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The model analyzed in this paper extends the classical secretary problem to the situation in which the decision maker wants to select the two best objects. Exactly two selections are permitted, and a selection strategy is defined by a pair of stopping times. Characterization of the optimal strategy is cumbersome, but useful asymptotic representations are obtained for long sequences.

KEY WORDS: Secretary problem, sequential decision process

1. INTRODUCTION

In the classical model of the secretary problem, we shall observe a sequence of n objects. The number, n, of objects is known. Although we may completely rank order any observed subsequence of objects, we possess no prior knowledge about the distribution of those qualities and attributes on which our preference ordering is based. At each stage of the sequence, we must decide, in real time, whether or not to select the object presented. Recall of a passed object is forbidden, and selection of any object terminates the process. No utility derives from any but the single best object - that ranked number 1 according to our preference ordering. Consequently, the objective is to maximize the probability of selecting the best object.

Comprehensive results for the classical model, and also for some interesting generalizations thereof, are given by Gilbert

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and Mosteller (1966). Variations of the classical model have been contributed by a host of researchers. Rasmussen (1975) invokes a payoff function that is more general than "all or nothing," and he and Robbins (1975) assume that n is random. Stewart (1978a) considers a statistical model, in which his state of information, about the true underlying distribution of the object population, is updated at each stage. Stewart (1978b) also considers a multicriterion version of the problem, and Albright (1976) investigates a Markov chain version.

In the present paper, we must select two objects from the sequence, and only the best pair yield any return. The model and notation are formalized in the next section. Section 3 considers the optimal timing of the second choice, given that one object has already been chosen. In section 4, we proceed to investigate when the first choice should be taken. Asymptotic results, for large n, are obtained in section 5 (Only after completing this work and submitting it for publication did this author become aware of the paper by Nikolaev (1977), who obtained identical asymptotic results via different methods.)

2. MODEL FORMULATION

Let $X_1, \ldots, X_n$ be a random permutation of $1,2,\ldots,n$; $P(X_1=i_1,\ldots, X_n=i_n) = 1/n!$, for all permutations $(i_1,\ldots,i_n)$. Then, X_r denotes the rank, among all n objects, of the observation made at stage r. At stage r, however, we know only the relative rank,

$$A_r = 1 + \text{card}\{i: i<r, X_i<X_r\}, \quad r = 1,\ldots,n,$$

where card$\emptyset$=0 (throughout the paper, we adopt the convention that set functions are zero on the null set). Let d denote the number of objects which must be selected; d=1,2. At any stage r, we must decide, based on only the observed relative ranks, $A_1,\ldots,A_r$, whether to pass or to select the currently proffered object. Of course, the decision also depends on n, r, and d. The objective is to find a selection strategy which maximizes the probability of selecting the best pair of objects.

Let τ_1 and τ_2 be stopping times relative to the subsequences $A_1, \ldots, A_r$, $r = 1,\ldots,n$, with $1 \leq \tau_1 < \tau_2 \leq n$. Interpret τ_i as the random stage at which the i^{th} selection is made, $i=1,2$. Then, τ_1^* and τ_2^* are optimal if $P(X_{\tau_1^*} + X_{\tau_2^*} = 3)$ is maximized over all such stopping times, τ_1, τ_2. There is no question about the existence of optimal rules, since the number of these stopping times is finite. Henceforth, we presume that we are following an optimal strategy; our job is to characterize (one such) optimal strategy.

Fortunately, the class of selection procedures may be simplified. The decision to be made at stage r depends on the history $A_1, \ldots, A_r$ only through the conditional distribution of $X_r, \ldots, X_n$, given that history. Now, $A_1, \ldots, A_{r-1}$ are independent of the sequence $X_r, \ldots, X_n$. The reason is that, no matter which objects will be viewed in the first r-1 stages, every (r-1)-tuple of their relative ranks, $(a_1, \ldots, a_{r-1})$, is equally likely. Hence, the relative ranks of the first r-1 objects provide no information about those objects which are yet to be observed. Thus, we may restrict our attention to stopping rules τ_1 and τ_2 that are determined solely by the stage index, r, and the relative rank, A_r, observed at that stage, $r = 1,\ldots,n$.

The optimal return functions are given by

$$u_r(a,d) = P(X_{\tau_1^*} + X_{\tau_2^*} = 3 \mid A_r=a, r, d), \tag{2.1}$$

the conditional probability of "winning," given that at stage r, with d selections yet to be made, we observe $A_r=a$, $r=1,\ldots,n$,

$a=1,\ldots,r$, $d=1,2$. Also, let

$$v_{r-1}(d) = Eu_r(A_r,d), \tag{2.2}$$

the expected probability of winning, after r-1 stages have been decided upon and there remain d objects to be selected. Then, $P(X_{\tau_1^*} + X_{\tau_2^*} = 3) = v_0(2)$, the probability of winning prior to the start of the process.

Certainly the following hold :

$$A_{\tau_1^*}=1 \text{ on } \{\tau_1^*<n-1\}, \tag{2.3}$$

$$A_{\tau_2^*}=1 \text{ or } 2 \text{ on } \{\tau_2^*<n\}, \tag{2.4}$$

$$\tau_2^* \leq \min\{r\colon\ r>\tau_1^*,\ A_r=1\}. \tag{2.5}$$

If any of (2.3)-(2.5) were violated, then we would have selected an object worse than one already passed, so we couldn't possibly win.

3. CHOOSING THE SECOND OBJECT (d=1)

Proceed with the usual backward induction argument on the stage index r, $1<r\leq n$. According to (2.5), if $A_r=1$, then it must be selected immediately, and (2.1) and (2.3) give

$$\begin{aligned} u_r(1,1) &= P(X_{r+1}>2,\ldots,X_n>2) \\ &= r(r-1)/n(n-1), \end{aligned} \tag{3.1}$$

where the independence of $A_1,\ldots, A_r$ from $X_{r+1},\ldots,X_n$ is again manifest. Suppose $A_r=2$. If it is selected, then again $u_r(2,1)= r(r-1)/n(n-1)$. If $A_r=2$ is passed, then $u_r(2,1) = v_r(1)$. Hence, the optimality equations are

$$u_r(2,1) = \max\{r(r-1)/n(n-1),\ v_r(1)\}, \quad r\geq 2, \tag{3.2}$$

with $v_n(\cdot)\equiv 0$. Finally, if $A_r=a>2$, then (2.4) implies that

$$u_r(a,1) = v_r(1),\ a>2. \tag{3.3}$$

To evaluate $v_r(1)$, note that A_{r+1} is uniformly distributed over $\{1,\ldots,r+1\}$. Combining (2.2) and (3.1)-(3.3) gives

$$v_r(1) = \frac{r}{n(n-1)} + \frac{1}{r+1}\, u_{r+1}(2,1) + \frac{r-1}{r+1}\, v_{r+1}(1). \tag{3.4}$$

The following is analogous to Lemma 1 of DeGroot (1970, p.328).

LEMMA 1 If $u_r(2,1) = \frac{r(r-1)}{n(n-1)}$, then $u_{r+1}(2,1) = \frac{r(r+1)}{n(n-1)}$, $r=2,\ldots,n-1$.

PROOF If the lemma is false for some r, (3.2) yields $u_{r+1}(2,1) = v_{r+1}(1)$. Solving (3.4) gives $v_{r+1}(1) = \frac{r+1}{r}\left[v_r(1) - \frac{r}{n(n-1)}\right]$. By hypothesis, $v_r(1) \leq u_r(2,1) = \frac{r(r-1)}{n(n-1)}$. Thus, $u_{r+1}(2,1) \leq \frac{r+1}{r}\left[\frac{r(r-1)}{n(n-1)} - \frac{r}{n(n-1)}\right] < \frac{r+1}{r} \cdot \frac{r(r-1)}{n(n-1)} < \frac{r(r+1)}{n(n-1)}$, which is impossible, according to (3.2).

Lemma 1 is important for characterizing τ_2^*. It says that there is some smallest stage index, r^*, such that, if $A_r=2$ for $r \geq r^*$, then A_r should be chosen. To complete the characterization, we need the analog to DeGroot's Lemma 2 (1970, pp. 328-29).

LEMMA 2 If $u_r(2,1) = r(r-1)/n(n-1)$ for some $r=2,\ldots,n$, then

$$v_{r-1}(1) = \frac{2(r-1)(r-2)}{n(n-1)}\left[\frac{1}{r-2} + \ldots + \frac{1}{n-2}\right]. \tag{3.5}$$

PROOF Use backward induction on r. If r=n, then $v_{n-1}(1) = 1/n + 1/n\, u_n(2,1) = 2/n$, by (3.4) and the hypothesis respectively. Also, $2/n$ = RHS of (3.5). Assume now that Lemma 2 holds for $r=k+1$, and consider $r=k$. The hypothesis of Lemma 2 gives $u_k(2,1) = k(k-1)/n(n-1)$, so Lemma 1 yields $u_{k+1}(2,1) = k(k+1)/n(n-1)$. Therefore, we may invoke the induction hypothesis to compute $v_k(1)$.

From (3.4) and the preceding,

$$v_{k-1}(1) = \frac{k-1}{n(n-1)} + \frac{1}{k} \cdot \frac{k(k-1)}{n(n-1)} + \frac{k-2}{k}\left\{\frac{2k(k-1)}{n(n-1)}\left(\frac{1}{k-1} + \ldots + \frac{1}{n-2}\right)\right\}$$

$$= \frac{2(k-1)(k-2)}{n(n-1)}\left(\frac{1}{k-2} + \ldots + \frac{1}{n-2}\right).$$

Now, we may actually compute r^*. From (3.2) and (3.5), r^* is the smallest r such that $v_r(1) = \frac{2r(r-1)}{n(n-1)}\left(\frac{1}{r-1} + \ldots + \frac{1}{n-2}\right) \leq \frac{r(r-1)}{n(n-1)}$, or

$$\frac{1}{2} \geq \frac{1}{r-1} + \frac{1}{r} + \ldots + \frac{1}{n-2}. \tag{3.6}$$

For $n \geq 4$, r^* is well defined; and, for n=2 or 3, the problem is trivial anyway.

The optimal stopping time for obtaining the second object is now fully characterized. If τ_1 is any stopping time for which $A_{\tau_1}=1$, then

$$\tau_2^* = \min\left\{\{r:\ r>\tau_1, A_r=1\} \cup \{r:\ n>r\geq r^*;\ A_r=2\} \cup \{n\}\right\}. \tag{3.7}$$

After having selected one object, we stop with $A_r=1$. If no $A_r=1$ is encountered, then we select an $A_r=2$ after r^* stages are observed. Finally, we will be stuck with the last object in the sequence if neither $A_r=1$ nor, for $r\geq r^*$, $A_r=2$ is previously encountered.

The following two results will be useful in our selection of an initial object.

LEMMA 3 For $r=2,\ldots,r^*-1$,

$$v_r(1) = \frac{r(r^*-r-1)}{n(n-1)} + \frac{r}{(r^*-1)} v_{r^*-1}(1). \tag{3.8}$$

PROOF Let $2\leq r<r^*$ be given, and define $\tau = \min\{i:\ r<i\leq r^*-1,\ A_i=1\}$. Conditioning on τ, we have

$$v_r(1) = \sum_{i=r+1}^{r^*-1} u_i(1,1)P(\tau=i) + v_{r^*-1}(1)\ P(\tau=0). \tag{3.9}$$

From (3.1), $u_i(1,1) = i(i-1)/n(n-1)$. Also, $P(\tau=i) = P(A_{r+1}>1,\ldots,$ $A_{i-1}>1,\ A_i=1) = \frac{r}{r+1}\cdot\frac{r+1}{r+2}\cdot\ldots\cdot\frac{i-2}{i-1}\cdot\frac{1}{i} = \frac{r}{i(i-1)}$. Similarly, $P(\tau=0) = \frac{r}{r^*-1}$. Substitution into (3.9) yields (3.8).

LEMMA 4 The function $v_r(1)$ is strictly increasing on $r=2,\ldots,r^*-1$, achieves its maximum at $r=r^*-1$, and is strictly decreasing on $r=r^*,\ldots,n$.

PROOF For $r\geq r^*-1$, $v_r(1)$ is given in (3.5). Taking first differences yields $\Delta v_r(1) = v_{r+1}(1) - v_r(1) = \frac{2r}{n(n-1)}\left[2\left(\frac{1}{r} + \ldots + \frac{1}{n-2}\right) - 1\right]$. From (3.6), $\frac{1}{2}\geq\frac{1}{r} + \ldots + \frac{1}{n-2}$, so $\Delta v_r(1)\leq 0$. Furthermore, except for the trivial case when $n=4$ and $r^*=3$, strict inequality holds in (3.6). Consider now $r\leq r^*-1$. Combine (3.8) and (3.5) to obtain

$$\Delta v_{r-1}(1) = v_r(1) - v_{r-1}(1) =$$

$$= \frac{r^*-2r}{n(n-1)} + \frac{1}{r^*-1}\cdot\frac{2(r^*-1)(r^*-2)}{n(n-1)}\left(\frac{1}{r^*-2} + \ldots + \frac{1}{n-2}\right).$$ From (3.6), $\frac{1}{r^*-2} + \ldots + \frac{1}{n-2} > \frac{1}{2}$, so $\Delta v_{r-1}(1) > \frac{r^*-2r}{n(n-1)} + \frac{r^*-2}{n(n-1)} = \frac{2}{n(n-1)}(r^*-r-1)\geq 0$.

4. CHOOSING THE FIRST OBJECT (d=2)

From (2.3), we need to consider only those stages for which $A_r=1$. If $A_r=1$ is selected, when $d=2$, then the process continues just as if $A_r=1$ had been passed, with $d=1$, so $u_r(1,2) = v_r(1)$. If $A_r=1$ is passed, then $u_r(1,2) = v_r(2)$, so

$$u_r(1,2) = \max\{v_r(1),\ v_r(2)\}\ . \tag{4.1}$$

The next lemma focuses our attention upon those stages $r<r^*-1$.

LEMMA 5 If $r\geq r^*-1$, then $u_r(1,2) = v_r(1)$.

PROOF Suppose $u_r(1,2) = v_r(2) > v_r(1)$, and let τ denote the random stage index at which the first object is finally selected. Then, $r<\tau\leq n-1$, and $u_r(1,2) = Ev_\tau(1)$. By Lemma 4, $v_r(1) > v_\tau(1)$, so $v_r(1)\geq u_r(1,2)$, a contradiction.

Henceforth, we presume that $r<r^*-1$. Next, we prove the analog to Lemma 1 for the case d=2.

LEMMA 6 If $u_r(1,2) = v_r(1)$, then $u_{r+1}(1,2) = v_{r+1}(1)$.

PROOF If $u_{r+1}(1,2) = v_{r+1}(2) > v_{r+1}(1)$, then $v_r(2) = \frac{1}{r+1} u_{r+1}(1,2) + \frac{r}{r+1} v_{r+1}(2) = v_{r+1}(2) > v_{r+1}(1)$. By the hypothesis and (4.1), $v_r(1) = u_r(1,2) = \max\{v_r(1), v_r(2)\}$, so $v_r(1) \geq v_r(2) > v_{r+1}(1)$; but, this last inequality contradicts Lemma 4.

To compute $v_r(2)$, we consider the analog to Lemma 2.

LEMMA 7 If $u_r(1,2) = v_r(1)$ for some $r \leq r^*-1$, then

$$v_{r-1}(2) = (r-1)\sum_{i=r}^{n-1} \frac{v_i(1)}{i(i-1)}. \tag{4.2}$$

PROOF With some manipulative acrobatics, one can obtain (4.2) directly from (3.5) and (3.8). Consider instead the following argument. According to Lemma 6, we must select A_τ, where $\tau = \min\{i: i \geq r, A_i=1\}$. Then, $v_{r-1}(2) = E\, v_\tau(1) =$

$$\sum_{i=r}^{n-1} v_i(1)\, \frac{r-1}{r} \cdot \frac{r}{r+1} \cdot \ldots \cdot \frac{i-2}{i-1} \cdot \frac{1}{i} =$$

$$= (r-1) \sum_{i=r}^{n-1} v_i(1)/i(i-1).$$

The optimal stopping time, τ_1^*, for selection of the first object is computed in a manner similar to that of the preceding section for τ_2^*. Let s^* denote the smallest integer such that $v_s(1) \geq v_s(2)$, or

$$v_s(1) \geq s \sum_{i=s+1}^{n-1} v_i(1)/i(i-1). \tag{4.3}$$

That s^* exist and $s^* \leq r^*-1$ follow from Lemma 5. Then,

$$\tau_1^* = \min\left\{\{i: i \geq s^*, A_i=1\} \cup \{n-1\}\right\}$$

Numerical results are given in the table at the end of the paper, but our attempts to provide further characterization of s*, beyond (4.3), have been frustrating. However, when the number of objects is large, we obtain some interesting results.

5. ASYMPTOTIC RESULTS

Application of standard techniques - see DeGroot (1970, pp. 330-31) - to (3.6) and (3.5) yields

$$r^* \approx e^{-\frac{1}{2}} n \doteq .607n, \tag{5.1}$$

$$v_{r^*-1}(1) \approx e^{-1} \doteq .368. \tag{5.2}$$

Given that a first object has been selected prior to stage r*, we pass all $A_r=2$ until 60.7% of all the objects have been observed. During this interval, we wait for the occurrence of an $A_r=1$. Beginning with stage r*, however, we behave as if the first selection were best and we are willing to complement it with what we hope will be the next best object. Coincidentally, (5.2) is precisely the limiting probability of selecting the single best object in the classical model. In the present model, we wait longer (cf. $e^{-\frac{1}{2}}n$ to $e^{-1}n$) but we have, potentially, two objects to choose -- the best or next best.

The analysis for the initial selection is based on (4.2) and (4.3). Using (3.8) and (3.5) to evaluate $v_i(1)$ for $i<r^*-1$ and $i \geq r^*-1$, respectively, in (4.2), we get

$$v_{s^*-1}(2) = (s^*-1)/n(n-1)\left\{\left[r^* + 2(r^*-2)c(r^*)\right]\left[c(s^*) - c(r^*-1)\right] - r^* + s^* + 1 + 2\sum_{i=r^*-1}^{n-1} c(i)\right\}, \tag{5.3}$$

where $c(i) = 1/(i-1) + 1/i + \ldots + 1/(n-2)$. Noting that the direction of the inequality in (4.3) is reversed for $s=s^*-1$, we get

$$(r^*-2)c(r^*-1)\left[3 + 2c(r^*-1)\right] + 2r^* - 2\sum_{i=r^*-1}^{n-1} c(i) \le (r^*-2)c(s^*)\left[1 + 2c(r^*-1)\right] + 2s^* + 1. \qquad (5.4)$$

Now, return to (4.3), with $s=s^*$.

Modifying (5.3) accordingly, and denoting the LHS of (5.4) by L, we get

$$L \geq (r^*-2)\ c(s^*+1)\left[1+2c(r^*-1)\right] + 2s^* + 3. \qquad (5.5)$$

From (3.6), $c(r^*) \approx 1/2$, and (5.1) gives $r^*/n \approx e^{-\frac{1}{2}}$. Also, $\sum_{i=r^*-1}^{n-1} c(i) = n-r^*+1-(r^*-3)c(r^*-1)$, so

$$\lim_{n\to\infty} n^{-1} \sum_{i=r^*-1}^{n-1} c(i) = 1-3/2e^{-\frac{1}{2}}.$$

Let $\lambda = \lim_{n\to\infty} L/n$. From (5.4), we obtain $\lambda \le 2e^{-\frac{1}{2}}\ \underline{\lim}\ c(s^*) + 2\ \underline{\lim}\ s^*/n$, while, from (5.5), we obtain $\lambda \geq 2e^{-\frac{1}{2}}\ \overline{\lim}\ c(s^*) + 2\ \overline{\lim}\ s^*/n$. It follows that $\lim_{n\to\infty} c(s^*)$ and $\lim_{n\to\infty} s^*/n$ exist and equality holds, asymptotically, in (5.4) and (5.5).

If we let $\alpha = \lim_{n\to\infty} s^*/n$, then $\lim_{n\to\infty} c(s^*) = -\log\alpha$. Dividing (5.4) by n and taking the limit gives

$$\alpha - e^{-\frac{1}{2}}\log\alpha = 7/2e^{-\frac{1}{2}} - 1. \qquad (5.6)$$

Solving (5.6) numerically yields $\alpha \doteq .2291$; thus

$$s^* \approx .229n. \qquad (5.7)$$

Also, from (5.3), we obtain

$$v_{s^*-1}(2) \approx .225. \qquad (5.8)$$

After passing approximately 23% of the objects, we shall then select the first object which is better than all the preceding. There is an approximately 22½% chance that the best pair will be selected, for $v_0(2) = v_1(2) = \ldots = v_{s^*-1}(2)$.

The results (5.1) and (5.6) - (5.8) were obtained by Nikolaev (1977) in a different manner. Nikolaev develops an expression, equivalent to our (5.3), for the probability, P, of winning. Assuming that $\alpha = \lim_{n\to\infty} s^*/n$ and $\beta = \lim_{n\to\infty} r^*/n$ exist, he computes the limit of P as a function of α and β. The limit is maximized, subject to the constraints $0 \leqslant \alpha \leqslant \beta \leqslant 1$.

Now, compare the optimal rule given asymptotically by (5.1) and (5.7) with results obtained by Gilbert and Mosteller (1966). In one model, they are given two choices to select the single best object. Their results are

$$s_1^* \approx .223n, \; r_1^* \approx .368n,$$

where the subscript "1" merely denotes a different model. Thus, they make their first selection at about the same stage as we do. However, their second choice is exercised only on observations $A_r=1$, which we never pass once the first selection is made. Because there is no value to their also selecting the second best object, they are willing to pass the second occurrence of $A_r=1$ for $s_1^* < r < r_1^*$. Indeed, that just means their first choice isn't the best, so they are now following the optimal strategy for the classical, one-choice model.

In another model, Gilbert and Mosteller (1966) have just one chance to pick either of the two best objects. They get

$$s_2^* \approx .347n, \; r_2^* \approx .667n.$$

After passing approximately 34.7% of the sequence, they will select the first $A_r=1$ thereafter. It makes sense that they can afford to wait longer than our 23%, because their criterion is less demanding. In the event an $A_r=1$ doesn't occur between $s_2^* < r < r_2^*$, there is a danger that the very best object occurred previously. Thus, they will then select an $A_r=2$ after two-thirds of the sequence is passed, because the second best is just as good to them as the very best. Their $r_2^* \approx .667n$ compares closely to our $r^* \approx .607n$, and the interpretation is essentially the same - apparently we have captured the best, and now we want the second best. Nevertheless, $r_2^* > r^*$; they can still afford to be just a little bit more discriminating than we can.

NUMERICAL RESULTS

	n, number of objects						
	5	10	25	50	100	500	∞
s^*/n	.4	.3	.24	.24	.24	.23	.229
r^*/n	.8	.7	.64	.62	.62	.608	.607
$v_{s^*-1}(2)$	.333	.271	.241	.233	.229	.226	.225

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