1982

Optimal Sequential Selection Based on Relative Ranks with Renewable Call Options

John S. Rose
University of Richmond

OPTIMAL SEQUENTIAL SELECTION BASED ON

RELATIVE RANKS WITH RENEWABLE CALL OPTIONS

1982-3

JOHN S. ROSE*

Sequential sampling problems may be affected significantly by the presence of sampling costs and the ability to recall historical observations. In the context of the classical secretary problem, we incorporate these two notions into the decision maker's action set, thereby creating a stopped decision process. Whenever a desirable applicant appears, we may consider purchasing an option to recall it subsequently. The problem is solved for the best-choice criterion, reduced or discounted by the option costs incurred.

KEY WORDS: Optimal choice; Secretary problem; Costly recall; Stopped decision process

1. INTRODUCTION AND MOTIVATION

Let us review briefly the context of what we shall refer to as the classical model of the secretary problem. Sampling sequentially without recall from a finite population of applicants, we must select exactly one member. Except for its size, no information about the population is available a priori. In order to evaluate the applicants

* Associate Professor, Robins School of Business, University of Richmond, Richmond, VA 23173. Research supported in part by a grant from the DuPont Company and by a fellowship from the Committee on Faculty Research, University of Richmond.

we assume only that our preferences would induce a complete ordering on any subset of the population. Thus, at any given stage of the process, the decision to select or to reject an applicant must be based solely on its relative rank among those already observed. We seek a stopping rule which maximizes the probability of selecting the best applicant. Suppose now that we observe an applicant which is preferable to the previously sampled, and rejected, applicants. Then, this applicant is certainly a candidate for the best. If the candidate is selected (rejected), then we risk (not) finding a better applicant among those yet unsampled.

In order to mitigate this risk, we propose extending the classical model to allow another decision alternative - purchasing an option to call the given candidate at the time of the next observation. If the applicant observed at the next stage is better, let the option expire and decide among selecting, rejecting, or purchasing an option to call the new candidate. Suppose the next applicant is worse. All call options are assumed to be renewable, so an identical situation is encountered: exercise the option (select the prior candidate), let the option expire (reject the candidate), or renew it. Thus, the option alternative provides a hedge against the immediate future. For the sake of parsimony, we shall refer to this alternative as the holding action.

Two models, differing only in their reward structures, are considered. In the additive cost model, the option price is a fixed positive amount. For each stage at which we decide to hold a candidate, that charge is deducted from the expected return. In the other model, the expected return is discounted by a positive proper fraction, raised to a power equal to the number of holding stages. We suggest that the discounted return model may be applicable in situations where, in return for being on call, the candidates demand an equity interest in our expected reward.

Detailed analysis of the classical model of the secretary problem and several variations thereof appear in Gilbert and Mosteller (1966). The concept of backward solicitation in this setting seems to have originated with Yang (1974) and has since been extended by others; see Corbin (1980) or Petruccelli (1981). According to Yang's model, we may attempt to recall for selection a previously rejected applicant, who in turn may refuse our offer with some known probability, in which case we are free to select another applicant. There is no cost associated with such solicitation. The essential operational difference between Yang's model and ours is that we must decide at the time of observation whether or not we shall later have the option to recall the candidate.

Even when recall is available, as in Yang's (1974) model, the secretary problem is still a pure stopping problem, provided that no penalty is imposed for unsuccessful solicitation. At each stage, the decision is either to continue or to (attempt to) stop, and, if the latter, then the best available applicant is solicited. If the solicitation is unsuccessful, then the process continues anyway. Because of the holding alternative, our models are more appropriately classified as stopped decision processes, albeit simple ones. Another example of a three-action secretary problem is provided by Rubin and Samuels(1977) who suppose that memory capacity is limited. Then, at each stage, they must decide among selecting, forgetting, or remembering the proffered applicant. However, memory is rent free, and solicitation of remembered applicants is not permitted.

A discounted secretary problem without recall was studied by Rasmussen and Pliska (1976) who assume that the final payoff is discounted for each stage of the sampling process, while we discount only for the holding stages. Also without invoking recall, Lorenzen (1981) investigates an infinite secretary problem with sampling cost which again is a monotone nondecreasing function of the duration of the process. However, the payoffs which he considers are fairly general functions of the selected candidate's rank, whereas our return is nil unless the best is obtained. Dhariyal and Dudewicz (1981) obtain some numerical results for the (finite) secretary problem with sampling costs. Unfortunately, they did not pursue the limiting behavior of their solution, which might have provided an interesting comparison to the asymptotic results obtained here.

Our notation and terminology are presented in Section 2. The additive cost model is formulated as a stopped decision process and its optimality equations are derived. The solution of the model is obtained in Section 3, and its asymptotic behavior is specified in Section 4. In Section 5, we compute the optimal procedure for the discounted return model, and in Section 6 the asymptotic solution for large sample size and small discount rate is obtained. Throughout the paper, we found it informative if not essential to compare our results to the classical model and its solution.

2. THE ADDITIVE COST MODEL

Let $(X_1,\text{---},X_n)$ be a random permutation of $\{1,\text{---},n\}$, so $P(X_1=i_1,\text{---},X_n=i_n) = 1/n!$ for all permutations $(i_1,\text{---},i_n)$. We interpret X_r as the absolute rank within the entire population of the applicant observed at stage r. After r stages, however, we know only the relative rankings

$$Z_{jr} = \text{card}\{i:X_i \leq X_j, i \leq r\},\ 1 \leq j \leq r \leq n. \tag{2.1}$$

Thus, when the j^{th} applicant appears, its relative rank within the j-sample is Z_{jj}; and, after an additional r-j observations, its relative ranking within the r-sample becomes Z_{jr}. For convenience, we shall write $Z_j=Z_{jj}$; obviously, $Z_j \leq Z_{jr}, j \leq r$. Note that Z_{jr} is uniformly distributed on $\{1,\text{---},r\}$ and that the observed history, $(Z_1,\text{---},Z_r)$, and the future, $(Z_{r+1},\text{---},Z_n)$, are independent.

There is a unique random index, J(r), such that $J(r) \leq r$ and $Z_{J(r),r} = 1, r=1,\text{---},n$. We shall refer to the applicant observed at stage J(r) as the <u>candidate</u> for stage r. Let A = {<u>pass</u>,<u>hold</u>,<u>stop</u>}, the set of actions available at any stage, and let $a_r \varepsilon A$ denote the action taken at stage r. The candidate for stage r is said to be <u>available</u> if $a_{J(r)} = \text{---} = a_{r-1} =$ <u>hold</u>. If J(r)=r, then $Z_r=1$ and the candidate is available; otherwise, the candidate was held for r-J(r) stages. Let $Y_r=1$ if the candidate for stage r is available; otherwise, let $Y_r=2$, say. Then, Y_r is a function of the history, $H_r=(Z_1,a_1,\text{---},Z_{r-1},a_{r-1},Z_r)$, through r stages. As we shall see, the process $Y=\{Y_r,r=1,\text{---},n\}$ is a sufficient statistic for the additive cost model.

The model is formulated as a stopped decision process, which for our purposes consists of five elements: state space, transition probabilities, terminal reward function, one-stage holding costs, and admissible actions. Consider these elements in the order given. Our choice of action at stage r depends in general on the prior history, H_r, of observed ranks and actions taken. However, as we verify below, the probabilities, reward, costs, and admissible actions depend on H_r only through Y_r. Consequently, it is fairly straightforward to show that Y_r is sufficient (cf. Theorem 6.0, p. 37, Hinderer (1970), making allowance for the optional stopping in our model) for choosing a_r. Thus, the state is given by $Y_r \varepsilon \{1,2\}$, $r=1,\text{---},n$.

Next, the transition probabilities from $H_r \times A$ to H_{r+1} are trivially independent of H_r. The initial distribution on $H_1=Z_1$ is $p_o(\{1\}) = P(Z_1=1) = 1$. For any realization h_r of H_r, with $a_j \neq \underline{\text{stop}}$, $j=1,\text{---},r-1$, we get $p_r((h_r,a_r),(h_r,a_r,z)) = P(Z_{r+1}=z) = 1/(r+1)$, provided $a_r \neq \underline{\text{stop}}$. It remains to specify the transition probabilities for the Y-process. Let $p_r(y,a,x) = P(Y_{r+1}=x|Y_r=y,a_r=a)$, so

$$p_o(\{1\}) = P(Y_1=1) = 1,$$

$$p_r(y,\underline{\text{pass}},x) = \begin{cases} 1/(r+1), x=1 \\ r/(r+1),\ x=2 \end{cases},\ y\varepsilon\{1,2\}, \tag{2.2}$$

$$p_r(1,\underline{\text{hold}},1) = 1,\ p_r(2,\underline{\text{hold}},x) = p_r(y,\underline{\text{pass}},x), r=1,\text{---},n-1.$$

Now, consider the reward achieved if we decide to stop the process at stage r by selecting the j^{th} applicant. Recall that our objective is to select the best applicant, so the terminal reward is defined to be $P(X_j=1|H_r,a_r = \underline{stop})$. Modifying (2.1) slightly, we have $X_j=Z_{jn}=Z_{jr} + \text{card}\{i:X_i<Z_{jr},r<i\leq n\}$. Thus, X_j depends on H_r only through Z_{jr}, and the terminal reward may now be written as $P(X_j=1|Z_{jr})$. Clearly, $X_j\geq Z_{jr}$, so $P(X_j=1|Z_{jr}\geq 2) = 0$; also $P(X_j=1|Z_{jr}=1) = r/n$, independent of j. It follows that we should consider selecting only the candidate for stage r, and that is feasible only if the candidate is available. Consequently, the terminal reward, $u(\cdot)$, depends only on the state $Y_r\varepsilon\{1,2\}$;

$$u_r(y) = \begin{cases} r/n \ , \ y=1 \\ 0 \ , \ y=2 \end{cases} . \tag{2.3}$$

Specification of the remaining elements of the stopped decision process is easy. The one-stage holding costs depend only on the action taken. If a_r = hold, a cost, $c>0$, is incurred. If a_r = pass, there is no cost. Finally, the set of admissible acts, A_r, also depends on H_r only through Y_r. For all $r<n$, pass ε A_r, while $A_n \equiv \{\underline{stop}\}$. There is no point in holding a noncandidate; according to (2.3) it shouldn't be selected, according to (2.2) it has no effect on the arrival of the next candidate, and it is costly. Consequently, we shall assume that hold ε A_r if and only if $Y_r=1$, $r<n$.

Next, we define the set of plans according to which the actions $\{a_r\}$ are chosen. Following conventional terminology, an (n-1)-tuple

$f = (f_1, \text{---}, f_{n-1})$ is called a (deterministic, Markov) policy if $f_r:\{1,2\} \to \{\text{pass, hold}\}$. Invoking our admissibility assumption, we shall also require that $f_r(2) = \text{pass}$. Let τ be any Markov time relative to the sequences $(Y_1, \text{---}, Y_r)$, $r=1,\text{---},n-1$, with $\tau \le n$. The interpretation is that, if $\tau = r$, then $a_r = \text{stop}$. From (2.3), we may also require that $\{\tau=r\} \subset \{Y_r=1\}$, $r<n$. Any pair (f,τ) is called a plan. Let D denote the set of all plans, and let $D^r = \{(f,\tau)\varepsilon D: \tau \ge r\}, r=1,\text{---},n$. For any plan, $(f,\tau)\varepsilon D$, the stochastic evolution and termination of the process $Y_1, a_1, Y_2, a_2, \text{---}$ are well defined and satisfy (2.2). Let $P^{f\tau}$ and $E^{f\tau}$ denote the associated probability and expectation; if no ambiguity arises, we shall drop the superscripts. It will be convenient to characterize plans by their support sets for the holding and stopping actions, respectively. We shall write ambiguously $(f,\tau) = (H,B)$, meaning that $\tau = \min\{\{r: r\varepsilon B, Y_r=1\} \cup \{n\}\}$ and $H = \{r: r \notin B, f_r(1) = \text{hold}\}$. Then, $B \subset \{1,\text{---},n\}$, $H \subset \{1,\text{---},n-1\}$, $B \cap H = \phi$, and $r \notin B \cup H$ implies that $f_r(1) = \text{pass}$.

We conclude this section with a description of the total returns and the optimality equations. For any plan $(f,\tau)=(B,H)\varepsilon D$, let

$$R_r(f,\tau) = \sum_{i=r}^{n} [u_i(Y_i)1_{\{\tau=i\}} - c1_H(r)1_{\{Y_i=1\}}1_{\{\tau>i\}}], \tag{2.4}$$

the return from the r^{th} stage on, $r=1,\text{---},n$. Let

$$U_r(y) = \max_{(f,\tau)\varepsilon D^r} E^{f,\tau}[R_r(f,\tau) \mid Y_r=y], \tag{2.5}$$

the optimal return functions, $y\varepsilon\{1,2\}$. Because D is finite, the max is achieved in (2.5). For r=n, $R_n(\cdot,\cdot) = u_n(Y_n)$ on D^n, so $U_n(1) = 1$

and $U_n(2) = 0$. For $r<n$, let $v_r = E[U_{r+1}(Y_{r+1})|a_r=\underline{pass}]$. From (2.2),

$$v_r = U_{r+1}(1)/(r+1) + rU_{r+1}(2)/(r+1). \quad (2.6)$$

At last, the optimality equations are

$$U_r(1) = \max\{r/n, U_{r+1}(1)-c, v_r\}, \quad (2.7)$$

and $U_r(2) = v_r$, $r=1,---,n-1$.

Using a standard backward induction argument of dynamic programming, we can derive (2.7) directly from (2.3)-(2.5). Otherwise, apply more general theory for stopped decision processes, as in Rieder (1975, Section 6). The interpretation of (2.7) is certainly clear enough. If the candidate for stage r is available, we may stop, thereby receiving the expected terminal reward r/n, we may hold, paying the holding cost c and moving to stage r+1 with the candidate available, or we may pass, in which case our expected return is v_r. According to the criterion of optimality, (f^*,τ^*) is optimal if and only if it chooses the maximizing action in (2.7). We may also write $(f^*,\tau^*) = (H^*,B^*)$ to denote the optimal plan. In order to resolve ties, we shall always prefer stopping over passing, which in turn is preferable to holding.

3. SOLUTION OF ADDITIVE COST MODEL

Suppose that $Y_r=1$ and that we are undecided between stop and hold. The expected return for stopping is $u_r(1)=r/n$. If we hold for, say, k periods and then stop, we expect to receive $u_{r+k}(1) = (r+k)/n$, for which we pay ck. Thus, if $c \geq 1/n$, there seems to be no incentive to hold, whereas if $c<1/n$, then we might as well hold until the last stage. This line of reasoning motivates the first two theorems of this section.

First, we introduce some notation for the classical model, with which most of our results will be compared. Frequent use is made of the sum

$$C(r) = 1/r+\cdots+1/(n-1), r=1,\cdots,n-1. \tag{3.1}$$

A very important role is played by r^*, the smallest integer r satisfying $C(r) \leq 1$. The optimal stopping time for the classical problem is $\tau' = \min\{\{r: r \geq r^*, Z_r=1\} \cup \{n\}\}$, where we adopt a convention of using primes (') to indicate results computed under the classical model. Its optimality equations are $U_r'(1) = \max\{v_r', r/n\}$, $r=1,\cdots,n-1$, where $v_r' = (r/n)C(r)$ for $r \geq r^*-1$ and $U_1'(1) = v_1'(1) = \cdots = v'_{r^*-1}$. See Gilbert and Mosteller (1966) or De Groot (1970, pp.325-331) for details.

THEOREM 3.1 If $c \geq 1/n$, then the optimal plan is equivalent to the classical procedure, i.e., $B^* = \{r^*,\cdots,n\}$ and $H^* = \phi$.

PROOF Use backward induction on r, and refer to the optimality equations, (2.7), as necessary. Trivially, $n \in B^*$. If $n \geq 2$, then $n-1 \geq r^*$ and $U_{n-1}(1) = \max\{(n-1)/n, 1-c, 1/n\} = (n-1)/n$. Hence, $n-1 \in B^*$. Assume now that $\{k,\cdots,n\} \subset B^*, k>r^*$, and consider the case $r=k-1$. Then, $v_{k-1} = v'_{k-1} \leq (k-1)/n$ and $U_k(1) = k/n$, so $U_{k-1}(n) = \max\{(k-1)/n, k/n-c\} = (k-1)/n$.

Therefore, $k-1\epsilon B^*$, too, and the theorem holds for $r\geq r^*$.

Next, let $r=r^*-1$. Then, $v_{r^*-1} = v'_{r^*-1} = (r^*-1)C(r^*-1)/n>(r^*-1)/n$, so $r^*-1\notin B^*$. Since $r^*\epsilon B^*$, $U_{r^*}(1) = r^*/n$, so $U_{r^*-1}(1) = \max\{v_{r^*-1}, r^*/n-c\}$. We have $(r^*-1)/n\geq r^*/n-c$ and $C(r^*-1)>1$, so $v_{r^*-1}>r^*/n-c$ and $r^*-1 \notin H^*$. Now, assume that $r \notin B^*, r \notin H^*$, for $k\leq r<r^*$. For $r=k-1$, we have $v_{k-1} = U_k(1)$, by assumption. Thus, $v_{k-1}>U_k(1)-c$. Furthermore, $v_{k-1} = --- = v_{r^*-1}>(r^*-1)/n>(k-1)/n$. We conclude that $k-1 \notin H^*$, $k-1 \notin B^*$. □

THEOREM 3.2 If $c<1/n$, then $B^*\equiv\{n\}$. Furthermore, $r-1\epsilon H^*$ implies $r\epsilon H^*, r<n-1$.

PROOF With $c<1/n$ and $n\geq 2$, $U_{n-1}(1) = 1-c$, so $n-1\notin B^*$. The induction hypothesis is that $r\notin B^*, r=k,---,n-1$. From (2.7), $U_k(1)\geq U_{k+1}(1)-c\geq --- \geq 1-(n-k)c$, so $U_{k-1}(1)\geq 1-(n-k+1)c>1-(n-k+1)/n = (k-1)/n$. Thus, $k-1\notin B^*$, and the first assertion is proved. Suppose that the second assertion is false for some $r<n-1$. Then, we would have $U_{r-1}(1)-c>v_{r-1} \geq v_r\geq U_{r+1}(1)-c$, or $U_{r+1}(1)<U_r(1)$. It is nearly trivial to show that $U_{r+1}(1)\geq U_r(1)$ for all r, so a contradiction is obtained. □

As a consequence of Theorem 3.2, the optimal plan may be simply characterized by the single integer, $s^* = \min H^*$. The first s^*-1 applicants are passed. The first candidate (if any) to appear thereafter is held, and the candidate for each subsequent stage is held, until stage n, at which time the candidate is selected. Fortunately, it is not difficult to solve for s^*.

THEOREM 3.3 With $c<1/n$, s^* is the largest integer r such that $C(r-1)\geq 1/cn$.

PROOF On $r\geq s^*$, we get

$$U_r(1) = U_{r+1}(1)-c = \cdots = 1-(n-r)c. \qquad (3.2)$$

Then, using (2.6), $v_{r-1} = U_r(1)/r+(r-1)v_r/r$, which may be evaluated recursively by applying (3.2) $n-r$ times. We obtain

$$v_{r-1} = (r-1)[A(r,n)-cB(r,n)], \qquad (3.3)$$

where $A(r,n) = \sum_{x=r}^{n} 1/(x(x-1))$ and $B(r,n) = \sum_{x=r}^{n-1} (n-x)/(x(x-1))$. Taking anti-differences yields $A(r,n) = 1/(r-1)-1/n$ and $B(r,n) = n/(r-1)-C(r-1)-1$. By definition, s^* is the largest integer r satisfying

$$v_{r-1} \geq U_r(1)-c. \qquad (3.4)$$

The theorem is established by substituting (3.2) and (3.3) into (3.4). □

As a corollary, we see that candidates accepted by the classical procedure will always be held in our framework.

COROLLARY 3.4 If $c<1/n$, then $s^*\leq r^*$. In particular, if $c<1/(nC(1))$, then $s^*=1$, and if $c\geq 1/(nC(r^*-1))$, then $s^*=r^*$.

PROOF From the theorem, $C(s^*-1)\geq 1/cn>1$, so $C(s^*-1)>C(r^*)$, which implies $C(s^*-1)\geq C(r^*-1)$, and that proves the first assertion. For the rest, let $r=1$ and $r=r^*$ in the theorem. □

The maximal expected return is $U_1(1) = v_1 = \cdots = v_{s^*-1}$, which is obtained from (3.3) with $r=s^*-1$. It is

$$U_1(1) = 1-c(n-s^*)+(s^*-1)(cC(s^*)-1/n). \tag{3.5}$$

Note that as $c \to 1/n, U_1(1) \to (r^*-1)C(r^*-1)/n = U_1'(1)$, the expected return from the classical model. In light of Theorem 3.1, such behavior is expected. Also, as $c \to 0$, then $U_1(1) \to 1$. Writing $c = \gamma/n$, for different values of n and γ, $0<\gamma<1$, we have computed s^* and $U_1(1)$ numerically; see Table 1.

Table 1. Numerical Solution of Additive Cost Model ($c<1/n$)

Population Size (n)	Fraction (γ)	Cost ($c=\gamma/n$)	Optimal Plan (s^*)	Maximal Return ($U_1(1)$)
5	.1	.02	1	.92
	.3	.06	1	.76
	.7	.14	2	.5317
	.9	.18	2	.455
25	.1	.004	1	.904
	.3	.012	2	.7173
	.7	.028	7	.4788
	.9	.036	9	.4087
100	.1	.001	1	.901
	.3	.003	5	.7121
	.7	.007	25	.4704
	.9	.009	34	.3993
1000	.1	.0001	1	.9001
	.3	.0003	37	.7109
	.7	.0007	241	.468
	.9	.0009	330	.3966
∞	.1		$s^* \simeq .00005n$	.9
	.3		$s^* \simeq .0357n$	.7107
	.7		$s^* \simeq .2397n$	.4678
	.9		$s^* \simeq .3292n$	.3963

4. ASYMPTOTIC SOLUTION, ADDITIVE COST MODEL

In order to maintain the condition $c<1/n$ as n gets large, let $c=\gamma/n$, $0<\gamma<1$.

THEOREM 4.1 The limit $\alpha = \lim_{n\to\infty} s^*/n$ exists, and $\alpha = \exp(-1/\gamma)$.

PROOF From Theorem 3.3, $C(s^*)<1/cn = 1/\lambda$. Using an integral approximation for the sum $C(\cdot)$, we have $\int_{s^*}^{n} dx/x<C(s^*)$, so $\log n/s^*<1/\gamma$, and $\liminf_{n\to\infty} s^*/n\geq\exp(-1/\gamma)$. Also, $C(s^*-1)\geq 1/\gamma$, and we obtain similarly $\limsup_{n\to\infty} s^*/n\leq\exp(-1/\gamma)$. □

The limiting return is now easily obtained from (3.5), with

$$\lim_{n\to\infty} U_1(1) = 1-\gamma+\gamma \exp(-1/\gamma). \tag{4.1}$$

So long as $\gamma<1$, (4.1) exceeds $1/e$, the limit in the classical model. Also, by combining Corollary (3.4), (3.5), and (4.1), we get $\lim_{n\to\infty} \lim_{\gamma\to 1} U_1(1) = \lim_{\gamma\to 1} \lim_{n\to\infty} U_1(1) = 1/e$. The limiting solutions computed for several values of γ appear in the last rows of Table 1.

5. DISCOUNTED RETURN MODEL

Now, instead of deducting holding costs from the terminal reward, as done in (2.4), we multiply the reward by a discount factor, $\beta, 0<\beta<1$, for each stage at which the candidate is held. For $(f,\tau) = (H,B)\varepsilon D$, the discounted return functions are

$$R_r(f,\tau) = \sum_{i=r}^{n} u_i(Y_i) 1_{\{\tau=i\}}\ \beta^{\sum_{j=r}^{i-1} 1_H(j) 1_{\{Y_j=1\}}}, \tag{5.1}$$

$r=1,\ldots,n$. The expressions for $U_r(y)$ and v_r are still given by (2.5) and (2.6) respectively, and $U_r(2) = v_r$, as before. The optimality equations become

$$U_r(1) = \max\{r/n, \beta U_{r+1}(1), v_r\}, \tag{5.2}$$

$r=1,---,n-1$, and these can be rigorously derived by adapting the usual inductive argument to fit the discounted model.

The first theorem and its lemmas provide a convenient and useful characterization of the optimal stopping rule. From Lemma 1, we find that no candidate should be selected if it is rejected in the classical model. Let $t^* = \min B^*$, the first stage at which an available candidate should be selected.

LEMMA 5.1 The optimal stopping time has the lower bound, $\tau^* \geq r^*$.

PROOF Obviously, $v_r \geq v'_r$ for all r. Suppose that $t^* < r^*$. From (5.2), the optimality of τ^*, and the definition of r^*, we get $t^*/n \geq v_{t^*} \geq v'_{t^*} = v'_{r^*-1} > (r^*-1)/n$, so $t^* > r^*-1$, a contradiction. □

LEMMA 5.2 The optimal stopping set is connected; $B^* = \{t^*, t^*+1, \ldots, n\}$.

PROOF Show that $r+1 \varepsilon B^*$ follows from $r \varepsilon B^*$. We rely on (5.2). If $r \varepsilon B^*$, then $r/n \geq \beta U_{r+1}(1) \geq \beta(r+1)/n$, so

$$\beta \leq r/(r+1). \tag{5.3}$$

Suppose now that $r+1 \notin B^*$. Because $n \varepsilon B^*$, there exists an integer k, $k \geq r+1$, such that $k \notin B^*$ and $k+1 \varepsilon B^*$. Furthermore, since $v_i \geq v_{i+1}$ for all i, we get $k/n > r/n = U_r(1) \geq v_r \geq v_k$. It follows that $k \varepsilon H^*$, the optimal holding set, so $U_k(1) = \beta U_{k+1}(1) = \beta(k+1)/n > k/n$. Thus, $\beta > k/(k+1)$, which violates (5.3). □

THEOREM 5.3 Let k be the unique positive integer which satisfies $(k-1)/k < \beta \leq k/(k+1)$. Then,

$$t^* = \begin{cases} r^* & , k \leq r^* \\ k & , r^* < k < n \\ n & , k \geq n \end{cases}, \tag{5.4}$$

and t^* is monotone nondecreasing in β.

PROOF Suppose that $t^*=t$, with $r^*<t<n$. Then, $t-1\notin B^*$, and $U_{t-1}(1)=\max\{\beta t/n, v_{t-1}\}>(t-1)/n$. From Lemma 5.2, $v_{t-1}=v'_{t-1}$, so $v_{t-1} = (t-1)C(t-1)/n$ and $C(t-1)\leq C(r^*)\leq 1$. Thus, $v_{t-1}\leq(t-1)/n$, so $U_{t-1}(1)=\beta t/n>(t-1)/n$; hence, $\beta>(t-1)/t$. That $\beta\leq t/(t+1)$ follows from (5.3). Now, suppose that $\beta\leq k/(k+1)$, $k\leq r^*$. If $t>r^*$, then the preceding argument gives a contradiction. From Lemma 5.1, we conclude that $t=r^*$. Finally, if $\beta>(n-1)/n$, then $U_{n-1}(1)=\max\{(n-1)/n,\beta,1/n\} = \beta$, so $n-1\in H^*$. Since B^* is connected, $t^*=n$. □

Now, we proceed to investigate the optimal holding set, H^*; it, too, is connected.

LEMMA 5.4 If $r-1\in H^*$, then $r\in H^*, r=2,\ldots,t^*-1$.

PROOF By hypothesis, $U_{r-1}(1) = \beta U_r(1)>v_{r-1}$. Because $r<t^*$, $U_r(1) = \max\{\beta U_{r+1}(1),v_r\}$. Suppose that $U_r(1) = v_r$, meaning $f_r^*(1)$ = <u>pass</u>. Then, $\beta v_r>v_{r-1}$, and $v_{r-1} = U_r(1)/r+(r-1)v_r/r = v_r$, from which we conclude $\beta>1$. □

As before, let $s^* = \min H^*$. The next theorem shows how to compute s^*, but we can now describe the optimal plan. Pass the first s^*-1 applicants and hold the first candidate, if any, observed at a stage between s^* and t^*-1, inclusive. Hold the candidate until stage t^*, when it will then be selected, or until it is superseded at some stage prior to t^* by yet another candidate, which should also be held. Beginning with stage t^*, stop and select the first available candidate.

THEOREM 5.5 For all $r \varepsilon H^*$,

$$(rt^*/n) \sum_{i=r+1}^{t^*-1} \beta^{t^*-i}/i(i-1)+(r/n)C(t^*-1) < \beta^{t^*-r}t^*/n. \tag{5.5}$$

PROOF If $r \varepsilon H^*$, then $\{r+1,r+2,---,t^*-1\} \subset H^*$, by the preceding lemma, so $U_r(1) = \beta U_{r+1}(1) = \beta^2 U_{r+2}(1) = \cdots = \beta^{t^*-r}U_{t^*}(1) = \beta^{t^*-r}t^*/n$, the right side of (5.5). Substituting for U.(1) in (2.6), we evaluate v_r recursively, obtaining $v_r = t^*\beta^{t^*-(r+1)}/(n(r+1))+rv_{r+1}/(r+1) = --- = (rt^*/n) \sum_{i=r+1}^{t^*-1} \beta^{t^*-i}/i(i-1)+rv_{t^*-1}/(t^*-1)$. Now, $v_{t^*-1} = v'_{t^*-1} = (t^*-1) \times C(t^*-1)/n$, so v_r equals the left side of (5.5). For $r \varepsilon H^*$, we must have $v_r < U_r(1)$, which is (5.5). □

Then, s* is simply the smallest positive integer, r, for which (5.5) holds. Of course, there is no assurance that $H^* \neq \phi$. If the terminal reward is too deeply discounted, we may never want to hold a candidate.

THEOREM 5.6 The optimal holding set is null if and only if

$$\beta \le \min\{(r^*-1)C(r^*-1)/r^*, r^*/(r^*+1)\}. \tag{5.6}$$

PROOF If $H^*=\phi$, then the optimal plan is the classical procedure, with $t^*=r^*$. By Theorem 5.3, $\beta \le r^*/(r^*+1)$. Furthermore, $U_{r^*-1}(1) = v'_{r^*-1} = (r^*-1)C(r^*-1)/n \ge \beta U_{r^*}(1) = \beta r^*/n$, and (5.6) holds. On the other hand, if (5.6) holds, then $t^*=r^*$, by Theorem 5.3. According to Lemma 5.4, if $H^* \neq \phi$, then $r^*-1 \varepsilon H^*$. However, substituting $r=r^*-1$ and $t^*=r^*$ into (5.5) contradicts (5.6). We conclude that $H^*=\phi$. □

The next proposition complements Lemma 5.1 and asserts that we should never reject a candidate which would be selected under the classical model.

PROPOSITION 5.7 If $H^* \neq \phi$, then $s^* < r^*$.

PROOF The proof is by induction on k, as given in Theorem 5.3. Suppose first that $k=r^*$, so $t^*=r^*$. If $H^* \neq \phi$, the previous theorem stipulates that $\beta > (r^*-1)C(r^*-1)/r^*$, so $r=r^*-1$ satisfies (5.5) and $s^* < r^*$. Now, assume that $r^*-1 \varepsilon H^*$ for $t^*=r^*,---,k$, and consider the case $t^*=k+1$. Rewriting (5.5) slightly, and letting $r=r^*-1$, the induction hypothesis becomes

$$\sum_{i=r+1}^{k} \beta^{r-i}/i(i-1)+\beta^{r-k}C(k)/k < 1/r, \tag{5.7}$$

$(k-1)/k < \beta \leq k/(k+1)$. Then (5.5) will hold with $t^*=k+1$ provided that

$$\lambda^{r-k-1}/(k+1) \leq \beta^{r-k}/k, \tag{5.8}$$

for $1 > \lambda > k/(k+1) \geq \beta$. The left (right) side of (5.8) is continuous and decreasing in λ (β), so it is sufficient to consider $\lambda=\beta=k/(k+1)$, which yields equality in (5.8). □

As β increases, there is less discount and we should be more inclined to hold the candidate for any given stage. Use the notation $H^*(\cdot)$ to denote explicitly the dependence of the optimal holding set on the discount factor.

PROPOSITION 5.8 If $\lambda > \beta$, then $H^*(\beta) \subset H^*(\lambda)$.

PROOF Let $r \varepsilon H^*(\beta)$ and essentially replicate the argument of the previous proposition to verify also that $r \varepsilon H^*(\lambda)$. □

Writing $\beta=(n-L)/n$, the numerical solution to the discounted return model is displayed in Table 2 for certain values of n and L. For comparative purposes, we also provide there the solution to the classical model.

Table 2. Numerical Solution of Discounted Return Model ($\beta=(n-L)/n$)

Pop'n Size(n)	Discount (L)	Factor (β)	Optimal Solution s^*	t^*	$U_1(1)$	Classical Solution r^*	$U_1(1)'$
5	.1	.98	1	5	.9224	3	.4333
	.5	.9	1	5	.6561		
	1	.8	2	4	.4793		
	2	.6	3	3	.4333		
	2.5	.5	3	3	.4333		
25	.1	.996	1	25	.9083	10	.3809
	.5	.98	3	25	.6299		
	1	.96	6	24	.4536		
	2	.92	10	12	.3817		
	2.5	.9	10	10	.3809		
100	.1	.999	1	100	.9057	38	.3710
	.5	.995	7	100	.6270		
	1	.99	21	99	.4502		
	2	.98	36	49	.3728		
	2.5	.975	38	39	.3710		
1000	.1	.9999	1	1000	.9049	369	.3682
	.5	.9995	64	1000	.6261		
	1	.999	201	999	.4493		
	2	.998	350	499	.3702		
	2.5	.9975	367	399	.3682		
∞	.1		.0001n	n	.9048	.3679n	.3679
	.5		.0634n	n	.6261		
	1		.1998n	n	.4492		
	2		.3493n	.5n	.3700		
	2.5		.3665n	.4n	.3679		

6. ASYMPTOTIC SOLUTION, DISCOUNTED RETURN

In view of Theorem 5.6, as n gets large, we must also allow β to approach unity; otherwise, the classical solution is obtained. Subsequent analysis is facilitated if we assume that $n(1-\beta)$ converge to a positive limit, say L, and for simplicity we write $\beta = \beta(n) = (n-L)/n$. The solution is surprisingly extremely sensitive to L, as we shall see. From Theorem 5.3, we get

$$\lim_{n\to\infty} t^*/n = \begin{cases} 1/e & , L>e \\ 1/L & , e\geq L\geq 1 \\ 1 & , L<1. \end{cases} \tag{6.1}$$

Henceforth, we shall require $L\leq e$, and let $\upsilon = \lim_{n\to\infty} t^*/n = \min\{1,1/L\}$.

We want to examine the limiting behavior of $s^* = s^*(n)$. Consider first the case $e\geq L\geq 1$. Let $\mu = \liminf_{n\to\infty} s^*/n$. Also, let $r = s^*$ and $t = t^*$ in (5.5), and take the limit. Dealing first with the summation, we get $(rt/n)\sum_{i=r+1}^{t-1}\beta^{t-i}/i(i-1)>(\beta rt/n)\int_{r+1}^{t}\beta^{t-x}dx/x^2 = (\beta rt/n^2)\times \int_{(r+1)/n}^{t/n}\beta^{t-ny}\,dy/y^2$, to which we may apply Fatou's lemma. First note that $\beta^t = [(1-L/n)^{n/L}]^{tL/n} \to 1/e$, while $\beta^{-ny} = [(1-L/n)^{n/L}]^{-yL} \to \exp(yL)$, as $n\to\infty$. Thus,

$$\liminf_{n\to\infty}(rt/n)\sum_{i=r+1}^{t-1}\beta^{t-i}/i(i-1) \geq (\mu\upsilon/e)\int_{\mu}^{\upsilon}\exp(yL)dy/y^2. \tag{6.2}$$

Integrating by parts, we get

$$\int_{\mu}^{\upsilon}\exp(yL)dy/y^2 = \exp(\mu L)/\mu - e/\upsilon + LI(\mu,\upsilon), \tag{6.3}$$

where $I(a,b) = \int_a^b \exp(y/b)dy/y$. For the second term on the left side of (5.5), we have $C(t-1) = \sum_{i=t-1}^{n-1} 1/i \simeq \int_t^n dx/x \simeq -\log\upsilon$, so

$$\liminf_{n\to\infty}(r/n)C(t-1) = -\mu\log\upsilon. \tag{6.4}$$

Finally, we have $\liminf_{n\to\infty}\beta^{-r} = \liminf_{n\to\infty}[(1-L/n)^{n/L}]^{-rL/n} = \exp(\mu/\upsilon)$, so

$$\liminf_{n\to\infty}\beta^{t-r}t/n = \upsilon\exp(\mu/\upsilon-1), \tag{6.5}$$

the limit of the right side of (5.5). Combining (5.5) with (6.2)-(6.5), we get $I(\mu,\upsilon)/e-\log\upsilon\leq 1$. Now, let $r = s^*-1$ in (5.5), thereby reversing the inequality. Let $\bar{\mu} = \limsup_{n\to\infty} s^*/n$ and modify the preceding argument appropriately. We obtain $I(\bar{\mu},\upsilon)/e-\log\upsilon\geq 1$. Since $I(\cdot,\upsilon)$ is monotone decreasing on $(0,\upsilon]$, it follows that $I(\bar{\mu},\upsilon)\leq I(\mu,\upsilon)$. Hence, we have proved

THEOREM 6.1 For $\beta = 1-L/n$, $1\leq L\leq e$, the limit $\mu = \lim_{n\to\infty} s^*/n$ exists and satisfies

$$(1/e) \int_{\mu}^{\upsilon} \exp(y/\upsilon)dy/y - \log \upsilon = 1 \tag{6.6}$$

In order to compute μ from (6.6), we may expand $I(\mu,\upsilon)$ as a power series, $I(\mu,\upsilon) = -\log(\mu/\upsilon)+ \sum_{k=1}^{\infty} [1-(\mu/\upsilon)^k]/(kk!)$, and include as many terms as necessary to achieve a desired level of accuracy. The last portion of Table 2 gives selected values to four decimal places.

Now, return to (5.5) and evaluate the limiting return. We easily obtain $\lim_{n\to\infty} U_1(1) = \lim_{n\to\infty} v_{s^*-1} = \upsilon \exp(-1+\mu/\upsilon)$, which we rewrite as

$$\lim_{n\to\infty} U_1(1) = \upsilon \exp(-L(\upsilon-\mu)), \tag{6.7}$$

and (6.7) holds for the case $L<1$ as well as $1\leq L\leq e$.

The corollary indicates that the classical procedure is a good approximation for the optimal plan when $\beta\leq 1-e/n$.

COROLLARY 6.2 If $\beta = 1-L/n$, $L\geq e$, then $\lim_{n\to\infty} s^*/n = \lim_{n\to\infty} U_1(1) = 1/e$.

PROOF Let $L=e$ in (6.6). Then, $\upsilon=1/e$, so $-\log \upsilon=1$; thus, $I(\mu,\upsilon)=0$. The integrand is positive, so $\mu=\upsilon$. Now, substitute $\upsilon=\mu=1/e$ in (6.7), and obtain $U_1(1)\simeq 1/e$, too. For $L>e$, just apply Proposition 5.8. □

There remains the case $L<1$. Analysis similar to that yielding Theorem 6.1 gives

$$\exp(-L)L\int_{\mu}^{1}\exp(yL)dy/y = 1, \quad (6.8)$$

where again $\mu = \lim_{n\to\infty} s^*/n$. The strategy of always holding the candidate is a good approximation to the optimal plan when $\beta \ge 1-L/n$ and L is nearly zero. The following corollary follows almost immediately from (6.7) and (6.8).

COROLLARY 6.3 For $\beta = 1-L/n$, $\lim_{L\to 0}\lim_{n\to\infty} s^*/n = 0$ and $\lim_{L\to 0}\lim_{n\to\infty} U_1(1) = 1$.

REFERENCES

CORBIN, R.M. (1980), "The Secretary Problem as a Model of Choice," Journal of Mathematical Psychology, 21, 1-29.

DHARIYAL, I.D., and Dudewicz, E.J. (1981), "Optimal Selection from a Finite Sequence with Sampling Cost," Journal of the American Statistical Association, 76, 952-959.

GILBERT, J.P., and Mosteller, F. (1966), "Recognizing the Maximum of a Sequence," Journal of the American Statistical Association, 61, 35-73.

DE GROOT, M. (1970), Optimal Statistical Decisions, New York: McGraw-Hill.

HINDERER, K. (1970), Foundations of Non-stationary Dynamic Programming with Discrete Time Parameter, Berlin: Springer-Verlag.

LORENZEN, T.J. (1981), "Optimal Stopping with Sampling Cost: The Secretary Problem," Annals of Probability, 9, 167-172.

PETRUCCELLI, J.D. (1981), "Best-choice Problems Involving Uncertainty of Selection and Recall of Observations," Journal of Applied Probability, 18, 415-425.

RASMUSSEN, W.T., and Pliska, S.R. (1976), "Choosing the Maximum from a Sequence with a Discount Function," Applied Mathematics and Optimization, 2, 279-289.

RIEDER, U. (1975), "On Stopped Decision Processes with Discrete Time Parameter," Stochastic Processes and their Applications, 3, 365-383.

RUBIN, H., and Samuels, S.M. (1977), "The Finite-memory Secretary Problem," Annals of Probability, 5, 627-635.

YANG, M.C.K. (1974), "Recognizing the Maximum of a Random Sequence Based on Relative Rank with Backward Solicitation," Journal of Applied Probability, 11, 504-512.