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Lyapunov Functionals that Lead to Exponential Stability and Instability in Finite Delay Volterra Difference Equations

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LYAPUNOV FUNCTIONALS THAT LEAD TO EXPONENTIAL STABILITY AND INSTABILITY IN FINITE DELAY VOLTERRA DIFFERENCE EQUATIONS

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Abstract. We use Lyapunov functionals to obtain sufficient conditions that guarantee exponential stability of the zero solution of the finite delay Volterra difference equation

$$x(t+1) = a(t)x(t) + \sum_{s=t-r}^{t-1} b(t,s)x(s).$$

Also, by displaying a slightly different Lyapunov functional we obtain conditions that guarantee the instability of the zero solution. The highlight of the paper is relaxing the condition $|a(t)| < 1$. Moreover we provide examples in which we show that our theorems provide an improvement of some of the recent literature.

1. Introduction

In this paper we consider the scalar linear difference equation with multiple delays

$$x(t+1) = a(t)x(t) + \sum_{s=t-r}^{t-1} b(t,s)x(s), \ t \geq 0, \tag{1.1}$$

where $r \in \mathbb{Z}^+$, $a : \mathbb{Z}^+ \to \mathbb{R}$ and $b : \mathbb{Z}^+ \times [-r, \infty) \to \mathbb{R}$. In this paper $\mathbb{R}$ denotes the set of real numbers and $\mathbb{Z}^+$ denote the set of positive integers. We will use Lyapunov functionals and obtain some inequalities regarding the solutions of (1.1) from which we can deduce exponential asymptotic stability of the zero solution. Also, we will provide a criteria for the instability of the zero solution of (1.1) by means of Lyapunov functionals.
Due to the choice of the Lyapunov functionals, we will deduce some inequalities on all solutions. As a consequence, the exponential stability of the zero solution is concluded. Consider the kth-order scalar difference equation

$$x(t+k) + p_1x(t+k-1) + p_2x(t+k-2) + \cdots + p_kx(t) = 0, \tag{1.2}$$

where the p_i's are real numbers. It is well known that the zero solution of (1.2) is asymptotically stable if and only if $|\lambda| < 1$ for every characteristic root λ of (1.2). There is no easy criteria to test for exponential stability

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of the zero solution of equations that are similar to (1.2) for variable coefficients. This itself highlights the importance of the creativity of constructing a suitable Lyapunov function that leads to the exponential stability. When using Lyapunov functionals, one faces the difficulties of relating the constructed Lyapunov functional back to the solution x so that stability can be deduced. This task is tedious and we did overcome it. The authors have done an extensive literature search and could not find any papers that dealt with exponential stability of Volterra equations of the form of (1.1). This paper offers easily verifiable conditions that guarantee exponential stability. Moreover, at the end of this manuscript we give criteria for the instability of the zero solution. Most importantly, our results will hold for $|a(t)| \geq 1$. We will illustrate our theory with several examples and numerical simulations.

Throughout this paper we use the convention that $\sum_{s=a}^{b} w(s) = 0$, if $a > b$.

Let $\psi : [-h, 0] \to (-\infty, \infty)$ be a given bounded initial function with

$$||\psi|| = \max_{-h \leq s \leq 0} |\psi(s)|.$$

It should cause no confusion to denote the norm of a function $\varphi : [-h, \infty) \to (-\infty, \infty)$ with

$$||\varphi|| = \sup_{-h \leq s < \infty} |\varphi(s)|.$$

The notation x_t means that $x_t(\tau) = x(t + \tau), \tau \in [-h, 0]$ as long as $x(t + \tau)$ is defined. Thus, x_t is a function mapping an interval $[-h, 0]$ into $\mathbb{R}$. We say $x(t) \equiv x(t, t_0, \psi)$ is a solution of (1.1) if $x(t)$ satisfies (1.1) for $t \geq t_0$ and $x_{t_0} = x(t_0 + s) = \psi(s)$, $s \in [-h, 0]$.
In preparation for our main results, we let

$$A(t, s) = \sum_{u=t-s}^{r} b(u + s, s). \tag{1.3}$$

Note that

$$A(t, t - r - 1) = 0.$$

Then, (1.1) is equivalent to

$$\triangle x(t) = \big(a(t) + +A(t + 1, t) - 1\big)x(t) - \triangle_t \sum_{s=t-r-1}^{t-1} A(t, s)x(s). \tag{1.4}$$

In [6], the author used the same method to study the exponential stability and instability of the zero solution of

$$x(t + 1) = a(t)x(t) - b(t)x(t - h).$$

We end this section with the following definition.

Definition 1.1. The zero solution of (1.1) is said to be exponentially stable if any solution $x(t, t_0, \psi)$ of (1.1) satisfies

$$|x(t, t_0, \psi)| \leq C\Big(||\psi||, t_0\Big)\zeta^{\gamma(t-t_0)}, \quad \text{for all } t \geq t_0,$$

where ζ is constant with $0 < \zeta < 1$, $C : \mathbb{R}^+ \times \mathbb{Z}^+ \to \mathbb{R}^+$, and γ is a positive constant. The zero solution of (1.1) is said to be uniformly exponentially stable if C is independent of t_0.

2. Exponential Stability

Now we turn our attention to the exponential decay of solutions of equation (1.1). For simplicity we let

$$Q(t) = a(t) + A(t + 1, t) - 1.$$

Assume

$$\triangle_t A^2(t, z) \leq 0, \quad \text{for all } t + s + 1 \leq z \leq t - 1. \tag{2.1}$$

Lemma 1. *Assume* (1.3) *and that for* $\delta > 0$ *the inequality*

$$-\frac{\delta}{(\delta + 1)r} \leq Q(t) \leq -r\delta A^2(t + 1, t) - Q^2(t) \tag{2.2}$$

holds. If

$$\begin{aligned} V(t) &= \left[x(t) + \sum_{s=t-r-1}^{t-1} A(t, s)x(s)\right]^2 \\ &+ \delta \sum_{s=-r}^{-1} \sum_{z=t+s}^{t-1} A^2(t, z)x^2(z), \end{aligned} \tag{2.3}$$

then along the solutions of (1.1) we have

$$\triangle V(t) \leq Q(t)V(t).$$

Proof. First we note that due to condition (2.2), $Q(t) < 0$ for all $t \geq 0$. Also, we use the fact that if $u(t)$ is a sequence, then $\triangle u^2(t) = u(t + 1)\triangle u(t) + u(t)\triangle u(t)$. For more on the calculus of difference equations we refer the reader to [3] and [4]. Let $x(t) = x(t, t_0, \psi)$ be a solution of (1.1) and define $V(t)$ by (2.3). Then along solutions of (1.4) we have

$$\begin{aligned} \triangle V(t) &= \left[x(t + 1) + \sum_{s=t-r}^{t} A(t + 1, s)x(s)\right] \triangle_t \left[x(t) + \sum_{s=t-r-1}^{t-1} A(t, s)x(s)\right] \\ &+ \left[x(t) + \sum_{s=t-r-1}^{t-1} A(t, s)x(s)\right] \triangle_t \left[x(t) + \sum_{s=t-r-1}^{t-1} A(t, s)x(s)\right] \\ &+ \delta\triangle_t \sum_{s=-r}^{-1} \sum_{z=t+s}^{t-1} A^2(t, z)x^2(z) \end{aligned} \tag{2.4}$$

We note that

$$\begin{aligned} x(t+1)+\sum_{s=t-r}^{t} A(t+1,s)x(s) &= (Q(t)+1)x(t) - \triangle_t \sum_{s=t-r-1}^{t-1} A(t,s)x(s) \\ &+ \sum_{s=t-r}^{t} A(t+1,s)x(s) \\ &= (Q(t)+1)x(t) + \sum_{s=t-r-1}^{t-1} A(t,s)x(s) \\ &= (Q(t)+1)x(t) + \sum_{s=t-r}^{t-1} A(t,s)x(s), \end{aligned}$$

since $A(t,t-r-1)=0$. With this in mind, (2.4) reduces to

$$\begin{aligned} \triangle V(t) &= \left[(Q(t)+1)x(t) + \sum_{s=t-r}^{t-1} A(t,s)x(s)\right] Q(t)x(t) \\ &+ \left[x(t) + \sum_{s=t-r}^{t-1} A(t,s)x(s)\right] Q(t)x(t) \\ &+ \delta\triangle_t \sum_{s=-r}^{-1}\sum_{z=t+s}^{t-1} A^2(t,z)x^2(z) \\ &= Q(t)V(t) + \big(Q^2(t)+Q(t)\big)\big)x^2(t) \\ &- \delta Q(t) \sum_{s=-r}^{-1}\sum_{z=t+s}^{t-1} A^2(t,z)x^2(z) \\ &+ \delta\triangle_t \sum_{s=-r}^{-1}\sum_{z=t+s}^{t-1} A^2(t,z)x^2(z) \\ &- Q(t)\left(\sum_{s=t-r}^{t-1} A(t,s)x(s)\right)^2. \end{aligned} \tag{2.5}$$

Also, using (1.3), we arrive at

$$\begin{aligned}
\triangle_t \sum_{s=-r}^{-1}\sum_{z=t+s}^{t-1} A^2(t,z)x^2(z) &= \sum_{s=-r}^{-1}\sum_{z=t+s+1}^{t} A^2(t+1,z)x^2(z) \\
&- \sum_{s=-r}^{-1}\sum_{z=t+s}^{t-1} A^2(t,z)x^2(z) \\
&= \sum_{s=-r}^{-1}\Big[A^2(t+1,t)x^2(t) + \sum_{z=t+s+1}^{t-1} A^2(t+1,z)x^2(z) \\
&- \sum_{z=t+s+1}^{t-1} A^2(t,z)x^2(z) - A^2(t,t+s)x^2(t+s)\Big] \\
&= \sum_{s=-r}^{-1}\Big(A^2(t+1,t)x^2(t) - A^2(t,t+s)x^2(t+s)\Big) \\
&+ \sum_{s=-r}^{-2}\sum_{z=t+s+1}^{t-1} \triangle_t A^2(t,z)x^2(z) \\
&= rA^2(t+1,t)x^2(t) - \sum_{s=-r}^{-1} A^2(t,t+s)x^2(t+s) \\
&+ \sum_{s=-r}^{-2}\sum_{z=t+s+1}^{t-1} \triangle_t A^2(t,z)x^2(z) \\
&\le rA^2(t+1,t)x^2(t) - \sum_{s=-r}^{-1} A^2(t,t+s)x^2(t+s) \qquad (2.6)
\end{aligned}$$

With the aid of Hölder's inequality, we have

$$\left(\sum_{s=t-r}^{t-1} A(t,s)x(s)\right)^2 \le r\sum_{s=t-r}^{t-1} A^2(t,s))x^2(s). \qquad (2.7)$$

Also,

$$\sum_{s=-r}^{-1}\sum_{z=t+s}^{t-1} A^2(t,z)x^2(z) \le r\sum_{s=t-r}^{t-1} A^2(t,s)x^2(s). \qquad (2.8)$$

By invoking (2.2) and substituting expressions (2.6), (2.7), and (2.8) into (2.5), we obtain

$$\begin{aligned}
\triangle V(t) &\le Q(t)V(t) + \big(Q^2(t) + Q(t) + r\delta A^2(t+1,t)\big)x^2(t) \\
&+ [-\big(\delta+1\big)rQ(t) - \delta]\sum_{s=t-h}^{t-1} A^2(t,s)x^2(s) \\
&\le Q(t)V(t). \qquad (2.9)
\end{aligned}$$

Theorem 2.1. *Assume the hypothesis of Lemma 1 holds and suppose there exists a number* $\alpha < 1$ *such that* $0 < a(t) + +A(t+1,t)) \leq \alpha$. *Then any solution* $x(t) = x(t, t_0, \psi)$ *of (1.1) satisfies the exponential inequality*

$$|x(t)| \leq \sqrt{\frac{r+\delta}{\delta} V(t_0) \prod_{s=t_0}^{t-1} \big(a(s) + +A(s+1,s)\big)} \tag{2.10}$$

for $t \geq t_0$.

Proof. First we note that condition (2.2) implies that there exists some positive number $\alpha < 1$ such that $|a(t) + +A(t+1,t))| < \alpha$. Now by changing the order of summation we have

$$\begin{aligned} \delta \sum_{s=-h}^{-1} \sum_{z=t+s}^{t-1} A^2(t,z)x^2(z) &= \delta \sum_{z=t-r}^{t-1} \sum_{s=-r}^{z-1} A^2(t,z)x^2(z) \\ &= \delta \sum_{z=t-r}^{t-1} A^2(t,z)x^2(z)(z-t+r+1) \\ &\geq \delta \sum_{z=t-r}^{t-1} A^2(t,z)x^2(z), \end{aligned}$$

where we have used the fact that $t-h \leq z \leq t-1 \implies 1 \leq z-t+h+1 \leq h$. Also

$$\left(\sum_{z=t-r}^{t-1} A(t,s)x(s) \right)^2 \leq r \sum_{z=t-r}^{t-1} A^2(t,s)x^2(s),$$

and hence,

$$\delta \sum_{s=-r}^{-1} \sum_{z=t+s}^{t-1} A^2(t,z)x^2(z) \geq \frac{\delta}{r} \left(\sum_{z=t-r}^{t-1} A(t,z)x(z) \right)^2.$$

Let $V(t)$ be given by (2.3). Then

$$\begin{aligned} V(t) &= \big[x(t) + + \sum_{s=t-r-1}^{t-1} A(t,s)x(s)\big]^2 + \delta \sum_{s=-r}^{-1} \sum_{z=t+s}^{t-1} A^2(t,z)x^2(z) \\ &\geq \big[x(t) + + \sum_{s=t-r-1}^{t-1} A(t,s)x(s)\big]^2 + \frac{\delta}{r} \left(\sum_{z=t-r}^{t-1} A(t,z)x(z) \right)^2 \\ &\geq \frac{\delta}{r+\delta} x^2(t) + \left[\sqrt{\frac{r}{r+\delta}} x(t) + + \sqrt{\frac{r+\delta}{r}} \sum_{z=t-r}^{t-1} A(t,z)x(z) \right]^2 \\ &\geq \frac{\delta}{r+\delta} x^2(t). \end{aligned}$$

Consequently,

$$\frac{\delta}{r+\delta}x^2(t) \leq V(t).$$

From (2.9) we get

$$V(t) \leq V(t_0)\prod_{s=t_0}^{t-1}\big(a(s) + +A(s+1,s)\big).$$

Consequently, we arrive at

$$|x(t)| \leq \sqrt{\frac{r+\delta}{\delta}V(t_0)\prod_{s=t_0}^{t-1}\big(a(s) + +A(s+1,s)\big)}$$

for $t \geq t_0$. This completes the proof.

Corollary 1. *Assume the hypothesis of Theorem 2.1 holds. Then the zero solution of* (1.1) *is exponentially stable.*

Proof. From inequality (2.10) we have that

$$\begin{aligned} |x(t)| &\leq \sqrt{\frac{r+\delta}{\delta}V(t_0)\prod_{s=t_0}^{t-1}\big(a(s) + +A(s+1,s)\big)} \\ &\leq \sqrt{\frac{r+\delta}{\delta}V(t_0)\alpha^{t-t_0}} \end{aligned}$$

for $t \geq t_0$. The proof is complete since $\alpha \in (0,1)$.

Next we give a criterion for instability via Lyapunov functionals.

3. Criterion For Instability

In this section we use a non-negative definite Lyapunov functional and obtain criteria that can be easily applied to test for instability of the zero solution of (1.1).

Theorem 3.1. *Let $H > r$ be a constant. Assume $Q(t) > 0$ such that*

$$Q^2(t) + Q(t) - HA^2(t+1,t) \geq 0. \tag{3.1}$$

If

$$\begin{aligned} V(t) &= \left[x(t) + + \sum_{s=t-r-1}^{t-1}A(t,s)x(s)\right]^2 \\ &- H\sum_{s=t-r-1}^{t-1}A^2(t,s)x^2(s), \end{aligned} \tag{3.2}$$

then along the solutions of (1.1) we have

$$\triangle V(t) \geq Q(t)V(t).$$

Proof. Let $x(t) = x(t, t_0, \psi)$ be a solution of (1.1) and define $V(t)$ by (3.2). Since the calculation is similar to the one in Lemma 1, we arrive at

$$\begin{aligned} \triangle V(t) &= Q(t)V(t) + \big(Q^2(t) + Q(t) - HA^2(t+1,t)\big)x^2(t) \\ &+ Q(t)(H-r)\left(\sum_{s=t-r-1}^{t-1} A^2(t,s)x^2(s)\right)^2 \\ &\geq Q(t)V(t), \end{aligned} \tag{3.3}$$

where we have used

$$\left(\sum_{s=t-r}^{t-1} A(t,s)x(s)\right)^2 \quad \leq \quad r\sum_{s=t-r}^{t-1} A^2(t,s)x^2(s)$$

and (3.1). This completes the proof.

We remark that condition (3.1) is satisfied for $Q(t) \geq \dfrac{-1 + \sqrt{1 + 4HA^2(t+1,t)}}{2}$.

Theorem 3.2. *Suppose the hypothesis of Theorem 4.1 holds. Then the zero solution of* (1.1) *is unstable, provided that*

$$\prod^{\infty} \big(a(s) + +A(s+1,s)\big) = \infty. \tag{3.4}$$

Proof. From (3.3) we have

$$V(t) \geq V(t_0) \prod_{s=t_0}^{t-1} \big(a(s) + +A(s+1,s)\big). \tag{3.5}$$

Let $V(t)$ be given by (3.2). Then

$$\begin{aligned} V(t) &= x^2(t) + +2x(t)\sum_{t-r-1}^{t-1} A(t,s)x(s) + \left[\sum_{t-r-1}^{t-1} A(t,s)x(s)\right]^2 \\ &- H\sum_{t-r-1}^{t-1} A^2(t,s)x^2(s). \end{aligned} \tag{3.6}$$

Let $\beta = H - r$. Then from

$$\left(\frac{\sqrt{r}}{\sqrt{\beta}}a - \frac{\sqrt{\beta}}{\sqrt{r}}b\right)^2 \geq 0,$$

we have

$$2ab \leq \frac{r}{\beta}a^2 + \frac{\beta}{r}b^2.$$

With this in mind we arrive at

$$\begin{aligned}2x(t)\sum_{t-r-1}^{t-1}A(t,s)x(s) &\leq 2\,|x(t)|\left|\sum_{t-r-1}^{t-1}A(t,s)x(s)\right| \\ &\leq \frac{r}{\beta}x^2(t)+\frac{\beta}{r}\left[\sum_{t-r-1}^{t-1}A(t,s)x(s)\right]^2 \\ &\leq \frac{r}{\beta}x^2(t)+\frac{\beta}{r}r\sum_{t-r}^{t-1}A^2(t,s)x^2(s).\end{aligned}$$

A substitution of the above inequality into (3.6) yields

$$\begin{aligned}V(t) &\leq x^2(t)+\frac{r}{\beta}x^2(t)+(\beta+r-H)\sum_{t-r-1}^{t-1}A^2(t,s)x^2(s) \\ &= \frac{\beta+r}{\beta}x^2(t) \\ &= \frac{H}{H-r}x^2(t).\end{aligned}$$

Using inequality (3.5), we get

$$\begin{aligned}|x(t)| &\geq \sqrt{\frac{H-r}{H}}\,V^{1/2}(t) \\ &= \sqrt{\frac{H-r}{H}}\,V^{1/2}(t_0)\left(\prod_{s=t_0}^{t-1}\big(a(s)++A(s+1,s)\right)^{\frac{1}{2}}.\end{aligned}$$

This completes the proof.

4. Applications and Numerical Evidence

In this section we provide examples that illustrate our theoretical results in two instances: when the coefficients $a(t)$ and $b(t,s)$ are constant, and when they are functions.

First, if $a(t)=a$ and $b(t,s)=b$ $(a,b\in\mathbb{R})$ we have $A(t,s)=\displaystyle\sum_{u=t-s}^{r}b$. Then A(t,s) = b(r+1-t+s). Hence, $\triangle_t A^2(t,s)=b^2(r-t+s)^2-b^2(r+1-t+s)^2\leq 0$ and thus condition (1.3) holds. Also $A(t+1,t)=br$, and hence condition (2.3) becomes

$$-\frac{\delta}{(\delta+1)r}\leq a+br-1\leq-\left[\delta b^2r^3+(a+br-1)^2\right]. \tag{4.1}$$

It is obvious from (4.1) that when $a=1$, b has to be negative.

Next we give four examples where the emphasis is on $|a| \geq 1$.

Example 1. Let $a = r = 1, b = -0.3$ and $\delta = 0.5$. Then one can easily verify that (4.1) is satisfied. Hence the zero solution of the delay difference equation

$$x(t+1) = x(t) - 0.3x(t-1) \tag{4.2}$$

is exponentially stable.

Example 2. Let $a = 1.2, b = -0.3, r = 1$, and $\delta = 0.5$. Then one can easily verify that (4.1) is satisfied. Hence the zero solution of the delay difference equation

$$x(t+1) = 1.2x(t) - 0.3x(t-1) \tag{4.3}$$

is exponentially stable as illustrated in Figure 1(a).

Example 3. Let $a = 1.29, b = -0.6, r = 1$, and $\delta = 0.5$. With these values (4.1) is satisfied, and therefore the zero solution of the delay difference equation

$$x(t+1) = 1.29x(t) - 0.6x(t-1)$$

is exponentially stable as shown in Figure 1(b).

Example 4. Let $a = 1.125, b = -0.15, r = 2$, and $\delta = \frac{2}{3}$. Then one can easily verify that (4.1) is satisfied. Hence the zero solution of the delay difference equation

$$x(t+1) = 1.125x(t) - 0.15\big(x(t-1) + x(t-2)\big)$$

is exponentially stable as shown in Figure 1(c).

It is worth mentioning that in both papers [5] and [7] it was assumed that

$$\prod_{s=0}^{t-1} a(s) \to 0, \text{ as } t \to \infty$$

for the asymptotic stability.

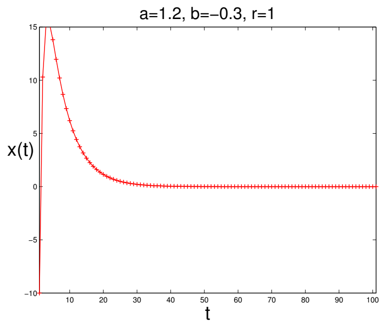
Example 5. Let $a = 1.3, b = -0.2, r = 1$ and $H = 1.1$. Then $Q(t) = 0.1 > 0$. Moreover $Q(t) \geq \dfrac{-1 + \sqrt{1 + 4HA^2(t+1,t)}}{2} = 0.0422$. Thus conditions (3.1) and (3.2) are satisfied and the zero solution of

$$x(t+1) = 1.3x(t) - 0.2x(t-1) \tag{4.4}$$

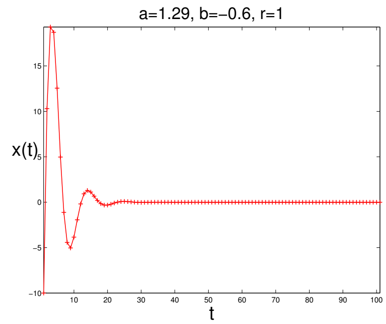
is unstable. Actually, all its solutions become unbounded for large t. Figure 2 shows a trajectory for the above equation with initial condition $x(0) = -10$ and $x(1) = -1.3$.

Remark: When $a(t)$ and $b(t,s)$ are constant the solution $x(t)$ of the delay difference equation (1.1) is the same as the sequence $(x_n)_{n \in \mathbb{N}_0}$ defined recursively as

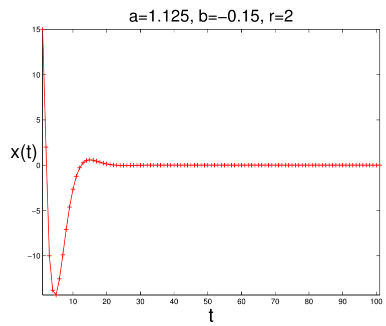
$$x_{n+r+1} = ax_{n+r} + b\left(x_{n+r-1} + \cdots + x_n\right), n \in \mathbb{N}_0, \tag{4.5}$$



(a)



(b)



(c)

FIGURE 1. Trajectories of (1.1) when $a(t)$ and $b(t, s)$ are constant. Figure 1(a) refers to Example 2 where $a = 1.2, b = -0.3$ and $r = 1$ with initial condition $x(0) = -10$ and $x(1) = 10.3$. Figure 1(b) refers to Example 3 where $a = 1.29, b = -0.6$ and $r = 1$ with initial condition $x(0) = -10$ and $x(1) = 10.3$. Figure 1(c) refers to Example 4 where $a = 1.125, b = -0.6$ and $r = 2$ with initial condition $x(0) = 15, x(1) = 2$ and $x(3) = -10$.

and for which the general solution can be obtained analytically. For $r = 1$ in particular, the general solution to (4.5) is easily calculated. For instance, the exact solution to (4.2) in Example 1 is

$$x(t) = \left(\frac{\sqrt{30}}{10}\right)^t \left(x(0)\cos(t\theta) + \frac{\frac{10x(1)}{\sqrt{30}} - x(0)\cos\theta}{\sin\theta}\sin(t\theta)\right),$$

where $\theta = \arctan\left(\frac{\sqrt{5}}{5}\right)$. Since $\left|\frac{\sqrt{30}}{10}\right| < 1$ we see that $\lim_{t\to+\infty} |x(t)| = 0$ with an exponential convergence.

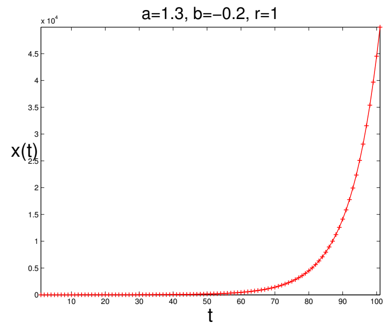


FIGURE 2. Trajectories of (1.1) when $a(t)$ and $b(t,s)$ are constant. This graph corresponds to Example 5 where $a = 1.3, b = -0.2$ and $r = 1$ with initial condition $x(0) = -10$ and $x(1) = -1.3$.

The exact solution to (4.3) in Example 2 is

$$\begin{aligned} x(t) &= \frac{1}{2}\left[\left(x(0) + 10\frac{x(1) - \frac{6x(0)}{10}}{\sqrt{6}}\right)\left(\frac{6+\sqrt{6}}{10}\right)^t \right. \\ &+ \left.\left(x(0) - 10\frac{x(1) - \frac{6x(0)}{10}}{\sqrt{6}}\right)\left(\frac{6-\sqrt{6}}{10}\right)^t\right]. \end{aligned}$$

Since $\left|\frac{6\pm\sqrt{6}}{10}\right| < 1$, we see that the solution $x(t)$ of Example 2 converges exponentially to zero. Similar calculations can be done for Examples 3 and 4.

Finally, the exact solution to (4.4) in Example 5 is

$$\begin{aligned} x(t) &= \frac{1}{2}\left[\left(x(0) + 20\frac{x(1) - \frac{13x(0)}{20}}{\sqrt{89}}\right)\left(\frac{13+\sqrt{89}}{20}\right)^t \right. \\ &+ \left.\left(x(0) - 20\frac{x(1) - \frac{13x(0)}{20}}{\sqrt{89}}\right)\left(\frac{13-\sqrt{89}}{20}\right)^t\right]. \end{aligned}$$

Since $\left|\frac{13+\sqrt{89}}{20}\right| > 1$, we see that $\lim_{t\to+\infty} |x(t)| = +\infty$.

We now give two examples that illustrate the exponentially stable and unstable case when $a(t)$ and $b(t,s)$ are functions. We corroborate our results with numerical simulations.

Example 6. Let $a(t) = d^{2t+1} + \frac{2}{3}$ and $b(t,s) = -d^{t+s}$ for $d \in \mathbb{R}$. Then $A(t,s) = -d^{2s}\displaystyle\sum_{u=t-s}^{r} d^u$, and therefore $A(t+1,t) = -d^{2t}\displaystyle\sum_{u=1}^{r} d^u = -d^{2t+1}$ for $r = 1$. We can show that $\Delta_t A^2(t,z) \le 0$ for all $t+s+1 \le z \le t-1$. If we take $r = 1$ and $\delta = 1$, we obtain $Q(t) = -\dfrac{1}{3}$. With these choices we see that the left inequality of condition (2.2) is trivially satisfied. To obtain the right inequality, we need to choose d such that $\left(d^2(d^4)^t + \frac{1}{9}\right) \leq -Q(t) = \frac{1}{3}$ for t large enough. It is therefore sufficient to choose $d \in (0, 1)$. In that case, $\lim\limits_{t\to+\infty}(d^4)^t = 0$ which implies that the right inequality of condition (2.2) will eventually be satisfied. Note that condition (2.2) is satisfied for all $t \geq 0$ when $d \in (0, \frac{\sqrt{2}}{3}]$. With these choices for the parameters d, δ and r, we can conclude that the zero solution of the delay difference equation

$$x(t+1) = \left(d^{2t+1} + \frac{2}{3}\right) x(t) - d^{2t+1}x(t-1)$$

is exponentially stable. We plotted two of its trajectories in Figure 3.

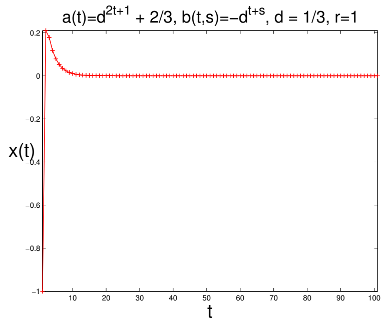
Example 7. Let $a(t) = d^{2t+1}+1.1$ and $b(t, s) = -d^{t+s}$. Then from Example 6 we have $A(t + 1, t) = -d^{2t+1}$ when $r = 1$. In that case choosing $H = 1$ yields $Q(t) = 0.1 > 0$. With these choices we see that condition (3.1) is satisfied if $d \in (0, 1)$ and hence the zero solution of the delay difference equation

$$x(t+1) = \left(d^{2t+1} + 1.1\right) x(t) - d^{2t+1}x(t-1)$$

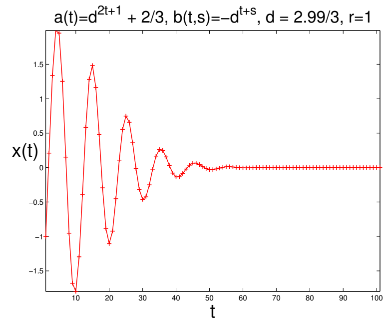
is unstable as illustrated in Figure 4. In fact, the zero solution is unstable for all choices of $a(t) = d^{2t+1} + \nu$ with $\nu > 1$. We note that with these choices of $a(t)$ and $b(t, s)$ we have

$$\prod^{\infty} \left(a(s) + +A(s+1, s)\right) = \prod^{\infty} \nu = +\infty,$$

and hence (3.4) is verified.



(a)



(b)

FIGURE 3. Trajectories of (1.1) when $a(t) = d^{2t+1} + \frac{2}{3}$ and $b(t,s) = -d^{t+s}$. These plots refer to Example 6 with $r = 1$. The initial condition was taken to be $x(0) = -1$ and $x(1) = 0.21$. In Figure 3(a) we plotted the trajectory obtained with $d = \frac{2}{3}$, and in Figure 3(b) we plotted the trajectory with $d = \frac{2.99}{3}$. In the latter case, since condition (2.2) is verified only after a certain value of t, the first few terms of the trajectory $x(t)$ are not converging to zero until condition (2.2) is satisfied.

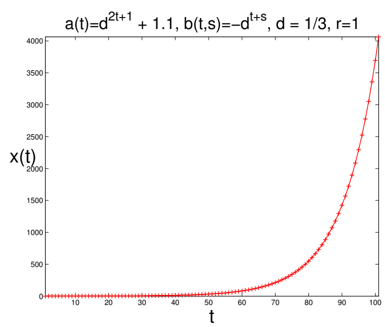


FIGURE 4. Trajectories of (1.1) when $a(t) = d^{2t+1} + 1.1$ and $b(t,s) = -d^{t+s}$. This graph corresponds to Example 7 with $r = 1$ and initial condition $x(0) = -1$ and $x(1) = 0.21$.

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