1-1-2017

To Each Their Own: Students Asking Questions Through Individualized Projects

S. A. Cook
University of Alaska, Anchorage, sam.cook@uaa.alaska.edu

James Hartman
The College of Wooster

Pamela B. Pierce
The College of Wooster, ppierce@wooster.edu

N. S. Seaders
Linn Benton Community College

To Each Their Own: Students Asking Questions Through Individualized Projects

S.A. Cook
Department of Mathematics and Statistics
University of Alaska Anchorage
Anchorage, AK 99508, USA
sam.cook@uaa.alaska.edu

J. Hartman
Department of Mathematics and Computer Science
The College of Wooster
Wooster, OH 44691
hartman@wooster.edu

P. Pierce
Department of Mathematics and Computer Science
The College of Wooster
Wooster, OH 44691
ppierce@wooster.edu

N.S. Seaders
Department of Mathematics
Oregon State University
Corvallis, OR 97330, USA
seadersn@math.oregonstate.edu

To Each Their Own: Students Asking Questions Through Individualized Projects

Abstract: As mathematics educators we want our students to develop a natural curiosity that will lead them on the path toward solving problems in a changing world, in fields that perhaps do not even exist today. Here we present student projects, adaptable for several mid- and upper-level mathematics courses, that require students to formulate their own questions and to begin to develop the basic research skills needed to answer these questions. These projects, where each student is given an individualized object to study, allow students to take ownership over their own learning while introducing them to the joy and challenge of discovery and research. Each student is directed to use the concepts and techniques presented in class as a set of tools to guide the investigation of their object. We discuss our experiences–both positive and negative–with these inquiry-based projects.

Keywords: Mathematical Inquiry, Project, Undergraduate Research, Linear Algebra, Abstract Algebra, Topology, Differential Geometry, Calculus

1 INTRODUCTION

In 2013, the Association of American Colleges and Universities (AAC&U) undertook an online survey of employers asking what skills and dispositions they found important as they were hiring recent graduates. Their report, *It Takes More Than a Major: Employer Priorities for College Learning and Student Success*, showed that employers are seeking much more than disciplinary knowledge. Employers want students who are poised for leadership in an ever-changing work environment. The report states that the "consensus among employers is that innovation, critical thinking, and a broad skill set are important for taking on complex challenges in the workplace." The report goes on to say that "Employers endorse several educational practices as potentially helpful in preparing college students for workplace success. These include practices that require students to a) conduct research and use evidence-based analysis; b) gain in-depth knowledge in the major and analytic, problem solving, and communication skills; and c) apply their learning in real-world settings [12]."

Suddenly, creating a syllabus for a single class can seem like a daunting task. As educators, we now need to make sure that we are contributing to the development of these skills and habits of mind within our students. After all, we want our students to find success in graduate programs and in employment outside of academia. How do we help our students to develop skills in inquiry and investigation, necessary to achieve goals like these, in courses throughout the mathematics major [4]? The projects presented here give a possible answer to this question. These projects ask students to investigate open-ended problems in a setting without clearly defined paths to pursue. This is exactly the type of task that will be before them as they begin any type of research project in graduate school or many projects in industry. By requiring this type of investigative project, we hope to develop a natural curiosity among our students that will benefit them throughout their careers [2]. We would like to give students a measure of control over their educational experience [13]. We also hope to develop the skills and dispositions necessary to pursue research in any area, and lead students on the path toward becoming lifelong learners (see Part III of [7]).

2 THE PROJECT

The beauty of these assignments lies in their simplicity. Give each student their own individual object of study, relevant to the course, and ask them to discover whatever they can about their object. Partway through the semester in a sophomore level linear algebra course, we give each student several personal matrices that they are to investigate during the semester. Toward the end of the course, the students must turn in a written document describing everything that they were able to discover about their matrices. Analogously, in an abstract algebra course, we assign to each student a multiplication table for a group of order 16 at the beginning of the term. By the end of the term, students will give a presentation detailing the properties and unique features of their group. Ideally, as students learn about new concepts in each course, they will have their assignment in mind and formulate questions about the object of study they have been given.

These projects were developed with several important learning goals in mind. First, we want to give students an opportunity to engage in self-directed discovery, which encourages exploration and experimentation, so that students may develop those habits of mind that lead to true inquiry. We also give students a means by which to take ownership of their learning by making the object of their study unique to them. By asking students to look at the various properties of a group or a matrix over a period of time, we provide them with an opportunity to engage continuously with the course material and to integrate concepts from the course that may at first seem very disjoint. Because the students are graded on a formal presentation of their findings, we expect that these projects will help develop the skills of oral and written communication. If the instructor encourages discussion and collaboration, then students learn how to exchange ideas and work together collegially. The projects hope to develop and encourage good mathematical behavior generally and they serve as an on-ramp to research for some students.

3 TWO PRIMARY EXAMPLES-IMPLEMENTATION

Linear Algebra

In Linear Algebra, each student is given two to four matrices (depending on the instructor and the class size in a particular semester) so that no two students have the same matrix. Behind the scenes, we generate specific types of matrices that can lead to potentially interesting investigations. Some examples of the types of matrices that we give are Toeplitz matrices, Hankel matrices, circulant matrices, magic matrices, and Pascal matrices. In the course, students have encountered upper and lower triangular matrices, diagonal matrices, and symmetric matrices, but they have no particular experience with these other types of matrices. Some of our students will correctly identify a matrix as 'circulant' or 'Hankel', for example, but this is not required. Regardless of whether the students are able to correctly identify the type of matrix they have been given, it is our hope that students who are given a matrix with a particular property will consider the broader class of all matrices with that property.

Students are instructed to use the tools from the course to tell the full story of the matrices in question. We tell the students up front that they will certainly want to give some basic properties of each matrix. Finding the determinant, the rank, the eigenvalues and eigenvectors are all considered basic, obvious pieces of the story. We have refrained from providing a laundry list of questions that must be answered, hoping that student curiosity will help the project take on a life of its own. Ultimately, we hope that students will begin to wonder about specific patterns that are evident in the entries of the matrix, that they will make a conjecture about a class of matrices that have a particular property, and prove that conjecture. We ask that students write five to ten pages (typeset, double-spaced) about each of their matrices. A sample project guide is provided in Appendix B.

We provide a LaTeX template to students containing the beginnings of a good submission. At the very least, the template helps the students to format a matrix in LaTeX, and see how sub-headings can be used to structure a paper into an organized presentation. We give a copy of the grading rubric to the students in advance so that they can see how they will be assessed on the project. The project itself accounts for approximately 10% of the course grade; 28% of the project grade is the writing component while the other 72% of the grade is the mathematical component. Within the mathematical component, students can see that we award points for correctly calculating the determinant, rank, row space, column space, eigenvalues, and eigenvectors of the matrix. They can also see that points are awarded for investigating a broader class of matrices, and that there are points awarded for the depth and creativity of the investigation. A sample rubric is provided in Appendix D.

Abstract Algebra

In Abstract Algebra, each student is assigned their own multiplication table for a group of order 16. Students are asked to give their group a name and begin a semester long investigation, recording their investigations in a research journal with periodic summaries, new questions, and conjectures. For this three credit class, students are expected to spend nine hours a week outside of class studying and doing homework for the course. We specifically state in the project assignment that students should devote two to three of those hours each week to investigating their group. The project counts for 15% of their course grade.

Eleven of the fourteen groups of order 16 are used for these individual assignments, and some groups are assigned to more than one student by relabeling the elements of the group to disguise the multiplication table. The cyclic group and the group whose elements are all of order two are not assigned. The remaining group is the dihedral group, which the instructor uses as an example throughout the semester to illustrate new course concepts and model inquiry on her individual object. We try to give the more interesting and complicated groups to students with more mathematics experience.

Each student is given a project guide introducing them to their journals (see Appendix C). Here we let the students know that each entry should include a date, a heading, and an explanation of the day's investigation. Each research session should end with a summary of results, any new questions or conjectures, as well as a list of what was proved and still needs to be proved. We have found through experience that the investigations are deeper and the journals are easier to read with this guidance.

Every two weeks students are required to submit their journal, which is roughly assessed based on the amount of research done in the previous two weeks. Initially students will get one or two paragraphs of comments, giving them more guidance early in the term, while later in the term they will see strategically located post-it notes with questions or comments. When we see something promising in a journal, we try to encourage students to probe a bit more deeply. If a journal does not have sufficient content, we take the opportunity to remind students of the amount of time they should be spending on their journal. We praise diligent work, even if it leads to a dead-end, and encourage flashes of insight.

At the end of the term, students write a 'Conclusions' section to their journal. This section includes a summary of any discoveries they made about their group, general results they proved, and interesting open questions that remain. During the last few days of class, each student gives an 8-10 minute presentation on their findings, and we follow these presentations with a class discussion of the groups of order 16.

The course uses Joseph Gallian's textbook [3] and we make sure to go over Gallian's classification of groups of order 8. After this discussion, it should be clear to students that they now fully understand their subgroups up to isomorphism. At the end of the student presentations, students are asked to draw their subgroup lattices on the whiteboards around the room and to determine how many groups of order 16 there are, using the Fundamental Theorem of Finite Abelian Groups to include any missing abelian groups. Our experience is that the class readily engages with this activity, and that they can easily 'discover' that there are 14 distinct groups of order 16. Students will often ask questions of one another to better understand the different groups and to get a feel for the landscape of the groups. Eventually each group gets expressed as some form of direct product or semi-direct product. The concluding discussion is a highlight of the course and a great way to finish the semester.

4 ADAPTABILITY

This type of project can be implemented in a wide variety of courses, by choosing an appropriate object to study and asking students to discover as much as they can about their individualized object. Hereafter we will call such an object a "gadget," a small device or tool that is often interesting or unfamiliar. In the examples above, the gadgets were matrices (typically 3×3 or 4×4) and groups of order 16. The authors have implemented a version of this project in the following additional courses: differential calculus, topology, and differential geometry, with corresponding gadgets being functions, topological spaces, and plane and space curves; for more information, see Appendix A. Colleagues of the authors have adapted it to analytic geometry and graph theory, where the gadgets were equations of quadric surfaces built from the students' ID numbers and well known graphs disguised in some way.

When choosing gadgets for your class, the basic idea is to look for mathematical objects that will provide a challenge for your students and that they can fruitfully experiment with over the term, while avoiding objects that students find too difficult to study. In addition, gadgets should integrate well with the course and its themes. For instance, a primary motivation for choosing groups of order 16 is to give students a chance to work on a classification problem, which reinforces lessons taught during the semester.

Clearly, the gadget will vary with the course content, and we observe that the benefits of different types of gadgets vary as well. For instance, courses like abstract algebra and topology naturally provide gadgets with rich internal structure, where the objects themselves can be fruitfully studied throughout the term. We have found that when the gadgets are rich enough to support study in their own right, collaboration among the students is very beneficial, resulting in a deeper understanding of the object by the student. On the other hand, other gadgets lend themselves to mathematical generalizations. In the case of linear algebra, students are encouraged to study not only the matrices (gadgets) they are given, but the class(es) to which those matrices be-long. Finally, initial gadgets can be assigned so that students can build upon the initial object to create new gadgets throughout the course. For instance, in differential geometry, from a given plane curve students can build a surface of revolution, or similarly from a space curve they could later construct a ruled surface, either of which provide an introduction to surface theory and an example of the important interplay between surfaces and curves that lie on them.

5 CHALLENGES AND SOLUTIONS

Over time, while the basic idea of the project has not changed, the specific features of the implementations have. These have evolved over time in response to various challenges that have presented themselves. For instance, ideally the project instructions would be "Here is an *Individualized Mathematical Gadget* chosen just for you. Learn as much as you can about your *Gadget* this term using what you learn in the course and present your results by *this date*." Needless to say, with this approach most students would not know where to begin or when they had "done enough". In this section, we share some of the stumbling blocks we have run into, and offer some potential solutions.

This project is, by design and necessity, very open-ended. Students may immediately need some help getting started on their inquiry, and as the course progresses, they may well need help figuring out what to do next, or what to do when they get stuck, which is not uncommon in this type of inquiry. To address these concerns, instructors have had success trying a couple of different approaches.

In the linear algebra course, a student's final results are presented in a written paper. Since the matrices given in the project are not pervasive throughout the entire course, the project has a duration of six weeks, and is assigned during the second half of the semester. To get students started we give a more detailed project guide (please see Appendix B), giving students their individualized matrices, listing a few specific questions that we want students to investigate about their matrices, including finding a broader class of matrices that includes their own matrix, as well as directions to “play” and “explore”. We post a “sample project” using a matrix with rank 1 so that students can observe what explorations one might undertake with such a matrix. In addition, we developed a grading rubric that is shared with the students that does not provide too much of a checklist and does not take away from the open-ended and student-driven aspects of the investigation. The rubric addresses both exposition as well as mathematical content and accuracy. Included in the rubric is a place to indicate whether specific properties of the matrix have been presented, a place to indicate whether the student examined a broader class of matrices, and a place to indicate the depth of the “creative explorations”. The rubric may be found in Appendix D.

In the abstract algebra course, groups are a pervasive topic and so students can explore their groups all semester long. The project guide directs students to apply or attempt to apply every new concept, topic, definition or theorem to their group, rather than giving them a list of questions to answer. The students are then required to keep track of their investigations in a research journal, as described above. In addition, the instructor is “assigned” her own mathematical object, in this case the dihedral group of order 16, and uses it whenever possible as an example in class, even when the property or the theorem does not apply. For example, the instructor will “check”, using only the multiplication table, to see if the group is abelian; find the order of each element, list all of the subgroups, find those that are normal, etc. and then report those “findings” to the class as each topic arises. This serves to give students a daily starting point outside of class and to show them that the work involved is not necessarily straightforward (not every theorem and definition applies to their object). The use of the research journals and an instructor example also helps to alleviate another problem encountered in courses where the duration of the project is the entire term. We find these additions help keep students motivated and keep students working regularly with their gadgets[1] (and helps keep students

[1]Somehow missing the memo, one student asked if the journal assessment counted toward his grade midway through the semester. After hearing an affirmative his jour-

from waiting until the last minute to complete their projects.)

As faculty members, we are all busy people, and the impact of any course design element on one's workload should be considered when designing a course. We have found that a primary consideration in this regard is class size.

Abstract algebra is usually a relatively small class, and in this case, the research journals are quite feasible. Students are able to investigate any line of inquiry related to the course that they found interesting (the only specific assignment all students have to accomplish is finding a subgroup lattice for their group). Unfortunately, even for diligent students, inquiries might lead to dead-ends rather than to results worthy of their final presentation, but students record their negative results as well as their successes in their journals. A full evening of work (3–4 hours) is required of the instructor every couple of weeks in order to adequately review and comment on them, but the last author reports that although this is a nontrivial amount of time, it is always the most rewarding part of the course.

Other courses tend to be much larger, possibly running to multiple sections in a semester. In these courses, it will most likely not be feasible to use tools like the research journals to facilitate the project. For example, when the matrix project was introduced for the first time, the instructor gave students five matrices at the beginning of the semester and asked students to keep journal entries for each matrix throughout the semester and update their journals regularly. As the instructor had a total of about 60 students in two sections of the course, the task of checking journal entries required more than one night of work, and it was not possible to return them in a timely manner so students could continue to update them regularly. One option that has been tried with some success in a differential calculus course is to put questions on the exams relating to the students' gadgets. An exam problem might ask for the student to give their gadget, and one or two basic properties they have discovered about it.

Since it is a larger class, in the linear algebra course the project is

nal experienced a dramatic shift upward in quantity and quality of his investigations.

facilitated by requiring a paper from each student about their gadget. In order to monitor and encourage depth of work in the course, the instructor ideally could read a draft of each student's project, comment on both its mathematical and grammatical content, and return it to the student to make revisions and improvements on depth of the mathematical exploration. The instructor could also suggest avenues for a student to pursue. Again, due to the large number of students in the course, the instructor has only offered to look at drafts in the presence of any student who desired this. Some students request this, but many do not.

One of the biggest challenges with the linear algebra project is to get students to go beyond simple calculations. Students often struggle with seeing their matrix as a single example of a more general type of matrix, and then discovering properties of that more general type. For example, a student might identify that the row sums, column sums, diagonal sum, and cross-diagonal sum of the matrix $\begin{bmatrix} 4 & 0 & 5 \\ 4 & 3 & 2 \\ 1 & 6 & 2 \end{bmatrix}$ are all 9, but never go on to consider a general form, $\begin{bmatrix} \frac{s}{3}+a & \frac{s}{3}-a-b & \frac{s}{3}+b \\ \frac{s}{3}+b-a & \frac{s}{3} & \frac{s}{3}-b+a \\ \frac{s}{3}+-b & \frac{s}{3}+a+b & \frac{s}{3}-a \end{bmatrix}$, of such a matrix. If a student is able to generalize to this form, then they may be able to see relationships between the entries of a matrix and other properties of the matrix. In the matrix above, for example, there is a nice relationship between the entries of the matrix and its eigenvalues and eigenvectors.

Some of our top students will be able to find a general form of their matrix. If students ask us what more they can do with this general form, we suggest that they try to discover properties of matrices of this form. We could get each student to probe more deeply if we did more individual coaching, but that would detract from the goal of independent and creative learning, and it would certainly add to the instructor workload. Truthfully, some students never get to this general form and simply focus on the computations involving their own matrix. Even for these students, the project allows them to review the fundamental matrix concepts found in the course.

6 BENEFITS AND OUTCOMES

In reflecting upon the two projects described above, we must ask whether or not our goals have been accomplished and whether there are particular benefits that derive from them. As stated in section 2, our goals are to introduce students to the process of inquiry, give students a means by which to take ownership of their learning, engage continuously with the course material and integrate concepts from the course, develop student skills of oral and written communication, and encourage and develop good mathematical behavior generally. Moreover, in an adaptation of the project that encourages student collaboration, additional goals include students learning to exchange ideas and work together collegially. While extensive assessment of learning outcomes associated with these projects has not been done, we do have anecdotal evidence that the goals for these projects have been met. It is worth noting that professors using the project at different schools and in several different classes have had similar experiences suggesting that these goals have been met.

Using this project, every student engages in self directed exploration and discovery of course topics, helping to develop those habits of mind necessary for true inquiry to take place. Even when modeling examples presented in class or in the homework problems, students are asking their own questions because they have thought to ask it about their individual gadget. They sometimes use tools other than those seen in class to answer those questions. One such student wrote a computer program that took in a multiplication table for a group and computationally determined properties of the group including all of the subgroups. In the linear algebra course, which primarily uses $\mathbb{R}$ as the scalar field, many students diagonalized their gadgets even when they had complex valued eigenvalues and corresponding eigenvectors.

Before the end of the term, many students ask questions about their object outside of what they see in class or in homework. In abstract algebra, one student developed a diagram to represent the group multiplication on the generators of her group, similar to the diagram used for the quaternions. After the presentations, the same student commented with surprise at the diversity of topics her fellow students had investigated, having expected them to investigate questions more or less the same as her own.

Along the way many students also recognize that their gadget fits into a larger class of objects and answer questions about that larger class. Upon observing certain patterns in their matrices, many students investigated outside sources to discover that their matrix was, for instance, a circulant matrix or a Hankel matrix. In abstract algebra, a student developed a procedure to generate a graph from a group and attempted to use graph theory to determine the number of groups of order 16. Even after the course moved to studying rings and the professor was no longer using her group as an example in class, one student decided to continue to apply the concepts and definitions from class to discover that his abelian group had a ring structure and proceeded to prove that any finite abelian group had a ring structure.

Using the project, students have even anticipated upcoming results in class. One such student found the center of their group, used this subgroup to define an equivalence relation and proved that the cosets of the equivalence relation also had a group structure long before the abstract algebra course covered factor groups.

The project seems to be an easy first step for students towards doing mathematical research. In both the linear algebra classes and the abstract algebra classes, some students continued to investigate their object after the term was over. In these cases, the grade was irrelevant, and finding out more about matrices or groups was the pressing desire. For example, one student generalized his methods to other groups and continued investigating until he had general results which he summarized in a google document and shared with the instructor weeks after the course ended. Another student, after being given a circulant matrix to investigate in the course, went on to do a two-semester senior independent study project that examined the class of circulant matrices.

Students have also been successful in undergraduate mathematical research as a result of the project. After the first implementation in an abstract algebra course, a student excitedly came back from a summer REU having used similar methods to those developed in her research journal to contribute to the research that summer. Although the REU was asking geometric questions, this student, using tools she learned while studying her group, was able to identify a group structure and the REU investigations had an algebraic component after her contribution. She particularly wanted the instructor to share with her current class the importance of doing the research journal.

Similarly, in a linear algebra course, the randomly generated matrices within a given class may have properties that go beyond those devised by the instructor. In those cases, students have the opportunity to observe and discover behavior that goes beyond what the instructor knows. For example, one student began to observe the behavior of powers of their magic matrix. After the course was over, the instructor was able to hire this student as a sophomore research student. They met once a week for a semester to examine further what happens when taking powers of magic matrices and together they addressed the question of when will the powers of a magic matrix be magic. The instructor and student were able to analyze the situation for 3×3 and 4×4 matrices and obtain some partial general results. This collaboration between student and instructor, sparked by the project, eventually led to a published paper [5]. The examples in the above three paragraphs show how this creative, open-ended and individualized assignment can help to instill in a student the necessary curiosity and persistence required in mathematical research.

Students do seem to take ownership of their individualized gadget. Many students use creative descriptions in their projects. For example, one student wrote "A Tale of Two Matrices", another "Group! (There it is)", a third wrote a "Dear Function" letter, and a fourth coined the label "funky-fresh" matrices for a class of matrices. Throughout the course, students talk about "their" matrices, "their" group or "their" function. In the abstract algebra course, students seemed to take pride in their research journals, requesting their return from the professor to show to visiting family as well as mentioning them in summaries of their accomplishments during job interviews and when requesting letters of recommendation. Some students even continue investigating their gadget after their course ends, a result covered in more detail above.

The design of the project encourages students to engage continuously with the course material. In addition, the project gives them an incentive to engage more deeply with the material and integrate course concepts into a more holistic understanding.

In the linear algebra course, all students find the standard properties of their own matrices and, for the most part, indicate that they have integrated various concepts in the course through their discussion of the particular properties of their matrices. A final exam question asks students to explain all the information that can be discovered from the reduced row echelon form of the matrix, something they have already done in their project. A number of former students commented that the project allowed them to see linear algebra in a more holistic manner.

In the abstract algebra course, students come to office hours seeking a deeper understanding of course concepts covered weeks earlier in order to apply this information to their group. Many students are willing to engage in tedious calculations in their research journal in order to understand their group more fully, which in turn helps reinforce many of the concepts of the class, *e.g.* many students examined their group as a permutation group via Cayley's Theorem and searched for automorphisms of their group by hand. Some students also spend time investigating questions with an obvious answer before finally realizing why it is indeed 'obvious', indicating that the research journals are helping students master the course content in a deeper way, *e.g.* a few students 'proved' that for their group the center was a normal subgroup before realizing why the result was true in general. One student even commented that she came to class to learn new things she could apply to her group. The examples in this and the previous paragraph indicate that the projects give students incentive to deepen their understanding of course content.

The project provides students an opportunity to develop their communication skills. In the linear algebra course, this assignment demands written work that not only contains mathematics, but explains that mathematics in a coherent way. It requires students to present written mathematics, integrating equations and prose, in a way which might be new to them. In the abstract algebra course, the presentations have been handled in two different ways. In the first implementation, students were encouraged to be creative and even artistic with their presentation[2]. In a subsequent implementation, students were asked to articulate any major results as well as share something they found interesting or intriguing in their research. Although the presentations in the first instance were more creative and fun, the presentations for the second instance were more substantial mathematically and modeled a mini-conference for the class where interesting insights were shared among colleagues. Students responded favorably, were engaged, and asked questions at the end of each 'talk'.

The project can also provide students an entry into professional collaboration. In the abstract algebra course, several students observed a classmate's investigations and attempted a similar investigation on their own group. Other students commented on similarities between their group and a classmate's group, even when their groups were not isomorphic. To capitalize on this, the instructor chose a class day close to the end of the term as a 'collaboration day', where every student brought their research journal and were told to discuss what they were doing with their classmates. Without any additional guidance, students began showing each other their research journals and discussing their discoveries. Students with the same group discovered each other and compared notes. In the presentations at the end of the term, students gave credit to their classmates for results or inspiration.

[2]Presentations included one rap, a representation of the cosets of a normal subgroup constructed from building blocks of a game, as well as a three-dimensional subgroup lattice with clay and sticks.

7 DISCUSSION

We have presented a project idea that is designed to encourage students to engage in individual inquiry while enhancing their understanding of course material. This project is highly adaptable, as evidenced by the variety of experiences of the authors, and offers numerous benefits. As we all seek continually to improve our teaching, we comment here briefly on some ways the project may be enhanced or changed to fit the needs of different instructors.

The duration of the project could be very brief, so that it is used as a comprehensive assignment at the end of a course, or the project can be ongoing throughout the semester. The instructor could change the expectations of the project, so that the scope of the project is very narrow, or it could be left very open-ended and broad. Students might work alone or they may be asked to work in teams. Instructors can choose the gadgets and the implementation that work best for their course and their situation. Whatever you choose, students should have the opportunity to formulate questions, explore, make conjectures, answer some of their own questions, and prove some of their own conjectures.

When the project concludes with oral presentations by the students, one could turn those presentations into a true mini-conference where students give 8-10 minute talks and a wider audience is invited. Given the rise of social media, and the apparent willingness of our students to use it, one could start a course blog, and have students post summaries of their results to it. This could lead to the same benefits as the collaboration day, but in “real time”, and it might encourage students to be talking about course concepts through the entire duration of the project.

Another possible enhancement of the project may come from the use of technology. Computational power is increasing, as is the ability for visualization using computers. To what degree new computational tools could be used to enhance a project like this remains unclear to the authors, but it is clear that the potential to employ them exists and is waiting to be exploited. While students in the linear algebra course do use technology to explore their matrices, there may be other ways in which that technology or others could be used.

In the end, the authors agree this project is a good and effective tool for exposing mid- and upper-division students to aspects of the mathematical enterprise that they might not otherwise see in a traditional course. Through such a project, students conduct a prolonged self-directed study, they work in a community engaged in similar endeavors, and they present their findings, all of which to some degree mimic the working life of a mathematician.

ACKNOWLEDGEMENTS

This paper started off as two distinct papers. We are thankful that the editors brought the four of us together to share our experiences using similar projects in our classes. The result is a stronger paper with more unified themes. Several reviewers also gave some valuable feedback, which was greatly appreciated. The first author would like to thank Mary Campbell of Linn Benton Community College for helpful feedback on how to improve the project, and for continued interest in the project. The first and fourth authors would like to thank Tevian Dray, Tom Dick, and Barbara Edwards of Oregon State University for their guidance throughout this process.

REFERENCES

[1] Open Source, *GAP (Groups, Algorithms, Programming)*, URL: http://www.gap-system.org, 2015.

[2] Dorff, Michael, *Nonacademic careers, internships, and undergraduate research*, Involve 7, No. 3, pp. 303-313.

[3] Gallian, J. A., *Contemporary Abstract Algebra*, 7th edition, Cengage Learning, Belmont, CA 2009.

[4] Coates, L., A. Fraser, S. Gray, and P. Pierce, *Developing Research Skills Across the Undergraduate Curriculum*, a book chapter appear-

ing in *Enhancing and Expanding Undergraduate Research: A Systems Approach*, 2015, pp. 145-168.

[5] Edwards, B. and J. Hartman, *Powers of Magic Matrices*, The Mathematical Gazette **95**, No. 533, 2011, pp. 284-292.

[6] Harward, D., *Engaged Learning and the Core Purposes of Liberal Education*, Liberal Education, Association of American Colleges and Universities, 2007.

[7] Minstrell, J and E. van Zee, ed., *Inquiring into Inquiry Learning and Teaching in Science*, American Association for the Advancement of Science, Washington, DC 2010.

[8] Munkres, J. R., *Topology: A First Course*, Prentice-Hall, Inc. Englewood Cliffs, NJ 1975.

[9] Seymour, E., A. Hunter, S. Laursen and T. Deantoni, Establishing the benefits of research experiences for undergraduates in the sciences: First findings from a three-year study. Science Education, **88**, 2004, pp. 493 534.

[10] Steen, L.A., and Seebach, J.A., *Counterexamples in Topology*, Dover Publications, New York 1995.

[11] College Learning for the New Global Century, National leadership Council for Liberal Education & Americas Promise. Association of American Colleges and Universities 2007.

[12] Hart Research Associates, *It Takes More Than a Major: Employer Priorities for College Learning and Student Success*, 2013. https://www.aacu.org/leap/presidentstrust/compact/2013SurveySummary, Accessed 15 April 2015.

[13] "Mathematics research by undergraduates: costs and benefits to faculty and the institution," report of the Subcommittee on Undergraduate Research, accepted by the MAA Committee on the Undergraduate Program in Mathematics, August 11,

2006. http://www.maa.org/sites/default/files/pdf/CUPM/CUPM-UG-research.pdf, Accessed 25 October 2015.

APPENDIX A: ASSIGNED GADGETS

Linear Algebra

In the linear algebra course students have been assigned anywhere from two to five matrices to explore. This has varied from year-to-year based upon the instructor, other elements in the course, and the scope of the project as part of the course.

While most times matrices are generated using a random number generator, they have often been generated using particular patterns. For example, magic matrices and circulant matrices seem to work quite well because there are interesting "patterns" in those matrices and exploration of eigenvalues and eigenvectors can lead to interesting properties. Other matrices have been used with some success, such as Hankel matrices and Toeplitz matrices. Others that have been tried without as much success are $0-1$ matrices and matrices in which all entries are generated randomly. One set of non-random matrices that have been used for selected students are Pascal matrices that come in three types: lower triangular, upper triangular, and symmetric.

The random generation of matrices has been done primarily with Maple and the code for a 4x4 magic matrices is given below.

Maple Code for 4x4 Magic Matrices

```
randomize(): #Seed the random number generator
with(LinearAlgebra): #Bring in the LinearAlgebra package in Maple
#Define a basis for 4x4 magic matrices with magic sum 0
m1:=<<-1,0,0,1> | <2,0,-1,-1> | <-1,0,1,0> | <0,0,0,0>>:
m2:=<<0,1,-1,0> | <1,-1,0,0> | <-1,0,1,0> | <0,0,0,0>>:
m3:=<<0,0,0,0> | <2,-1,-1,0> | <-2,1,1,0> | <0,0,0,0>>:
m4:=<<1,0,-1,0> | <0,-1,1,0> | <0,0,0,0> | <-1,1,0,0>>:
m5:=<<1,0,-1,0> | <-1,0,1,0> | <1,0,-1,0> | <-1,0,1,0>>:
m6:=<<0,0,0,0> | <1,0,0,-1> | <-1,0,0,1> | <0,0,0,0>>:
m7:=<<0,0,0,0> | <0,0,1,-1> | <1,0,-1,0> | <-1,0,0,1>>:
#Define the magic matrix that will allow any magic sum
m :=<<1,1,1,1> | <1,1,1,1> | <1,1,1,1> | <1,1,1,1>>:
```

```
#Defines the range of coefficients for the linear combination of the matrices above
ran := rand(-9..9):
N:=4: #Defines the number of magic matrices desired
#Loop to generate the desired number of matrices
for j from 1 to N do
ran()*m1+ran()*m2+ran()*m3+ran()*m4+ran()*m5+ran()*m6+ran()*m7+ran()*m;
od;
```

Abstract Algebra

There are 14 groups of order 16 and every one was assigned as an object to study except the cyclic group of order 16 and the abelian group with elements of order 2. The groups were given as multiplication tables in the second week of the term, right after the class introduced the multiplication table for a group. Below is a list of the groups assigned for student study, including the GAP small group library identification. (Note that the multiplication tables were generated using the computer algebra system GAP, *Groups, Algorithms, Programming* [1].)

Students were assigned duplicate groups 'disguised' by relabeling the elements. In both classes the instructor used the dihedral group of order 16 (without identifying it as such) as her group of study. In both classes students with less mathematical background were given abelian groups, which still gave rise to interesting questions and discoveries.

Abelian Groups:
1) $\mathbb{Z}_4 \times \mathbb{Z}_4$, SmallGroup(16,2)
2) $\mathbb{Z}_8 \times \mathbb{Z}_2$, SmallGroup(16,5)
3) $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, SmallGroup(16,10)

Non-abelian Groups:
4) $(\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes_3 \mathbb{Z}_2$, SmallGroup(16,3)
5) $\mathbb{Z}_4 \rtimes_3 \mathbb{Z}_4$, SmallGroup(16,4)
6) The modular group, M16, $\mathbb{Z}_8 \rtimes_5 \mathbb{Z}_2$, SmallGroup(16,6)
7) The semidihedral group, SD16, SmallGroup(16,8)

8) The generalized quaternion group, Q16, SmallGroup(16,9)
9) $D_8 \times \mathbb{Z}_2$, SmallGroup(16,11)
10) $Q_8 \times \mathbb{Z}_2$, SmallGroup(16,12)
11) The central product of D_8 and $\mathbb{Z}_4$, SmallGroup(16,13)

Instructors group: The dihedral group $D_{16} \cong \mathbb{Z}_8 \rtimes_{-1} \mathbb{Z}_2$, SmallGroup(16,7)

Unassigned groups: $\mathbb{Z}_{16}$, SmallGroup(16,1) and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, SmallGroup(16,14).

Differential Calculus

There are a wealth of choices here. Generally, this course will feature a textbook section concerning "Curve Sketching", where a lot of the preceding course material is put together. The main thing is to avoid choosing a function for your students that appears as a worked example in this section. Most of the problems encountered by students doing this project in calculus arise from analyzing the second derivative, so an additional thing you may wish to consider when choosing your gadgets is how difficult the second derivative is to compute and analyze. (You would like a function that provides a challenge, but is not impossible to analyze.)

We have found the relatively simple function, $Instructor(x) = \frac{x-1}{x+1}$ serves quite well as the instructor's function. A surprisingly good function to assign to students is $JohnDoe(x) = e^{1/x}$. We have also found the function $f(x) = \frac{\sin x}{x}$ is very difficult for students.

Although the calculus projects had their merits, we believe that this general idea works better in mid-and upper-level courses in the major. In these post-calculus courses, there are richer examples that students can study in much greater depth and with greater independence.

Topology

Below are the topological spaces assigned to the students for their study. The spaces were assigned to the students about one third of the way through the term, when general topological spaces were introduced (right after the class' study of metric spaces was concluded). Following each is the number of the space in Steen and Seebach [10], if it can be found there, or a reference to another source for information. If it is a named space, that is noted.

1) Let $X = \{x \in \mathbb{Z}^+ | x \geq 2\}$. Consider the topology on X generated by the sets $U_n = \{x \in \mathbb{Z}^+ | x \text{ divides } n\}$,for $n \geq 2$. This is S & S # 57, the *Divisor Topology.*

2) Let P be the open upper half plane with the Euclidean topology τ, and L be the real axis. On the set $X = P \cup L$, a set is open if either it is in τ, or it is of the form $\{x\} \cup (D \cap U)$, where $x \in L$ and U is a Euclidean neighborhood of x in the plane. This is S & S # 78, the *Half-Disc Topology.*

3) On the set $[-1, 1]$, generate a topology from sets of the form $(a, 1]$ with $a < 0$ and $[-1, b)$ with $b > 0$. This is S & S # 53, the *Overlapping Interval Topology.*

4) Let X be the set of rational numbers, and m be any point in X. Define a topology on X by declaring open any set whose complement is either finite or includes m. This is S & S # 23, the *Countable Fort Space.*

5) Let X be the usual Euclidean plane. Let $\mathcal{S}$ consist of all open discs in the plane with all of the horizontal diameters, except the centers, excluded. Then $\mathcal{S}$ forms a subbasis for a topology on X. This is S & S # 76, the *Deleted Diameter Topology.*

6) Let $X = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq 1\}$, the unit square in the plane. Order M using the lexicographic order. Place the order topology on X. This is S & S # 48, the *Lexicographic Ordering on the Unit Square.*

7) Let X be the set of all prime ideals of integers; that is, X is the set of all ideals P in $\mathbb{Z}$ whose complement is multiplicatively closed. Define a topology on X by taking as a basis all sets $V_x = \{P \in X | x \notin P\}$, for all $x \in \mathbb{Z}^+$. This is S & S # 56, the *Prime Ideal Topology.*

8) Let $X = \{(x, \sin(\frac{1}{x})) | 0 < x \leq 1\}$, considered as a subset of the Euclidean plane with the induced topology. The *Topologist's Sine Curve* is the set $S^* = S \cup \{(0, 0)\}$. This is S & S # 116.

9) Let $K = \{\frac{1}{n} | n \in \mathbb{N}\}$, and define the *Comb Space* to be the set

$$C = (\{0\} \times [0,1]) \cup (K \times [0,1]) \cup ([0,1] \times \{0\})$$

considered as a subspace of $\mathbb{R}^2$. This is the *Comb Space*, described in Munkres [8].

Typical Presentation: The student "John Doe" would be assigned space # 9 as follows

Let $J = \{\frac{1}{n} | n \in \mathbb{N}\}$, and define *Doe Space* to be the set

$$D = (\{0\} \times [0,1]) \cup (J \times [0,1]) \cup ([0,1] \times \{0\})$$

considered as a subspace of $\mathbb{R}^2$.

Remark: For philosophical reasons, the instructor did not assign himself a space to study. However, in future offerings of the course, the instructor will assign himself one. The most appropriate choice might be the topologist's sine curve. It provides a great example of the distinction between connectedness and path-connectedness.

Differential Geometry

On the first day of class, each student was assigned a plane and a space curve of their own to study. Only the parametric formulas were given. Students were asked immediately to sketch the graphs of their plane curves, noting what domain they used, and discussing such things as continuity and injectivity. They were asked to think about whether their curves had "curvature", and if so, "how much" curvature they had. A similar exercise was performed with the space curves a little later in the term. They were tasked with using their curves as examples when studying the material from class, and in preparation for part of a report about their gadgets. (Much later in the term, the initial information was returned to the students so they could compare their initial observations with what they now knew about their curves.)

When surface theory was begun later in the term, students were tasked with forming a surface of revolution from their plane curves, and the tangent surface from their space curves, though the instructor suggested they should experiment with other constructions.

Plane Curves	Space Curves
$\alpha(t) = (t^2 - 3, 1 - 3t)$	$\beta(t) = (\cos t. \sin t, t)$
$\alpha(t) = (\sin t, \sin 2t)$	$\beta(t) = (t \cos t, t \sin t, t)$
$\alpha(t) = (e^t - 1, e^{2t})$	$\beta(t) = (2t \cos t, 2t \sin t, t^2)$
$\alpha(t) = (e^t \cos t, e^t \sin t)$	$\beta(t) = (t, \sin t, e^t)$
$\alpha(t) = (-\cos t, 2 \sin t)$	$\beta(t) = (t, \sin t, t)$
$\alpha(t) = (\sin t, \frac{1}{2} \cos 2t - \frac{1}{2}))$	$\beta(t) = (t/2 + \sqrt{t^2 + 1}/2, 1/(2t + 2\sqrt{t^2 + 1}), \ln(t + \sqrt{t^2 + 1})/2))$
$\alpha(t) = (1 + \sqrt{t}, t^2)$	$\beta(t) = ((1 - \cos t) \cos t, (1 - \cos t) \sin t, \sin t)$
$\alpha(t) = (\cos t, t^2)$	$\beta(t) = (t, e^t, \sqrt{t})$

Remark: Appropriate choices for instructor assignment would be some parametrization of a parabola in the plane, and the helix in space.

APPENDIX B: LINEAR ALGEBRA PROJECT GUIDE

Linear Algebra 2013
Matrix Project

Student Name

Due Date: Tuesday, November 26, 2013 at 4:00 pm (the start of your Thanksgiving break)

Directions: Below are two matrices for you to get to know. You will be writing a paper that investigates the properties of each of these matrices. Basically, your charge is to tell me everything that you can about the matrices. This maybe one of your first experiences with research, because the question here is very open-ended. Of course, there are certain specific things that I will be looking for in your paper. I will expect you to investigate the following:

- Find some basic properties of the matrix: determinant, rank, row space, etc.
- Find any eigenvalues and eigenvectors of the matrix. If the eigenvalues are complex(imaginary) numbers, you do not need to investigate the eigenvectors.
- Examine any interesting arithmetic properties of the matrix. You can look at the numbers within the matrix to see if they have interesting properties. You can also look to see what happens when this matrix interacts with other matrices.
- Ask whether this matrix fits into some broader class of matrices (e.g. nilpotent, orthogonal, symmetric, etc.) If so, find the properties of that class. Determine whether the class forms a subspace of M_{nn}, the class of all $n \times n$ matrices. If the class forms a subspace, you should prove that it is a subspace. If it is not a subspace, a counterexample to show why it fails to be a subspace would be helpful.

A partial example of the kind of work I would like to see will be sent to you via email. Your paper should be written as if it were going to be read by your fellow students. Thus you should assume they have the basic knowledge from this course, but nothing beyond. One goal of this assignment is to show that you understand the concepts that have been covered in this course. A second goal is for you to begin to explore properties and structures that you may not have encountered before and to begin to ask your own questions about what you are finding. This is a creative exercise and you may do anything (legal) to these matrices that you desire. Use your imagination.

Resources: You should feel free to use Maple or other tools to do computations for you. You can use textbook or online resources to help you learn about your matrix, but please do your own research and do not share what you find with others in the class. Let them find materials on their own. Please cite any references that you use.

Presentation: Your final paper should be completed in LaTeX or Word. It should be double-spaced and be of a length that is sufficient to describe the properties of each matrix. (As a guideline, please do not write more than 10 pages about each matrix). You are free to determine the organization of your paper, but should express your ideas in clear and concise ways with flawless grammar and spelling. You may wish to divide your exploration into sections and use sub-headings to delineate your work. This is both a writing assignment and a mathematics assignment. You are writing about the mathematics that you are exploring. I am willing to help you with an initial draft, but will do that only if you schedule an appointment with me so that we can examine your draft together. The paper will be graded based on the primary trait analysis provided. I would be happy to answer any questions you might have concerning this assignment.

Your Personal Matrices

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 3 & 0 \\ 3 & 1 & -5 \end{bmatrix}.$$

APPENDIX C: ABSTRACT ALBEGRA PROJECT GUIDE

Project and Research Journal

What is mathematical research? I wish I could tell you in a nice, neat paragraph what it is like to find a question, try to see what is going on, fiddle with a problem, bang your head against a wall, have a flash of insight, apply someone else's results, conjecture your own, trudge through endless calculations, try to find the right question to ask... Instead, you get to be involved in your own research investigation of the '___' -group (fill the name of your choice)! In the end, while the results may not be new to the mathematical community, they will be new to you, and allow you to experience some of the joys and frustrations of the research process in mathematics.

In the first week of class I will give you your very own group to study/investigate/explore. Over 2/3 of the course will be spent studying group theory: examples, structure, properties, etc. As we introduce each definition, theorem, or property, your job is to use this to study your group. Does your group have that property? How does the theorem apply? What new thing can you say about your group? Of course, if you claim something about the group, you should prove that claim (or at least make a very thorough attempt). Throughout the term I will be doing the same with my group and using it as an example in class. Some properties and investigations will be easy and clear for your group, other things will take time, some may even be extremely difficult, and perhaps a few impossible.

You should buy a notebook with at least 50 pages (as fancy or bland as you wish) in which to record your research. The notebook is your record/journal of your investigation into this group. In it you should neatly record calculations, propositions and proofs pertaining to your group, and questions for further investigation that occur to you as you work. Some things to remember as you journal.

- Each entry should include a date.
- Your entries should be legible.

- Before each calculation/investigation there should be a heading or explanation of what you are attempting.
- After each calculation/investigation there should be a conclusion.
- At the end of each research session you should summarize your results.
- At the end of each research session you should summarize new questions and conjectures or theorems.

Think of your journal as a source that you would want to use for future reference several years later if you were doing research. A nice way to conclude each research session is to answer the following questions:

1. What new conclusion do I have about my group (or groups in general)?
2. What new question do I have about my group (or groups in general)?
3. What statements did I prove (or give an outline/sketch of a proof)?
4. What still needs to be proved?

These will be collected every other Monday and (hopefully) returned Wednesday, so I can check and see your progress. Although I might give some feedback, I usually won't and it is up to you to discover any errors as you continue to investigate your group. Again, this project is for you to be involved in your own research investigation, which includes finding your own errors as well as discovering new things.

After we have finished the group theory portion of the course and have moved on to rings, I will no longer collect journals, but you will be working on a summary of your research findings for your group to turn in along with your journal at the end of the term.

APPENDIX D: LINEAR ALGEBRA GRADING RUBRIC

Primary Trait Analysis Grading Scale

Each Matrix Investigation will be assessed for mathematical content and also for writing style.

The **Mathematics** portion of your grade will be determined based on the following:

1. Basic Characteristics of the Matrix (determinant, rank, row space, column space, etc.)
2. Eigenvalues and eigenvectors
3. Powers and Invertibility
4. "Interaction" with other matrices
5. Examination of a broader matrix class.
6. Creative explorations

In each of the areas above, you will receive a score from 0(missing) to 6(exemplary), which will then be multiplied by 2 for a total of **72** possible points.

The **Writing** portion of your grade will be determined based on the following:

1. Audience Appropriate
2. Clarity of Exposition
3. Mathematical Accuracy of Statements
4. Editing (presence of typos and minor errors)

In each of these areas, you will receive a score from 0(bad or missing) to 7(exemplary), for a total of **28** possible points.

BIOGRAPHICAL SKETCHES

S.A. Cook received his undergraduate education at Reed College, and his PhD at Oregon State University. His mathematical interests include differential geometry and mathematical relativity. His non-mathematical interests include hiking and haiku.

J. Hartman received an undergraduate degree from Manchester College in Indiana and a Ph.D. from Michigan State University along with an M.S. in Statistics. His mathematical interests include linear algebra and matrix theory, operator theory, and real and complex analysis.

P. Pierce did her undergraduate work at Amherst College and completed her Ph.D. at Syracuse University. Her mathematical interests include real analysis, linear algebra,and geometry. Other interests include swimming, traveling, and choral music.

N.S. Seaders received her undergraduate education at Seattle Pacific University, and her PhD at Oregon State University. Her mathematical interests include group theory, topology and graph theory. Her non-mathematical interests include singing and backpacking.