  in  $\mathbb{N}$ ,  $\downarrow A_n$  is closed, then  $C$  is arrettract of  $P$ . This is the approach which will be taken next.

The first question which arises concerning this method is, Can a partially ordered set actually be partitioned in such a manner? The answer is that not all can be. For example, if a partially ordered set contains a maximal chain which has no countable cofinal subchain, then it cannot be the union of a countable number of lower ends generated by antichains. Such chains must therefore be avoided. Even so,

it is not certain that such partitions occur in general. A Souslin tree is a partially ordered set which cannot be partitioned in the desired way even though all of its chains and antichains are countable. The existence of a Souslin tree is independent of the usual axioms of set theory (see [D]), so it may be that the existence of such partitions is also independent.

DEFINITION. A partially ordered set,  $P$ , is layered if and only if there is a collection,  $\{A_n : n \in \mathbb{N}\}$ , of antichains of  $P$  such that  $P$  is equal to  $\bigcup_{n \in \mathbb{N}} \downarrow A_n$ .

THEOREM 4. If every antichain of a layered partially ordered set,  $P$ , generates a closed lower end, then any unbounded chain in  $P$  of order type  $\omega$  is a retract of  $P$ .

PROOF. Let  $\{A_n : n \in \mathbb{N}\}$  be a collection of antichains of  $P$  such that  $P$  equals  $\bigcup_{n \in \mathbb{N}} \downarrow A_n$ . First, assume that for every  $n$

in  $\mathbb{N}$ ,  $\downarrow A_n$  is a subset of  $\downarrow A_{n+1}$ . Let  $C = \{c_n : n \in \mathbb{N}\}$  be an unbounded chain in  $P$  and assume that  $C$  is not contained in  $\downarrow A_n$  for any  $n$ . Set  $d_n$  equal to the largest element of  $C$  contained in  $\downarrow A_n$ , if there are any elements of  $C$  in  $\downarrow A_n$ , and  $c_1$  if there are not. Define a function,  $f: P \rightarrow C$ , by

$$f(x) = \begin{cases} c_1 & \text{if } x \in \downarrow c_1 \\ c_n & \text{if } x \in (\downarrow c_n) - (\downarrow c_{n-1}) \text{ and } n > 1 \\ d_1 & \text{if } x \in (P - \downarrow C) \cap (\downarrow A_1) \\ d_n & \text{if } x \in (P - \downarrow C) \cap ((\downarrow A_n) - (\downarrow A_{n-1})) \text{ and } n > 1 \end{cases}$$

Let  $x$  and  $y$  be elements of  $P$  such that  $x \leq y$ . If  $y$  is an

element of  $\downarrow C$ , then  $x$  is an element of  $\downarrow C$  and  $f(x) \leq f(y)$ . If  $y$  is an element of  $(P - \downarrow C) \cap ((\downarrow A_n) - (\downarrow A_{n-1}))$ , where  $\downarrow A_{n-1}$  is empty if  $n$  is 1, then  $x$  is an element of  $\downarrow A_n$  and  $f(x) \leq d_n = f(y)$ .

Let  $X$  be a nonempty countable chain in  $P$  which has a supremum in  $P$ . If  $\sup(X)$  is an element of  $\downarrow c_n$  for some  $n$ , then  $X$  is a subset of  $\downarrow c_n$ . Furthermore, from some point on, all the elements of  $X$  must be in  $(\downarrow c_n) - (\downarrow c_{n-1})$ . Thus,  $f(\sup(X)) = c_n = \sup(f(X))$ . Assume that  $\sup(X)$  is an element of  $(P - \downarrow C) \cap ((\downarrow A_n) - (\downarrow A_{n-1}))$ , where  $\downarrow A_{n-1}$  is again empty if  $n$  is 1. Then, since  $\downarrow A_{n-1}$  is closed,  $X$  may be assumed to be a subset of  $(\downarrow A_n) - (\downarrow A_{n-1})$ . In fact, it may be assumed that  $X$  is a subset of  $(P - \downarrow C) \cap ((\downarrow A_n) - (\downarrow A_{n-1}))$ , since if it were not it would be a subset of  $\downarrow d_n$ , which is closed. Therefore,  $f(\sup(X)) = d_n = \sup(f(X))$ . Hence  $C$  is a retract of  $P$ .

If there is an  $n$  in  $\mathbb{N}$  such that  $C$  is a subset of  $\downarrow A_n$ , then it may be assumed that  $C$  is a subset of  $\downarrow A_n$  for every  $n$  in  $\mathbb{N}$ . Set  $D_1$  equal to  $\bigcup_{n \in \mathbb{N}} (\downarrow A_n - (\uparrow c_2))$ . For all  $m > 1$ , set  $D_m$  equal to  $(\bigcup_{n \in \mathbb{N}} ((\downarrow A_n - (\uparrow c_{m+1})) \cap (\uparrow c_m))) - \bigcup_{j < m} D_j$ . Let  $x$  be an element of  $P$ ,  $n$  an element of  $\mathbb{N}$ , and  $a$  an element of  $A_n$  such that  $x$  is less than  $a$ . Now  $a$  is not an upper bound of  $C$ , so either there is an  $m$  in  $\mathbb{N}$  such that  $a$  is an element of  $(\downarrow A_n - (\uparrow c_{m+1})) \cap (\uparrow c_m)$  or it is an element of  $\downarrow A_n - (\uparrow c_2)$ . Thus, either  $x$  is in  $D_m$  or  $x$  is in  $D_1$ . Hence  $P$  equals  $\bigcup_{n \in \mathbb{N}} D_n$ .

Let  $X$  be a nonempty countable chain in  $D_m$  which has a supre-

mum in  $P$ . Since  $P$  equals  $\bigcup_{n \in \mathbb{N}} \downarrow A_n$ , there is a  $k$  in  $\mathbb{N}$  such that  $X$  is contained in  $\downarrow A_k$ . Then  $X$  is contained in  $(\downarrow A_k) \cap D_m$  which is equal to  $\downarrow((A_k - (\uparrow c_{n+1})) \cap (\uparrow c_n))$  for some  $n > 1$  or  $\downarrow(A_k - (\uparrow c_2))$ . In either case,  $X$  is a subset of a lower end generated by an antichain. But such a lower end is closed, so  $\sup(X)$  is an element of  $D_m$ . Thus,  $D_m$  is closed. Let  $x$  and  $y$  be elements of  $P$  such that  $x$  is less than  $y$ . Assume that  $x$  is an element of  $D_m$  and that  $y$  is an element of  $D_n$ . Clearly, if  $y$  is an element of  $\downarrow((A_k - (\uparrow c_{n+1})) \cap (\uparrow c_n))$ , then  $x$  must be also. Therefore,  $m$  is less than or equal to  $n$ . By Lemma 1,  $C$  is a retract of  $P$ .

The requirement that every antichain of  $P$  generate a closed lower end is not necessary for the first part of the previous proof. There it is only needed that each of the  $A_n$ 's generate a closed lower end. The requirement is necessary in the second part, however, as is shown by the following example.

EXAMPLE 5. Let  $G$  be the collection of infinite countable subsets of  $\omega_1$  which don't have a largest element. Let  $P_5$  be the set  $\omega_1 \cup \left( \bigcup_{\alpha \in \omega_1 - \omega} \bigcup_{A \in G} (\alpha \times A \times \{A\}) \right)$ . Let  $\{X_\alpha : \alpha \in \omega\}$  be a partition of  $\omega_1 - \omega$  such that, for every  $\alpha$ ,  $X_\alpha$  is infinite. If  $x$  and  $y$  are elements of  $P_5$ , set  $x \leq y$  if and only if one of the following conditions holds.

- 1)  $x \in \omega$ ;  $y \in X_\alpha$  for some  $\alpha \in \omega$ ; and  $x \leq \alpha$  in  $\omega$
- 2)  $x = (\alpha, \beta, A) \in \alpha \times A \times \{A\}$ ;  $y = (\alpha, \delta, A) \in \alpha \times A \times \{A\}$ ; and  $\beta \leq \delta$

- 3)  $x=(\alpha, \beta, A) \in \alpha \times A \times \{A\}$  and  $y=\alpha$  or  $y=\beta$
- 4)  $x, y \in \omega$  and  $x \leq y$  in  $\omega$
- 5)  $x=0$  (the least element of  $\omega$ )

Clearly,  $\omega$  is an unbounded chain in  $P$ ;  $\omega_1^{-\omega}$  is an anti-chain; and  $\downarrow(\omega_1^{-\omega})$  is all of  $P_5$  and is closed. Let  $f$  be an  $\omega$ -chain continuous function from  $P_5$  to itself and assume that  $f(P_5)$  is a subset of  $\omega$ . Then there is an element,  $m$ , of  $\omega$  such that  $f^{-1}(m)$  is infinite. Let  $A$  be an element of  $G$  which is contained in  $f^{-1}(m)$ . Then for every  $\alpha$  in  $\omega_1^{-\omega}$ ,  $f(\alpha \times A \times \{A\})$  is a subset of  $\{1, 2, \dots, m\}$ . Since  $\alpha$  is the supremum of  $\alpha \times A \times \{A\}$ , it must be less than or equal to  $m$ . Hence for every  $n \geq m$ ,  $f(n)$  is less than or equal to  $m$ . Thus, no cofinal subchain of  $\omega$  is a retract of  $P_5$ .

In fact,  $P_5$  has the least fixed point property for  $\omega$ -chain continuous functions. Assume that  $f(0)$  is not 0. Since  $P^{-\omega}$  is  $\omega$ -chain complete, it may be assumed that  $\{f^n(0) : n \in \mathbb{N}\}$  is contained in  $\omega$ . By the argument above, it may also be assumed that there is an  $\alpha$  in  $\omega_1^{-\omega}$  such that  $f(\alpha)$  is an element of  $\omega_1^{-\omega}$ . Let  $f(\alpha)$  be an element of  $X_m$ . For every  $n \geq m$ , let  $A_n$  be the element of  $G$  contained in  $X_n$ . Since  $\alpha$  is the supremum of  $\alpha \times A_n \times \{A_n\}$ , there is an element,  $\beta$ , of  $A_n$  such that  $f((\alpha, \beta, A))$  equals  $f(\alpha)$ . Therefore,  $f(\beta)$  equals  $f(\alpha)$  and  $f(n)$  is less than or equal to  $m$ . It follows that  $f$  has a least fixed point.

**THEOREM 5.** If every antichain of a partially ordered set,  $P$ , generates a closed lower end; every lower end of  $P$

which contains a countable chain but none of its upper bounds is layered; and  $P$  has the least fixed point property for  $\omega$ -chain continuous functions, then  $P$  is  $\omega$ -chain complete.

PROOF. Assume that  $P$  contains a nonempty countable chain,  $C$ , which does not have a supremum. By Theorem 4 and Lemmas 2 and 3, it may be assumed that every countable chain in  $P$  is bounded: that the set,  $U$ , of upper bounds of  $C$  has a set,  $M$ , of minimal elements; that every element of  $U$  is above an element of  $M$ ; and that no two elements of  $M$  have a common upper bound.

Let  $D$  be the set of all the elements,  $x$ , of  $P$  such that  $\uparrow x$  is not contained in  $(\downarrow(\uparrow m)) \cup \downarrow(\uparrow n)$  for any  $m$  in  $M$ . Let  $X = \{x_n : n \in \mathbb{N}\}$  be a chain in  $D$  which has a supremum in  $P$ . Assume that for every  $k$  in  $\mathbb{N}$ , there is a  $j \geq k$ , elements  $m_{r_j} \neq m_{s_j}$  of  $M$ , and  $y_{r_j}$  of  $\uparrow m_{r_j}$  and  $y_{s_j}$  of  $\uparrow m_{s_j}$  such that  $x_j$  is less than both  $y_{r_j}$  and  $y_{s_j}$ . Let  $Y_r$  be the set of all the  $y_{r_j}$ 's and  $Y_s$  the set of all the  $y_{s_j}$ 's.

Assume that both  $Y_r$  and  $Y_s$  are infinite and that  $Y_r$  contains an infinite antichain,  $A_r$ . Since the lower end generated by an antichain is closed,  $\sup(X)$  is below all but a finite number of the elements of  $A_r$ . Let  $Z_s$  be those elements of  $Y_s$  which are paired with the elements of  $A_r$ . If  $Z_s$  has an infinite antichain,  $A_s$ , then  $\sup(X)$  is below all but

a finite number of the elements of  $A_s$ . Thus, there is a  $j$  in  $\mathbb{N}$  such that  $\text{sup}(X)$  is less than  $y_{r_j}$  and  $y_{s_j}$ . Then  $\text{sup}(X)$  is an element of  $D$ .

If every antichain of  $Z_s$  is finite, then it contains an infinite chain,  $C_s$ , which, by assumption, is bounded. Let  $y_s$  be an upper bound for this chain. Then  $\text{sup}(X)$  is less than or equal to  $y_s$ . Let  $B_r$  be the set of elements of  $A_r$  which are paired with the elements of  $C_s$ . Since  $B_r$  is an antichain and  $X$  is contained in  $\downarrow B_r$ ,  $\text{sup}(X)$  is in  $\downarrow B_r$ . Let  $y_r$  be an element of  $B_r$  which is above  $\text{sup}(X)$ . Then  $y_r$  and  $y_s$  are elements of  $U$  which are above distinct elements of  $M$  and are both greater than  $\text{sup}(X)$ . Thus  $\text{sup}(X)$  is in  $D$ .

Assume that all antichains in both  $Y_r$  and  $Y_s$  are finite. Let  $C_r$  be an infinite chain in  $Y_r$ ; let  $y_r$  be an upper bound of  $C_r$ ; and let  $m_r$  be the element of  $M$  less than or equal to  $y_r$ . Let  $Z_s$  be the set of elements of  $Y_s$  which are paired with the elements of  $C_r$ . Then no element of  $Z_s$  is greater than  $m_r$ . Since every antichain of  $Z_s$  is finite, it must contain an infinite chain,  $C_s$ . Let  $y_s$  be an upper bound of  $C_s$  and let  $m_s$  be the element of  $M$  less than or equal to  $y_s$ . Then  $m_r$  does not equal  $m_s$  and  $\text{sup}(X)$  is less than both  $y_r$  and  $y_s$ . Therefore,  $\text{sup}(X)$  is in  $D$ .

Assume that for every  $k$  in  $\mathbb{N}$ , there is a  $y_k \geq x_k$  such that  $y_k$  is not contained in  $\downarrow(\uparrow m)$  for any  $m$  in  $M$ . Let  $Y$  be the set of all the  $y_k$ 's. Assume that  $Y$  is infinite. If  $Y$



contains an infinite antichain, then  $\text{sup}(X)$  is in the lower end it generates and is therefore in  $D$ . If  $Y$  does not contain an infinite antichain, then it contains an infinite chain which is bounded. Let  $y$  be an upper bound for this chain. Then  $y$  is above  $\text{sup}(X)$  and is not in  $\downarrow(\uparrow m)$  for any  $m$  in  $M$ . Therefore,  $D$  is closed.

Since  $C$  is unbounded in  $D$ , there is, by Theorem 4, a retraction,  $f$ , of  $D$  onto  $C'$ , a cofinal subchain of  $C$  of order type  $\omega$ . Let  $m_1$  and  $m_2$  be distinct elements of  $M$ . Define a function  $g: P \rightarrow C' \cup \{m_1, m_2\}$ , by

$$g(x) = \begin{cases} m_1 & \text{if } \uparrow x \subset (\downarrow(\uparrow m_1)) - D \\ m_2 & \text{if } \uparrow x \subset (\downarrow(\uparrow m)) - D \text{ and } m \neq m_1 \\ f(x) & \text{if } x \in D \end{cases}$$

Then  $f$  preserves order. Since the sets  $(\downarrow(\uparrow m_1)) - D$  and  $\bigcup_{m \in M - \{m_1\}} ((\downarrow(\uparrow m)) - D)$  are closed and  $g$  is  $\omega$ -chain continuous on them and on  $D$ , it is  $\omega$ -chain continuous on  $P$ . Thus,  $C' \cup \{m_1, m_2\}$  is a retract of  $P$  and  $P$  cannot have the least fixed point property for  $\omega$ -chain continuous functions.

**COROLLARY 3.** If every antichain of a partially ordered set,  $P$ , generates a closed lower end and every lower end of  $P$  which contains a countable chain but none of its upper bounds is layered, then  $P$  is  $\omega$ -chain complete if and only if it has the least fixed point property for  $\omega$ -chain continuous functions.

The following theorem by Edwin Miller appears as Theorem

B in [Mi].

THEOREM 6. If every antichain of an uncountable partially ordered set,  $P$ , is finite, then  $P$  contains an uncountable chain.

COROLLARY 4. If every antichain of a partially ordered set,  $P$ , is finite and every chain contains a countable cofinal subchain, then  $P$  is layered.

PROOF. Assume that there is no countable collection,  $G$ , of antichains of  $P$  such that  $P$  equals  $\bigcup_{A \in G} A$ . Let  $A_1$  be a maximal antichain of  $P$ . If  $\beta$  is less than  $\omega_1$  and, for every  $\alpha$  less than  $\beta$ ,  $A_\alpha$  has been defined, then let  $A_\beta$  be a maximal antichain of  $P - \bigcup_{\alpha < \beta} A_\alpha$ . The set  $A = \bigcup_{\alpha < \omega_1} A_\alpha$  is an uncountable partially ordered set and, by Theorem 6, contains an uncountable chain,  $C$ . But  $C$  must intersect an uncountable number of  $A_\alpha$ 's and so cannot contain a countable cofinal subchain.

COROLLARY 5. If every antichain of a partially ordered set,  $P$ , is finite and every chain contains a countable cofinal subchain, then  $P$  is  $\omega$ -chain complete if and only if it has the least fixed point property for  $\omega$ -chain continuous functions.

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