



University of Richmond UR Scholarship Repository

Math and Computer Science Faculty Publications

Math and Computer Science

2008

Load Sharing Models

Paul H. Kvam

University of Richmond, pkvam@richmond.edu

Jye-Chyi Lu

Follow this and additional works at: <https://scholarship.richmond.edu/mathcs-faculty-publications>



Part of the [Applied Statistics Commons](#), and the [Mathematics Commons](#)

Recommended Citation

Kvam, Paul H. and Lu, Jye-Chyi, "Load Sharing Models" (2008). *Math and Computer Science Faculty Publications*. 212.
<https://scholarship.richmond.edu/mathcs-faculty-publications/212>

This Post-print Chapter is brought to you for free and open access by the Math and Computer Science at UR Scholarship Repository. It has been accepted for inclusion in Math and Computer Science Faculty Publications by an authorized administrator of UR Scholarship Repository. For more information, please contact scholarshiprepository@richmond.edu.

Load-Sharing Models

Introduction

Consider a system of components whose lifetimes are governed by a probability distribution. *Load sharing* refers to a model of stochastic interdependency between components that operate within a system. If components are set up in a parallel system (*see Parallel, Series, and Series-Parallel Systems*) for example, the system survives as long as at least one component is operating. In a typical load-sharing system, once a component fails, the remaining components suffer an increase in failure rate due to the extra “load” they must encumber due to the failed component.

Component interdependencies are discussed in **Dependence; Aging and Positive Dependence; Measures of Association**. Another model for component interdependency is called a *shock model*, such as Marshall and Olkin [1] bivariate exponential model, which enables the user to model stochastic dependencies between component lifetimes by incorporating latent variables to allow simultaneous component failures. For example, suppose two components in a system are characterized by failure processes Λ_1 and Λ_2 . In a shock model, we can add a new process Λ_3 that characterizes events causing both components to fail, hence the overall failure process for the components become $\Lambda_1 + \Lambda_3$ and $\Lambda_2 + \Lambda_3$. This is the basis for most dependent failure models in probabilistic risk assessment, including *Common Cause Failure* models used in the nuclear industry; see Chapter 8 of Bedford and Cooke [2].

In contrast to the shock model, the load sharing creates a *dynamic* reliability model in which component lifetime distributions may or may not change throughout the course of a system lifetime. Daniels [3] originally adopted the load-share model to describe how the strain on yarn fibers increases as individual fibers within a bundle break. In this case, reliability is measured from strength instead of time until failure. Freund [4] formalized the **probability theory** for a bivariate exponential load-share model.

In general, the shock model provides an easier avenue for multivariate modeling of system component lifetimes. However, dynamic models such as the load-share model are deemed more realistic in environments where a component’s performance can

change once another component in the system fails or degrades. In typical applications of the load-share model, the group of operating components share a load (e.g., current, stress, weight) and once a component within the system fails, the failure rate of the surviving components increase because there are fewer components to share the constant load.

The load-sharing framework can apply to general problems of detecting members of a finite population. If resources allocated for finding a finite set of items are defined globally, rather than assigned individually, then once items are detected, resources can be redistributed for the problem of detecting the remaining items. This action gives rise to a load-sharing model. For bridge construction, some beams connected to the girder are further supported by a set of welded joints. The failure of one or two welded joints can cause the increase of stress on the remaining joints, inducing a load-share model.

Load-Share Rules

Perhaps the most important element of the load-share model is the rule that governs how failure rates change after some components in the system fail. This rule depends on the reliability application and how the components within the system interact that is, through the structure function. For example, an *equal load-sharing* rule, illustrated in Figure 1, implies that the extra load caused by the failed component is shared equally among the surviving ones. A *local*

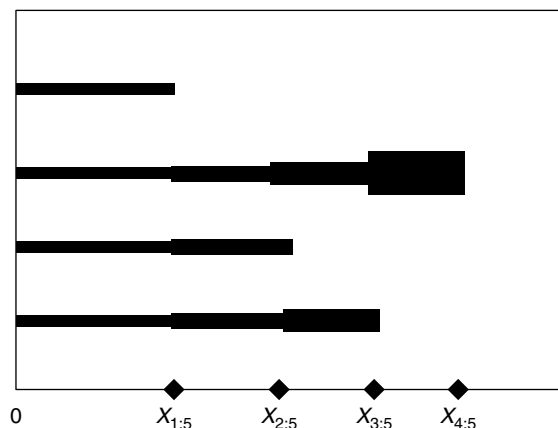


Figure 1 Illustrative load transfer with an equal load-sharing rule

2 Load-Sharing Models

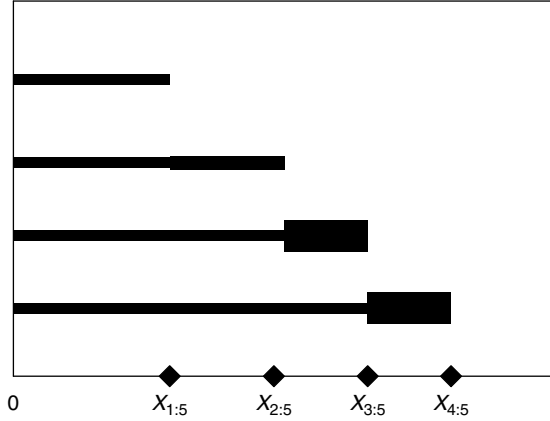


Figure 2 Illustrative load transfer with an local load-sharing rule, where the load from a failed component is transferred to its closest neighbors

load-sharing rule, illustrated in Figure 2, dictates that a failed component's load is transferred to adjacent components; the proportion of the load the surviving components inherit depends on their distance to the failed component.

More generally, a *monotone load-sharing* rule states that the load on the surviving components is nondecreasing with respect to the failure of other components in the system. Not all load-share rules need to be monotone. In finite population sampling, the event of finding one of the n observations might decrease the rate for finding the others. To model the detection rate in software debugging for example, the detection time for existing faults can depend on the number of other faults in the software that have already been found (*see Software Reliability Modeling and Analysis*). The discovery of a critical fault in the software might help reveal or conceal other yet undetected bugs. If other faults are more concealed, they have a decreased rate of discovery in the debugging process.

For assessing an unknown load-sharing rule, Kim and Kvam [5] model the dynamic nature of **component reliability** through proportional hazards. The baseline component failure time distribution is denoted by F . The hazard function (or cumulative hazard rate) corresponding to F is $R(x) = -\log(1 - F(x))$, and the hazard rate is $r(x) = f(x)[1 - F(x)]^{-1}$, where $f(x)$ is the density of F . Until the first component failure, the failure rate of each of k components in the system equals the baseline rate

$r(x)$. Upon the first failure within a system, the failure rates of the $k - 1$ remaining components jump to $\gamma_1 r(x)$, and remain at that rate until the next component failure. After this failure, the failure rates of the $k - 2$ surviving components jump to $\gamma_2 r(x)$, and so on. The failure rate of the last remaining component is $\gamma_{k-1} r(x)$. An equal load-share rule is implied by the $k - 1$ parameters $\gamma_1, \gamma_2, \dots, \gamma_{k-1}$.

To illustrate the relationship of the load-share rule to the system failure process, we consider Daniels' original application for modeling fiber strength.

Stress-Strength Models

For researchers in the textile industry who investigate the reliability of composite materials, a bundle of fibers can be considered as a parallel system subject to a steady tensile load. The rate of failure for individual fibers depends on how the unbroken fibers within the bundle share the load of this overall stress. The load-share rule of such a system depends on the physical properties of the fiber composite. Yarn bundles or untwisted cables tend to spread the stress load uniformly after individual failures. This leads to an equal load-share rule, which implies the existence of a constant system load that is distributed equally among the working components.

In more complex settings, a bonding matrix joins the individual fibers as a composite material, and an individual fiber failure affects the load of certain surviving fibers (e.g., neighbors) more than others, dictated by a local load-sharing rule. If individual fiber strengths are independently and identically distributed with distribution function F , the (parallel) system strength is

$$Y_n = \max_n \left\{ \frac{n - k + 1}{n} X_{k:n}, k = 1, \dots, n \right\} \quad (1)$$

Unfortunately, the distribution F_{Y_n} for Y_n becomes intractable quickly as n increases. Suh *et al.* [6] developed a recursive formula for F_n

$$F_{Y_n} = \sum_{i=1}^n \binom{n}{i} (-1)^{i+1} (F(x))^i F_{n-i} \left(\frac{nx}{n-i} \right) \quad (2)$$

that allows algorithmic computation, but even this solution becomes computationally infeasible with large n .

As an alternative approach, consider a system with each fiber undergoing a positive stress level x . As

x increases, more and more individual fibers break. In this scenario, however, the system load actually decreases at the instance of failure because here the system load is measured as the total stress divided by the number of surviving fibers. Now the actual load in the system (per fiber) at stress level x is

$$L(x) = xn^{-1} \sum_{i=1}^n I(X_i > x) \quad (3)$$

Because $x^{-1}nL(x)$ is binomially distributed, $L(x)$ has an asymptotic normal distribution with mean $x(1 - F(x))$ and variance $x^2F(x)(1 - F(x))$. The mean is unimodal in x , so there is a stress value x_0 such that $L(x_0)$ represents the maximum load the system can endure. For example, if $F(x) = x$, $0 \leq x \leq 1$ (strength is uniformly distributed between 0 and 1) then $x_0 = 0.5$, the maximum value of $x(1 - x)$.

Various techniques have been used to approximate $P_n(x) = n^{-1} \sum I(X_i \leq n)$ in $L(x) = x(1 - P_n(x))$. These are summarized in Crowder *et al.* [7]; Barbour [8] approximation shows a slight improvement over the others based on $P_n(x) \approx \Phi(z_x)$, where

$$z_x = \frac{\sqrt{n}(x - L_0 - \lambda\Delta^{1/3}n^{-2/3})}{\sigma^2 + \gamma n^{-1/2}\Delta^{2/3}} \quad (4)$$

$L_0 = L(x_0)$, $\sigma^2 = x_0^2 F(x_0)(1 - F(x_0))$, and

$$\Delta = \frac{x_0^4 F'(x_0)^2}{2F'(x_0) + x_0 F''(x_0)} \quad (5)$$

(λ, γ) are constant with $\gamma \approx -0.317$ and $\lambda \approx 0.996$. McCartney and Smith [9] carefully evaluated several approximation techniques, noting that the Barbour technique outperform other methods in the lower tail of the distribution.

Time-to-Failure Models

The load-share model has found broad application in life testing and dependent systems analysis since first being developed for material strength models. In most examples, load-sharing models serve as a detriment to systems, especially **parallel systems**. Most research has emphasized parametric lifetime distributions for the dependent components (e.g., Exponential, Weibull) with an assigned load-sharing rule to characterize the dependence. Rydeń [10] extended the load-sharing framework to nonparametric **estimation** of component and

system lifetime using the same load-sharing rules of Daniels [3].

As a simple example, if the components of a parallel system have independent exponential distributions with failure rate λ , it is easy to show through spacings (the time between successive failures in the system) that the expected system lifetime is $\binom{n+1}{2}\lambda^{-1}$. However, if there is a load-sharing system in which the load of the failed component is added to the other components, the expected system lifetime is reduced to λ^{-1} . This is another consequence of the memoryless property that is, for $0 < a < t$, if $X \sim \text{Exp}(\lambda)$, $P(X > t | X > a) = P(X > t - a)$. In this case, it does not even matter if the load is shared equally by all the surviving components or whether the load is put only on one or two survivors.

Figure 3 shows the expected lifetime for parallel system of Weibull(a, b)-lifetime components. The scale parameter (a) equals one and the shape parameter (b) changes from one to five, so the left-most case refers to the exponential model, where the expected lifetime is 1.0. For the components, the failure rate is $r(t) = t^b$. Here, the load from a failed component is assumed to be distributed equally among the surviving components *via* the scale parameter as described by Kvam and Peña [11] and Liu [12].

With complex load-sharing rules, the number of parameters can escalate dramatically. Lynch [13] showed that complex systems of Weibull-distributed components, in a load-sharing model, can be modeled using a mixture of gamma-type distributions.

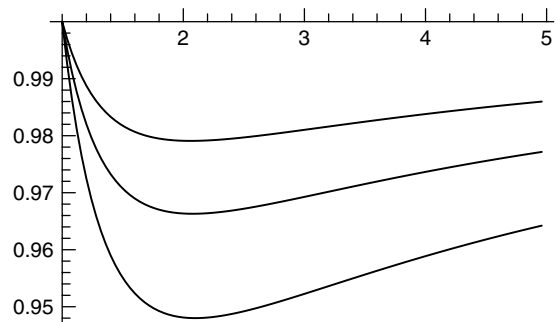


Figure 3 Expected parallel system lifetime versus shape parameter of Weibull-distributed components. The shape parameter changes from 1 (Exponential case) to 5. Curves are for cases $n = 20$ (highest), $n = 10$ (middle) and $n = 5$ (lowest)

Estimating the Load-Sharing Rule

Using nonparametric **maximum likelihood estimation** (MLE), Kvam and Peña [11] estimated a simple load-sharing rule based on independent identically distributed (i.i.d.) parallel systems for which the failure rates of all components were equal, but the change in rate after a component failure depends on the set of functioning components in the system. The proportional hazard model by Kim and Kvam [5] is applied to estimate the load-sharing rule; that is, failure rate changes are characterized by $k - 1$ unknown parameters $\gamma_1, \gamma_2, \dots, \gamma_{k-1}$ and the unknown baseline distribution or hazard function. In this case, estimating the underlying baseline functions F was of primary interest.

If X_{ij} represents the lifetime of the j th component in the i th parallel system, then the random spacing T_{ij} is the time between j th failure and $(j - 1)$ th failure for the i th system. Kim and Kvam [5] considered the exponential model. For this special case where $i = 1, \dots, n$ and $j = 1, \dots, k$, the likelihood function for the i th system is

$$L_i(\theta, \boldsymbol{\gamma}; t_{i1}, t_{i2}, \dots, t_{ik}) = k! \theta^k \prod_{j=1}^{k-1} \gamma_j \exp\left(-\theta \sum_{j=1}^k (k-j+1) \gamma_{j-1} t_{ij}\right) \quad (6)$$

Although the likelihood provides no closed form solution to obtaining MLEs, a Gauss-Seidel algorithmic method can be used to solve them. Kim and Kvam [5] also considered order restricted inference caused by monotone load sharing where $1 \leq \gamma_1 \leq \gamma_2 \leq \dots \leq \gamma_{k-1}$. In this case, a component failure causes the increase in the work-load of the other components, which can equate to an increase of failure rate.

Kvam and Peña [11] generalized the special case of Kim and Kvam [5] in which the components are distributed exponential. $S_{i,1} < S_{i,2} < \dots$ represents the successive component failure times for the i th system. With the counting processes

$$N_i(t) = \sum_{j=1}^k I(S_{i,j} \leq t), \quad i = 1, 2, \dots, n \quad (7)$$

$N_i(t)$ represents the number of component failures for the i th system that occurred by time t . We can express γ in terms of the counting processes

as $\gamma[N_i(w)] = \sum_{j=0}^{k-1} \gamma_j I(N_i(w) = j)$. To formulate the nonparametric likelihood in terms of stochastic processes, let

$$Y_i(w) = (k - N_i(w-)) I(\tau \geq w) \quad (8)$$

Define $\mathcal{F}_{it} = \sigma\{(N_i(w), Y_i(w+)); w \leq t\}$ to be the filtration generated by the i th system up to time t , and let $\mathcal{F}_i = \bigvee_{t=1}^n \mathcal{F}_{it}$. The load-share model can be described by specifying the failure intensities as

$$\begin{aligned} \Pr\{dN_i(t) = 1 | \mathcal{F}_{it-}\} \\ = r(t) Y_i(t) \gamma[N_i(t-)] dt, \quad i = 1, \dots, n \end{aligned} \quad (9)$$

Note that

$$M_i(t) = N_i(t) - \int_0^t \gamma[N_i(u-)] r(u) Y_i(u) du \quad (10)$$

is a vector of orthogonal square-integrable zero-mean martingales. See, for example, Andersen *et al.* [14]. In terms of $\{(N_i(w), Y_i(w)), 0 \leq w \leq \tau\}$: $i = 1, 2, \dots, n$, the likelihood is

$$\begin{aligned} L(R(\cdot), \boldsymbol{\gamma}) \\ = \left\{ \prod_{i=1}^n \prod_{0 \leq w \leq \tau} \pi [Y_i(w) \gamma[N_i(w-)] dR(w)]^{dN_i(w)} \right\} \\ \times \exp \left\{ - \int_0^\tau Y_i(w) \gamma[N_i(w-)] dR(w) \right\} \end{aligned} \quad (11)$$

We can estimate $R(\cdot)$ from the equation (10) by fixing $\boldsymbol{\gamma}$, in $\hat{R}(\cdot; \boldsymbol{\gamma})$. By plugging $\hat{R}(\cdot; \boldsymbol{\gamma})$ into the equation (10), we have the profile likelihood $L_p(\boldsymbol{\gamma})$ for $\boldsymbol{\gamma}$, which is then maximized in $\boldsymbol{\gamma}$ to obtain the estimator $\hat{\boldsymbol{\gamma}}$. The semiparametric estimator of $R(\cdot)$ is $\hat{R}(\cdot) = \hat{R}(\cdot; \hat{\boldsymbol{\gamma}})$. To estimate R this way, we define $J(w) = I(\sum_{i=1}^n Y_i(w) > 0)$, where $J(w) = 0$ indicates all nk components have already failed at time $w-$. If $\boldsymbol{\gamma}$ is known, by using the zero-mean property of the martingale $\sum_{i=1}^n M_i(\cdot)$, we have

$$\hat{R}(s; \boldsymbol{\gamma}) = \int_0^s \frac{J(w) dN(w)}{\sum_{i=1}^n Y_i(w) \gamma[N_i(w-)]} \quad (12)$$

This is analogous to the derivation of the Nelson-Aalen estimator in Aalen [15].

The estimator in the equation (12) is a generalized Nelson-Aalen estimator, and is similar in structure to the hazard function estimator for tensile strengths derived by Rydeń [10]. To obtain the estimator of $R(\cdot)$ for the more general case where $\boldsymbol{\gamma}$ is unknown, we first obtain the profile likelihood for $\boldsymbol{\gamma}$ by plugging in $\hat{R}(\cdot; \boldsymbol{\gamma})$ given in equation (12) into the likelihood function in equation (10). From equation (11) and equation (12) we obtain the profile likelihood to be

$$L_p(s; \boldsymbol{\gamma}) = \prod_{i=1}^n \prod_{0 \leq w \leq s} \mathcal{P} \left[\frac{Y_i(w) \boldsymbol{\gamma} [N_i(w-)]}{\sum_{l=1}^n Y_l(w) \boldsymbol{\gamma} [N_l(w-)]} \right]^{dN_i(w)} \tag{13}$$

This profile likelihood is maximized with respect to $\boldsymbol{\gamma}$ to obtain $\hat{\boldsymbol{\gamma}}$, which is then plugged in into $\hat{R}(\cdot; \boldsymbol{\gamma})$ to obtain the semiparametric estimator of R given by

$$\hat{R}(s) = \hat{R}(s; \hat{\boldsymbol{\gamma}}) \tag{14}$$

By virtue of the product representation of $\bar{F} = 1 - F$ given by $\bar{F}(s) = \mathcal{P}_{0 \leq w \leq s} [1 - R(dw)]$, we can estimate \bar{F} with

$$\hat{\bar{F}}(s) = \mathcal{P}_{0 \leq w \leq s} [1 - \hat{R}(dw)] \tag{15}$$

The solution to the nonparametric MLE is detailed in Kvam and Peña [11], and asymptotic properties for \hat{R} and $\boldsymbol{\gamma}$ are discussed in the section titled “Time-to-Failure Models” of that paper.

References

[1] Marshall, A.W. & Olkin, I. (1967). A multivariate exponential distribution, *Journal of the American Statistical Association* **62**, 30–44.
 [2] Bedford, T. & Cooke, R. (2001). *Probalistic Risk Analysis: Foundations and Methods*, Cambridge University Press.
 [3] Daniels, H.E. (1945). The statistical theory of the strength of bundles of threads, *Proceedings of the Royal Society of London. Series A* **183**, 405–435.
 [4] Freund, J. (1961). A bivariate extension of the exponential distribution, *Journal of the American Statistical Association* **56**, 971–977.
 [5] Kim, H.T. & Kvam, P.H. (2004). Reliability estimation based on system data with an unknown load share rule, *Lifetime Data Analysis* **10**, 83–94.

[6] Suh, M.W., Bhattacharyya, B.B. & Grandage, A. (1970). On the distribution and moments of the strength of a bundle of fibers, *Journal of Applied Probability* **7**, 712–720.
 [7] Crowder, M.J., Kimber, A.C., Smith, R.L. & Sweeting, T.J. (1991). *Statistical Analysis of Reliability Data*, Chapman & Hall.
 [8] Barbour, A.D. (1981). Brownian motion and a sharply curved boundary, *Advances in Applied Probability* **13**, 736–750.
 [9] McCartney, L.N. & Smith, R.L. (1983). Statistical theory of the strength of fiber bundles, *ASME Journal of Applied Mechanics* **105**, 601–608.
 [10] Rydeń, P. (1999). *Estimation of the reliability of systems described by the Daniels load-sharing model*, Licentiate Thesis, Department of Mathematical Statistics, UmeåUniversity, Sweden.
 [11] Kvam, P.H. & Peña, E.A. (2005). Estimating load-sharing properties in a dynamic reliability system, *Journal of the American Statistical Association* **100**, 262–272.
 [12] Liu, H. (1999). Reliability of a load-sharing k-out-of-n:G system: non-iid components with arbitrary distributions, *IEEE Transactions on Reliability* **47**, 279–284.
 [13] Lynch, J.D. (1999). On the joint distribution of component failures for monotone load sharing systems, *Journal of Statistical Planning and Inference* **78**, 13–21.
 [14] Andersen, P.K., Borgan, O., Gill, R.D. & Keiding, N. (1993). *Statistical Models Based on Counting Processes*, Springer-Verlag, New York.
 [15] Aalen, O.O. (1978). Nonparametric estimation for a family of counting processes, *Annals of Statistics* **6**, 701–726.

Further Reading

Gücer, D.E. & Gurland, J. (1962). Comparison of statistics of two fracture models, *Journal of the Mechanics and Physics of Solids* **10**, 365–373.
 Smith, R.L. (1982). The asymptotic distribution of the strength of a series-parallel system with equal load-sharing, *Annals of Probability* **10**, 137–171.

Related Articles

Competing Risks; Cumulative Distribution Function (CDF); Dependence; Markov Processes; Measures of Association; Parallel, Series, and Series-Parallel Systems; Renewal Theory; Software Failure Data Analysis; Software Reliability Modeling and Analysis; Stochastic Deterioration; Stochastic Orders and Aging; Stress-Strength Model.

PAUL H. KVAM AND JYE-CHYI LU