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Degradation Models and Implied Lifetime Distributions

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Abstract

In experiments where failure times are sparse, degradation analysis is useful for the analysis of failure time distributions in reliability studies. This research investigates the link between a practitioner's selected degradation model and the resulting lifetime model. Additive and multiplicative models with single random effects are featured. Simple, seemingly innocuous assumptions of the degradation path create surprising restrictions on the lifetime distribution.

Keywords: additive model, bathtub and increasing failure rates, crack growth, random effects, stochastic ordering

1 Introduction

Reliability testing based on time-to-failure measurements are often hampered by the lack of observed failures. Accelerated life testing (ALT) can hasten product failure during test intervals by stressing the product beyond its normal use. In many tests, the failure data is supplemented by *degradation data*, which refers to measurements of product wear available at one or more time points in the reliability test.

Recently, degradation data has become a necessity due to extremely high product reliability that yields sparse failure data in life tests. Meeker and Escobar [15] offer a comprehensive guide to degradation analysis for various life tests, including ALT, and show that degradation analyses have great potential to improve upon reliability analysis.

However, degradation analysis can also introduce the potential for inconsistency in the experimenter's treatment of the data. The key to the analysis is the perceived link between the degradation measurements and the failure time. The degradation model actually implies a lifetime distribution, but those distributions rarely conform to industry needs for lifetime data analysis. Typically, the resulting estimate of the lifetime distribution must be solved numerically with estimate uncertainty computed using simulation and intensive re-sampling methods such as bootstrap procedures.

In this article, we investigate the link between a chosen degradation model and the resulting lifetime model. On a smaller theoretical scale, this relationship has been studied in terms of stochastic processes (Aven and Jensen [1]). We consider additive and multiplicative models and seek degradation models that lead to particular families (*e.g.*, “bathtub” shaped failure rate) of lifetime distributions.

2 Degradation Models

Degradation models vary markedly across the fields of reliability modeling. Many practical problems can be modelled with a linear (or log-linear) rate of degradation, such as Lu, Park and Yang's ([14])

random effects model for semiconductor degradation. Bogdanoff and Kozin ([5]) employ both linear and more complex nonlinear models to characterize degradation in materials testing (e.g., crack growth). The random effects in the degradation model are the key link between the degradation function and the resulting lifetime distribution. Some degradation models employ a single random effect as an error term in an additive model. Modern degradation models are apt to consider several random effects that enter into the degradation function in nonlinear form, including multiplicative terms. For this research, basic additive and multiplicative models are considered, and the focus is on a single random effect.

2.1 Additive Degradation Model

Consider the general additive degradation model

$$\mathcal{D}(t; X, \Theta) = \eta(t; \Theta) + X, \quad (1)$$

where $\eta(t; \Theta)$ is a deterministic mean degradation path with fixed effect parameters Θ for time $t \geq 0$. We focus on η being monotonic since most degradation measurements have this quality. Bae and Kvam [2] consider the non-monotonic degradation of light displays, but this example is the exception to the norm. X represents random variation around a mean degradation level $\eta(t; \Theta)$ with a cumulative distribution function (cdf) G_X and a probability density function (pdf) g_X .

We assume that failure occurs when the test item's degradation level reaches at a pre-determined threshold value (\mathcal{D}_f). For a monotonically decreasing degradation path (DDP), a failure is defined as the time that the degradation level decreases below the threshold, *i.e.*, $\mathcal{D}(t; X, \Theta) < \mathcal{D}_f$ and $\mathcal{D}(t; X, \Theta) > \mathcal{D}_f$ for a monotonically increasing degradation path (IDP). Let $F_{AD}(t)$ denote the lifetime distribution generated by the DDP in the additive degradation model (1). Then

$$\begin{aligned} F_{AD}(t) &= Pr[\mathcal{D}(t; X, \Theta) < \mathcal{D}_f] = Pr[X < \mathcal{D}_f - \eta(t; \Theta)] \\ &= G_X(\mathcal{D}_f - \eta(t; \Theta)), \end{aligned} \quad (2)$$

with survival function $\bar{F}_{AD}(t) = 1 - F_{AD}(t) = 1 - G_X(\mathcal{D}_f - \eta(t; \Theta))$. For the IDP, the lifetime distribution is $F_{AI}(t) = 1 - G_X(\mathcal{D}_f - \eta(t; \Theta))$.

Note that (2) is a valid distribution only if $G_X(\mathcal{D}_f - \eta(0-; \Theta)) = 0$ and $G_X(\mathcal{D}_f - \eta(+\infty; \Theta)) = 1$. For the IDP, we require that $G_X(\mathcal{D}_f - \eta(0-; \Theta)) = 1$ and $G_X(\mathcal{D}_f - \eta(+\infty; \Theta)) = 0$. If $\eta(t; \theta)$ has finite asymptotes, for example, the additive degradation model will not necessarily produce a proper lifetime distribution function. Along with these constraints on G , we assume G (and hence F) is twice differentiable on $(0, \infty)$. Hereafter, we write $\eta(t)$ for $\eta(t; \Theta)$ for simplicity.

The failure rate corresponding to the DDP, $r_{AD}(t)$, is defined by

$$\begin{aligned} r_{AD}(t) &= \frac{f_{AD}(t)}{1 - F_{AD}(t)} = \frac{[G_X(\mathcal{D}_f - \eta(t))]' }{1 - G_X(\mathcal{D}_f - \eta(t))} \\ &= -\eta'(t) \cdot \frac{g_X(\mathcal{D}_f - \eta(t))}{1 - G_X(\mathcal{D}_f - \eta(t))} \\ &= -\eta'(t) \cdot r_X(\mathcal{D}_f - \eta(t)), \end{aligned} \quad (3)$$

where r_X denotes the failure rate of the degradation random variable X . If $\eta(t)$ is continuous and decreasing monotonically, $r_{AD}(t) \geq 0$ for $r_X(t) \geq 0$. Note that failure rate of the DDP is closely related to that of the random effect X and the functional form of $\eta(t)$. The cumulative failure rate is $R_{AD}(t) = \int_0^t r_{AD}(x)dx$, and can be expressed as

$$R_{AD}(t) = -\log \bar{F}_{AD}(t) = -\log [1 - G_X(\mathcal{D}_f - \eta(t))], \quad (4)$$

In slight contrast to the DDP, the lifetime failure rate corresponding to the IDP is

$$r_{AI}(t) = -\frac{[G_X(\mathcal{D}_f - \eta(t))]' }{G_X(\mathcal{D}_f - \eta(t))} = \eta'(t) \cdot \frac{g_X(\mathcal{D}_f - \eta(t))}{G_X(\mathcal{D}_f - \eta(t))}. \quad (5)$$

with

$$R_{AI}(t) = -\log [G_X(\mathcal{D}_f - \eta(t))]. \quad (6)$$

The p^{th} quantile of lifetime distribution with the DDP is the unique value t_p that satisfies $F_{AD}(t_p) = G_X(\mathcal{D}_f - \eta(t_p)) = p$, and assuming that η is invertible with inverse η^{-1} ,

$$t_{AD,p} = \eta^{-1}[\mathcal{D}_f - G_X^{-1}(p)], \quad (7)$$

and the p^{th} quantile of the IDP is

$$t_{AI,p} = \eta^{-1}[\mathcal{D}_f - G_X^{-1}(1 - p)]. \quad (8)$$

2.2 Multiplicative Degradation Model

In some cases, multiplicative models are exchangeable with additive models through a log transformation; for example, when X is a lognormal-distributed random variable in the multiplicative degradation model, $\log X$ has a normal distribution in the additive degradation model. In cases where η contains random effects, however, the multiplicative model is not interchangeable an additive model. The general multiplicative degradation model can be expressed as

$$\mathcal{D}(t; X, \Theta) = X \cdot \eta(t; \Theta). \quad (9)$$

A failure-time distribution corresponding to the DDP is

$$F_{MD}(t) = Pr[X \cdot \eta(t) < \mathcal{D}_f] = Pr\left[X < \frac{\mathcal{D}_f}{\eta(t)}\right] = G_X\left(\frac{\mathcal{D}_f}{\eta(t)}\right), \quad (10)$$

and for the IDP, the failure-time distribution is $F_{MI}(t) = 1 - G_X(\frac{\mathcal{D}_f}{\eta(t)})$. To satisfy $F_{MI}(0-) = 0$ for IDP in the multiplicative degradation model, we need $G_X(\frac{\mathcal{D}_f}{\eta(t)})|_{t \rightarrow 0-} \rightarrow 1$. The failure rate for F_{MD} in (10) is

$$r_{MD}(t) = \frac{\left[G_X\left(\frac{\mathcal{D}_f}{\eta(t)}\right)\right]'}{1 - G_X\left(\frac{\mathcal{D}_f}{\eta(t)}\right)} = -\left(\frac{\mathcal{D}_f}{\eta(t)}\right) \{\log[\eta(t)]\}' \cdot r_X\left(\frac{\mathcal{D}_f}{\eta(t)}\right), \quad (11)$$

and for the lifetime distribution based on the IDP, the failure rate is

$$r_{MI}(t) = -\frac{\left[G_X\left(\frac{\mathcal{D}_f}{\eta(t)}\right)\right]'}{G_X\left(\frac{\mathcal{D}_f}{\eta(t)}\right)} = \left(\frac{\mathcal{D}_f}{\eta(t)}\right) \{\log[\eta(t)]\}' \cdot \frac{g_X\left(\frac{\mathcal{D}_f}{\eta(t)}\right)}{G_X\left(\frac{\mathcal{D}_f}{\eta(t)}\right)}. \quad (12)$$

Similar to the additive model, the failure rates for the multiplicative model are scaled by the failure rate for the degradation model. The p^{th} quantile for $F_{MD}(t)$ is

$$t_{MD,p} = \eta^{-1}\left[\frac{\mathcal{D}_f}{G_X^{-1}(p)}\right], \quad (13)$$

and for $F_{MI}(t)$,

$$t_{MI,p} = \eta^{-1}\left[\frac{\mathcal{D}_f}{G_X^{-1}(1-p)}\right]. \quad (14)$$

3 Reliability Characteristics of Implied Lifetime Distributions

In this section, we investigate properties of lifetime distributions generated from the additive and multiplicative degradation models. The lifetime distribution is uniquely determined by the degradation distribution's failure rate, and accordingly, we provide basic definitions for the classes of a lifetime distribution in terms of failure rate.

Definition 3.1 *Lifetime distribution F is an increasing failure rate (IFR) distribution if its failure rate $r(t)$ increases monotonically over time; that is, $r'(t) \geq 0$ for all $t \geq 0$. Likewise, F has a decreasing failure rate (DFR) if $r'(t) \leq 0$.*

Definition 3.2 *Lifetime distribution F is defined as increasing (decreasing) failure rate average (IFRA) or (DFRA) if $-(1/t)\log \bar{F}(t)$ is nondecreasing (nonincreasing) in t on $\{t \in \mathbb{R}_+ : \bar{F}(t) > 0\}$.*

Definition 3.3 *Lifetime distribution F is bathtub (BT) or upside-down bathtub (UBT) shaped, if there exists $t^* > 0$ such that $r'(t) < (>) 0$ for all $t \in [0, t^*)$, $r'(t^*) = 0$, and $r'(t) > (<) 0$ for all $t > t^*$.*

Obviously, the *IFR* class is a subset of the *IFRA* class. A bathtub failure rate characterizes life tests in which some early failures occur for a short time until failure rate stabilizes, then eventually increases as the test item ages.

3.1 Additive Degradation Model

Firstly define $\alpha(t)$ as the reciprocal of the failure rate,

$$\alpha(t) = r(t)^{-1} = \frac{\bar{F}(t)}{f(t)}.$$

It follows that $\alpha(t)$ is positive, continuous, and twice differentiable on $(0, \infty)$. For the DDP in an additive degradation model, we have

$$\begin{aligned} \alpha'_{AD}(t) &= -\{[\eta'(t) \cdot r_x(\mathcal{D}_f - \eta(t))]^{-1}\}' \\ &= \alpha_{AD}(t) \left[\eta'(t) \xi_X(t) - \frac{\eta''(t)}{\eta'(t)} \right], \end{aligned} \quad (15)$$

where $\xi_X(t)$ is defined by

$$\xi_X(t) = \frac{r'_X(\mathcal{D}_f - \eta(t))}{r_X(\mathcal{D}_f - \eta(t))}. \quad (16)$$

It can be easily proven that the lifetime distribution derived from the DDP has DFR if $G_X(t)$ possesses DFR since $\xi_X(t) \leq 0$ for the increasing function $\mathcal{D}_f - \eta(t)$. Consequently, $\alpha'_{AD}(t) > 0$ for all $t \geq 0$, which, from (15), implies that $F_{AD}(t)$ is a DFR distribution. More conditions are necessary in order for $F_{AD}(t)$ to possess IFR.

Theorem 3.1 *For the additive degradation model with decreasing $\eta(t)$ with random error $X \sim G$, if $\xi_X(t)$ is bounded with lower limit $-(d/dt)\eta'(t)^{-1}$, then $F_{AD}(t)$ has increasing failure rate.*

Proof: $F_{AD}(t)$ has IFR $\iff \alpha'_{AD}(t) \leq 0 \iff \xi_X(t) \geq \eta''(t)[\eta'(t)]^{-2} = -(d/dt)\eta'(t)^{-1}$. ■

Theorem 3.2 *If $G_X(t)$ is an IFR distribution, then $F_{AD}(t)$ possesses IFRA.*

Proof: By **Theorem 4.1** in Barlow and Proschan [3], the IFR property implies that $R_X(T) = -\log \bar{G}_X(T)$ is convex, that is, $R_X(\gamma T) \leq \gamma R_X(T)$. Since $\mathcal{D}_f - \eta(t)$ is an increasing function, by taking $T = \mathcal{D}_f - \eta(t)$, $R_X(\gamma(\mathcal{D}_f - \eta(t))) \leq \gamma R_X(\mathcal{D}_f - \eta(t)) \iff R_{AD}(\gamma t) \leq \gamma R_{AD}(t) \iff \bar{F}_{AD}(\gamma t) \geq [\bar{F}_{AD}(t)]^\gamma$. This is equivalent to the IFRA property. ■

A bathtub-shaped failure rate for the lifetime distribution can be generated from the additive model. Following **Theorem 3.1**, it can be shown that $F_{AD}(t)$ holds BT (or UBT) shaped failure rate provided that there exists $t^* > 0$ such that $\xi_X(t) < (>) -(d/dt)\eta'(t)^{-1}$ for all $t \in [0, t^*)$, $\xi_X(t^*) = -(d/dt)\eta'(t)^{-1}|_{t=t^*} = 0$, and $\xi_X(t) > (<) -(d/dt)\eta'(t)^{-1}$ for all $t > t^*$.

Similar results hold if the degradation path is increasing, rather than decreasing. Consider the IDP in an additive degradation model. Combining (5) and (15),

$$\alpha'_{AI}(t) = \frac{G_X(\mathcal{D}_f - \eta(t))}{g_X(\mathcal{D}_f - \eta(t))} \left[-\frac{\eta''(t)}{[\eta'(t)]^2} + \delta(t) \right] - 1, \quad (17)$$

where

$$\delta(t) = g'_X(\mathcal{D}_f - \eta(t))/g_X(\mathcal{D}_f - \eta(t)).$$

Noting that $G_X(\mathcal{D}_f - \eta(t)) = - \int_{t_0}^t \eta'(y) \cdot g_X(\mathcal{D}_f - \eta(y)) dy$, where $t_0 = \inf\{t : Pr[g_X(\mathcal{D}_f - \eta(t)) = 0] \rightarrow 1\}$, $\alpha'_{AI}(t)$ can be represented as

$$\begin{aligned} \alpha'_{AI}(t) &= \int_{t_0}^t \frac{-\eta'(y)g_X(\mathcal{D}_f - \eta(y))}{g_X(\mathcal{D}_f - \eta(t))} \left[-\frac{\eta''(t)}{[\eta'(t)]^2} + \delta(t) - \delta(y) \right] dy \\ &\quad + \int_{t_0}^t \frac{-\eta'(y)g_X(\mathcal{D}_f - \eta(y))}{g_X(\mathcal{D}_f - \eta(t))} \cdot \delta(y) dy - 1. \end{aligned}$$

However,

$$\begin{aligned} \int_{t_0}^t \frac{-\eta'(y)g_X(\mathcal{D}_f - \eta(y))}{g_X(\mathcal{D}_f - \eta(t))} \cdot \delta(y) dy &= \frac{\int_{t_0}^t -\eta'(y)g'_X(\mathcal{D}_f - \eta(y)) dy}{g_X(\mathcal{D}_f - \eta(t))} \\ &= \frac{g_X(\mathcal{D}_f - \eta(t))}{g_X(\mathcal{D}_f - \eta(t))} = 1, \end{aligned}$$

and consequently,

$$\alpha'_{AI}(t) = \int_{t_0}^t \frac{\eta'(y)g_X(\mathcal{D}_f - \eta(y))}{g_X(\mathcal{D}_f - \eta(t))} \left[\delta(y) - \delta(t) + \frac{\eta''(t)}{[\eta'(t)]^2} \right] dy, \quad (18)$$

which leads to the following result.

Theorem 3.3 For the IDP $\eta(t)$, if $\delta(t) = g'_X(\mathcal{D}_f - \eta(t))/g_X(\mathcal{D}_f - \eta(t))$ decreases monotonically, then $F_{AI}(t)$ has DFR.

Proof: Based on (18), if $\delta'(t) < 0$, then $\alpha'_{AI}(t) > 0$ for all $t \geq 0$, which implies that $F_{AI}(t)$ is a DFR distribution. ■

The result in (18) will be also useful to define sufficient conditions for BT or UBT shaped failure rate of $F_{AI}(t)$.

It was proven in this section that properties of the lifetime distributions derived from additive degradation models are largely determined by the functional form of the mean degradation function $\eta(t)$, as well as the distribution of random effect X .

3.2 Multiplicative Degradation Model

The failure rates of lifetime distributions derived from the multiplicative degradation model, as given in (11) for the DDP and (12) for the IDP, depend directly on the deterministic degradation function $\eta(t)$. For example, consider the following DDP

$$\mathcal{D}(t; X, \Theta) = X \cdot (\theta_1 t + 1)^{-\theta_2}, \quad \theta_1, \theta_2 > 0, \quad (19)$$

where X follows a Weibull distribution with a scale parameter $\lambda > 0$, and a shape parameter $\kappa > 0$. As shown in the Figure 1, failure rates at fixed values of $\theta_1 = 0.1$ and $\lambda = 1$ are decreasing ($\theta_2 < 2.0$), constant ($\theta_2 = 2.0$), or increasing ($\theta_2 > 2.0$) even if X has DFR ($\kappa = 0.5$).

In this example, the failure rate of the IDP eventually decreases to zero, and this phenomenon can be easily explained in that $\eta(t) \gg \{\log[\eta(t)]\}'$, consequently

$$r_{MI}(t) \approx \frac{\{\log[\eta(t)]\}'}{\eta(t)} \cdot \frac{g_X\left(\frac{\mathcal{D}_f}{\eta(t)}\right)}{G_X\left(\frac{\mathcal{D}_f}{\eta(t)}\right)} \rightarrow 0$$

as $t \rightarrow \infty$. From simulation results, it could be observed that resulting failure rate from the IDP is decreasing or unimodal function according to the form of $\eta(t)$.

4 Examples

In this section we provide some illustrations of degradation path models that lead to specific lifetime distributions. Beyond the scope in Lu and Meeker [13], lifetime distributions with BT shaped failure rate are derived using the relationship between G_X and $\eta(t)$.

4.1 X is Weibull-Distributed

Consider the simple IDP $\mathcal{D}(t; X, \Theta) = \theta t$, where $X \equiv \theta (> 0)$ is the degradation rate. Assuming X is Weibull distributed with cdf

$$G_X(x) = 1 - \exp[-(\lambda x)^\kappa], \quad \lambda, \kappa > 0 \quad (20)$$

then the lifetime distribution is

$$F_{MI}(t) = Pr[\theta t \geq \mathcal{D}_f] = Pr\left[\theta \geq \frac{\mathcal{D}_f}{t}\right] = \exp\left[-\left(\frac{\lambda \mathcal{D}_f}{t}\right)^\kappa\right],$$

and by letting $\nu = \lambda \mathcal{D}_f$,

$$F_{MI}(t) = \exp[-(v/t)^\kappa]. \quad (21)$$

This is a cdf of the rarely applied *inverse-Weibull* distribution. Huang and Askin [10] applied the Weibull distribution to characterize item-to-item variability of electronic devices which degrade linearly, but the inverse-Weibull lifetime distribution (and its difficulty in use) is not discussed. The failure rate of inverse-Weibull distribution is given by

$$r_{MI}(t) = \frac{\kappa v^\kappa t^{-\kappa-1} \exp[-(v/t)^\kappa]}{1 - \exp[-(v/t)^\kappa]},$$

which has a unimodal failure rate with $\lim_{t \rightarrow 0} r_{MI}(t) = \lim_{t \rightarrow \infty} r_{MI}(t) = 0$, in strong contrast to the Weibull distribution.

For the DDP, we consider the model used by Fukada [8] to characterize the degradation of electronic devices:

$$\mathcal{D}(t; X, \Theta) = \theta_3 \cdot \exp[-\exp\{(\theta_1 t)^{\theta_2}\}]. \quad \theta_1, \theta_2 > 0, \theta_3 \geq 1 \quad (22)$$

After a logarithmic transformation,

$$\log \mathcal{D}(t; X, \Theta) = \log \theta_3 - \exp\{(\theta_1 t)^{\theta_2}\}.$$

If $X \equiv \log \theta_3 (\geq 0)$ is assumed to be a random effect that follows a Weibull distribution with cdf given in (20), then for $\eta(t) \equiv -\exp\{(\theta_1 t)^{\theta_2}\}$, which satisfies $\mathcal{D}_f - \eta(t) \geq 0$, we obtain the lifetime distribution

$$F_{AD}(t) = G_X(\mathcal{D}_f - \eta(t)) = 1 - \exp[-\{\lambda(\mathcal{D}_f + \exp\{(\theta_1 t)^{\theta_2}\})\}^\kappa].$$

In the case $\kappa = \lambda = 1$, (i.e., X has a standard exponential distribution), the survivor function is given by

$$\bar{F}_{AD}(t) = \exp[-\mathcal{D}_f - \exp\{(\theta_1 t)^{\theta_2}\}]. \quad (23)$$

Suppose that a failure is considered as $\mathcal{D}_f (= \log \mathcal{D}) = -1$, then Eq. (23) is reduced to the survivor function of an *exponential power* distribution with a failure rate $r_{AD}(t) = \theta_1 \theta_2 (\theta_1 t)^{\theta_2 - 1} \exp[(\theta_1 t)^{\theta_2}]$.

The exponential power distribution is one of the few tractable two-parameter distributions that possess a BT shaped failure rate (see Figure 2). Its failure rate is BT shaped when $\theta_2 < 1$, achieving a minimum at $[(1 - \theta_2)/(\theta_1\theta_2)]^{1/\theta_2}$. For $\theta_2 = 1$, the exponential power distribution is reduced to an extreme value distribution (Dhillon [6]).

Next, consider the following DDP where a random effect is entered into the model multiplicatively:

$$\mathcal{D}(t; X, \Theta) = \theta_3 \cdot \{\log[\theta_1 t + 1]\}^{-\theta_2}, \quad \theta_i > 0, \quad i = 1, 2, 3 \quad (24)$$

with deterministic degradation function $\{\log[\theta_1 t + 1]\}^{-\theta_2}$. Suppose that $X \equiv \theta_3$ follows a Weibull distribution with cdf given by (20). Then, combining (10) and (20), a lifetime distribution function derived from degradation model (24) is

$$F_{MD}(t) = 1 - \exp[-(\lambda \mathcal{D}_f)^\kappa \cdot \{\log[\theta_1 t + 1]\}^{\kappa\theta_2}]. \quad (25)$$

by In the case $\kappa = 1$, $\lambda \mathcal{D}_f = 1$, and $\theta_2 \geq 1$, the distribution (25) can be expressed as

$$F_{MD}(t) = 1 - \exp[-\{\log(\theta_1 t + 1)\}^{\tilde{\theta}_2+1}],$$

for $\tilde{\theta}_2 \geq 0$. This is called a *2-parameter distribution II* in Dhillon [6], and its failure rate

$$r_{MD}(t) = \frac{\theta_1(\tilde{\theta}_2 + 1) \cdot (\log[\theta_1 t + 1])^{\tilde{\theta}_2}}{\theta_1 t + 1}$$

exhibits increasing and decreasing failure pattern, as shown in Figure 3.

4.2 X is Gamma-Distributed

Suppose that the random effect X follows gamma distribution with a scale parameter $\lambda > 0$ and a shape parameter $\gamma > 0$. The pdf of X is

$$g_X(x) = \frac{\lambda^\gamma x^{\gamma-1} e^{-\lambda x}}{\Gamma(\gamma)}, \quad x \geq 0,$$

where $\Gamma(\cdot)$ denotes the well-known gamma function.

To investigate general characteristics of the lifetime distribution from a gamma-distributed degradation path, for example, we consider the following metal corrosion process. The rate of

metal corrosion decreases in time as $d\eta(t)/dt = \theta_2/(\theta_1 + t)$, where θ_1 and $\theta_2 > 0$ are material-specific constants, hence the mean degradation path is $\eta(t) = \theta_2 \log[\theta_1 + t]$ (Tomashov [19]). When we assume $X \equiv \theta_2$ is gamma-distributed, the failure-time distribution is

$$\begin{aligned} F_{MI}(t) &= 1 - G_X \left(\frac{\mathcal{D}_f}{\eta(t)} \right) = 1 - \int_0^{\frac{\mathcal{D}_f}{\eta(t)}} \frac{\lambda^\gamma x^{\gamma-1} e^{-\lambda x}}{\Gamma(\gamma)} dx \\ &= 1 - \int_0^{\frac{\mathcal{D}_f}{\log(\theta_1+t)}} \frac{\lambda^\gamma x^{\gamma-1} e^{-\lambda x}}{\Gamma(\gamma)} dx, \end{aligned} \quad (26)$$

and the pdf is obtained by differentiating (26)

$$f_{MI}(t) = \frac{(\lambda \mathcal{D}_f)^\gamma e^{-\frac{\lambda \mathcal{D}_f}{\log(\theta_1+t)}}}{\Gamma(\gamma) [\log(\theta_1 + t)]^{\gamma+1}} \cdot \frac{1}{\theta_1 + t}, \quad t \geq 0. \quad (27)$$

Its failure rate can be obtained by using (26) and (27). While the gamma distribution's failure rate can be increasing ($\gamma > 1$), constant ($\gamma = 1$) or decreasing ($\gamma < 1$), failure rates of the metal corrosion process with a gamma distributed item-to-item variability are always unimodal regardless of the shape parameter's value as shown in Figure 4.

4.3 X is Log-logistically Distributed

The pdf of a log-logistic or a Weibull-exponential distribution (Dubey [7]) is

$$g_X(x) = \frac{\beta e^\alpha x^{\beta-1}}{(1 + e^\alpha x^\beta)^2}, \quad x > 0, \beta > 0,$$

with cdf

$$G_X(x) = 1 - \frac{1}{1 + e^\alpha x^\beta}.$$

The log-logistic distribution is a special case of Burr's type XII family of distribution. Consider the following DDP:

$$\mathcal{D}(t; X, \Theta) = \theta_3(\theta_1 t)^{-\theta_2}, \quad \theta_i > 0, \quad i = 1, 2, 3,$$

with a random effect $X \equiv \theta_3$. If X follows a log-logistic distribution with parameters α and β , a lifetime distribution of \mathcal{D} is

$$F_{MD}(t) = 1 - \frac{1}{1 + e^\alpha [\mathcal{D}_f(\theta_1 t)^{\theta_2}]^\beta},$$

and by letting $b = e^{-\alpha/(\theta_2\beta)}\mathcal{D}_f^{-1/\theta_2}\theta_1^{-1}$,

$$F_{MD}(t) = 1 - \left[1 + \left(\frac{t}{b} \right)^{\theta_2\beta} \right]^{-1}, \quad b > 0, t \geq 0. \quad (28)$$

In the special case where $\theta_2\beta = 2$, the distribution (28) represents $b(F_{2,2}^{1/2})$, where $F_{2,2}$ denotes a (central) F random variable with (2,2) degrees of freedom. The pdf of $Z = bF_{2,2}^{1/2}$ is

$$f_Z(z) = \frac{(b/2)^{-1}(z/b)}{B(1,1)\{1 + (z/b)^2\}^2},$$

where $B(\cdot)$ denotes the beta function. The resulting failure rate

$$r_{MD}(t) = \frac{2t}{b^2 + t^2}$$

increases to time $t = b$ and then decreases; see Figure 5.

4.4 X is a Gaussian Process

In a large number of applications, the random variation confounded in a deterministic degradation path is time-dependent, so $X = \{X(t) : t \geq 0\}$ can be modelled as a stochastic process. Sobczyk and Spencer [18] provided broad applications of a stochastic process in modeling random fatigue growth. Apart from developing a theoretical framework, we focus on deriving the distribution of failures as the result of performance degradation using a stochastic process, specifically the *Gaussian process*. A Gaussian process is defined as the stochastic process where $X(t_1), \dots, X(t_n)$ has a multivariate normal distribution for any $t_i \in \mathbb{R}_+^n$.

A stochastic process $\{W(t) : t \geq 0\}$ is said to be a *Wiener process* if $Pr[W(0) = 0] = 1$, $\{W(t) : t \geq 0\}$ has stationary and independent increments, and for every $t > 0$, $W(t)$ is normally distributed with mean 0 and variance $\sigma_w^2 t$. The covariance of the Wiener process is defined as: $\gamma_w(s, t) = \sigma_w^2 \cdot (s \wedge t)$, where ' \wedge ' means the minimum value of the two.

In relating $F(t)$ to $G_X(t)$ where $X \equiv X(t)$ follows the Wiener Process, consider the IDP in an additive degradation model. The lifetime distribution is given by

$$F_{AI}(t) = 1 - G_X(\mathcal{D}_f - \eta(t)) = \frac{1}{\sqrt{2\pi t}\sigma_w} \int_{\mathcal{D}_f - \eta(t)}^{\infty} \exp\left[-\frac{x^2}{2t\sigma_w^2}\right] dx.$$

By letting $y = x(t\sigma_W)^{-1/2}$,

$$F_{AI}(t) = \frac{1}{\sqrt{2\pi}} \int_{\frac{\mathcal{D}_f - \eta(t)}{\sqrt{t}\sigma_W}}^{\infty} \exp\left[-\frac{y^2}{2}\right] dy = 1 - \Phi\left\{\frac{\mathcal{D}_f - \eta(t)}{\sqrt{t}\sigma_W}\right\},$$

where $\Phi\{\cdot\}$ denotes the cdf of the standard normal distribution. For example, if $\eta(t) = \theta t$ for $\theta > 0$, then

$$\begin{aligned} F_{AI}(t) &= 1 - \Phi\left\{\frac{\mathcal{D}_f}{\sqrt{t}\sigma_W} - \frac{\theta\sqrt{t}}{\sigma_W}\right\} \\ &= 1 - \Phi\left\{\frac{1}{\alpha}\left[\sqrt{\frac{\beta}{t}} - \sqrt{\frac{t}{\beta}}\right]\right\} = \Phi\left\{\frac{1}{\alpha}\left[\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right]\right\}, \end{aligned} \quad (29)$$

for $\alpha = \sigma_W(\mathcal{D}_f\theta)^{-1/2}$ and $\beta = \mathcal{D}_f/\theta$. It is noted that the derived lifetime distribution (29) is a Birnbaum-Saunders distribution [4] with a shape parameter $\alpha > 0$ and a scale parameter $\beta > 0$.

The failure rate of the Birnbaum - Saunders distribution is

$$r_{AI}(t) = \frac{\frac{\sqrt{\frac{t}{\beta}} + \sqrt{\frac{\beta}{t}}}{2\alpha t} \cdot \phi\left\{\frac{1}{\alpha}\left[\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right]\right\}}{1 - \Phi\left\{\frac{1}{\alpha}\left[\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right]\right\}}. \quad (30)$$

where $\phi\{\cdot\}$ is the pdf of the standard normal distribution. It is known that this failure rate is always unimodal with $\lim_{t \rightarrow 0} r_{AI}(t) = 0$, and $\lim_{t \rightarrow \infty} r_{AI}(t) = \frac{1}{2\alpha^2\beta}$.

For another example using a Gaussian process, we consider a fatigue crack growth model. The prediction of stochastic crack growth accumulation is crucial for the reliability and durability analysis of various material in manufacturing. Yang *et al.* [21] and Sobczyk [17] proposed the crack growth rate model

$$\frac{d\alpha(t)}{dt} = X(t) \cdot \eta(\Delta K(\alpha)), \quad (31)$$

where $\alpha(t)$ is the crack size at time t , $\Delta K(\alpha)$ is the stress intensity range which is a function of crack size α . Based on the principles of fracture mechanics, $\eta(\cdot)$ represents a mean crack growth rate, whereas the random process $\{X(t) : t \geq 0\}$ accounts for the statistical variability of the crack growth accumulation. A commonly used crack growth rate function is the Paris-Erdogan relationship [16], *i.e.*, $\eta(\Delta K(\alpha)) = C(\Delta K)^m$, where C and m are assumed constants for a given material.

Yang *et al.* [21] assumed that $X(t)$ is a stationary lognormal random process with a median value of unity, equivalently, the process $Z(t) = \log X(t)$ is a Gaussian process with zero mean. The stationary lognormal random process is defined by the autocovariance function $\gamma_X(\tau) = E[X(t)X(t + \tau)]$. Here, $\gamma_X(0) = Var[X] = \sigma_X^2$. The autocovariance function plays a significant role in random process analysis and specifies the statistical behavior of the random process. From a physical standpoint, the autocovariance function of the crack growth rate should decrease as the time difference τ increases. According to Yang *et al.* [21], the autocovariance function $\gamma_X(\tau)$ of the random fatigue crack growth process is an exponentially decaying function of time difference τ :

$$\gamma_X(\tau) = \sigma_X^2 \cdot \exp(-\zeta|\tau|), \quad \tau > 0 \quad (32)$$

where ζ^{-1} is a measure of the correlation between $X(t)$ and $X(t + \tau)$. This autocovariance scheme provides flexibility in covering a wide range of dispersion of crack growth accumulation through the correlation parameter ζ . Based on the mean crack growth rate $\eta(\Delta K(\alpha))$ and the lognormal random process $X(t)$ with the autocovariance (32), we can derive a lifetime distribution of a fatigue degradation process.

Denote a deterministic initial crack size as α_0 at $t_0 = 0$ and the crack size at service time τ as $\alpha(\tau)$. Then the mean service time to reach $\alpha(\tau)$ from α_0 is

$$\int_{\alpha_0}^{\alpha(\tau)} \frac{d\alpha}{\eta(\Delta K(\alpha))} = \int_0^\tau X(t)dt,$$

since $K(\alpha)$ is a function of the crack size α . Let $\omega(\tau) = \int_0^\tau X(t)dt$, then the crack size $\alpha(\tau)$ is a monotone increasing function of $\omega(\tau)$. The integration of the lognormal random process, $\{\omega(\tau) : \tau > 0\}$ is also a lognormal random process with mean

$$E[\omega(\tau)] = \int_0^\tau E[X(t)]dt = \mu_X\tau,$$

and variance

$$\begin{aligned} Var[\omega(\tau)] &= \int_0^\tau \int_0^\tau E[X(t_1)X(t_2)]dt_1dt_2 \\ &= \int_0^\tau \int_0^\tau \sigma_X^2 \cdot \exp[-\zeta(t_2 - t_1)]dt_1dt_2 \\ &= 2 \left(\frac{\sigma_X}{\zeta} \right)^2 \cdot (\exp(-\zeta\tau) + \zeta\tau - 1). \end{aligned}$$

For a fatigue process, lifetime is defined as the time that a crack size increases beyond a pre-determined threshold value α_f , therefore

$$Pr[\alpha(\tau) \geq \alpha_f] = Pr[\omega(\tau) \geq \omega_f] = 1 - \Phi \left[\frac{\log \omega_f - \mu_\omega(\tau)}{\sigma_\omega(\tau)} \right], \quad (33)$$

where $\omega_f = \int_{\alpha_0}^{\alpha_f} (\eta(\Delta K(\alpha)))^{-1} d\alpha$, and $\mu_\omega(\tau)$, $\sigma_\omega(\tau)$ can be obtained by solving the following equation: $E[\omega(\tau)] = \exp[\mu_\omega(\tau) + \sigma_\omega^2(\tau)/2]$ and $Var[\omega(\tau)] = \exp[2\mu_\omega(\tau) + \sigma_\omega^2(\tau)] \cdot (\exp[\sigma_\omega^2(\tau)] - 1)$. The lifetime distribution presented in (33) is flexible enough to cover a broad domain of fatigue growth failures by taking into account the correlation parameter ζ . In the case in which the correlation parameter ζ approaches zero, for example, from (32) the autocovariance function $\gamma_X(\tau)$ is independent of τ and (33) is reduced to a lognormal distribution for $\omega(\tau) = X\tau$. For the Paris-Erdogan relationship, the failure rate of (33) is decreasing in τ . It can also be observed that the failure rate decreases as the correlation parameter ζ increases (see Figure 6).

5 Stochastic Ordering of a degradation lifetime distribution

Denote the distribution function and survival function of a random variable X_i by F_i and \bar{F}_i respectively for $i = 1, 2$. In this section, we employ stochastic orders in the degradation distributions to derive properties for the implied lifetime distribution.

Definition 5.1 X_1 is said to be **stochastically smaller** than X_2 , written $X_1 \leq_{st} X_2$, if $\bar{F}_1(t) \leq \bar{F}_2(t)$ for all t ; or equivalently, if $E[\psi(X_1)] \leq E[\psi(X_2)]$ for all increasing function $\psi(\cdot)$ for which the integrals are well-defined.

Definition 5.2 X_1 is smaller than X_2 in the sense of **likelihood ratio**, written $X_1 \leq_{lr} X_2$ if, when X_1 and X_2 are absolutely continuous random variables with density $f_1(t)$ and $f_2(t)$,

$$\frac{f_1(t)}{f_2(t)} \text{ is nonincreasing in } t.$$

It is well-known that

$$X_1 \leq_{lr} X_2 \Rightarrow X_1 \leq_{st} X_2.$$

Theorem 5.1 Let X_1 and X_2 be random effects of corresponding degradation paths \mathcal{D}_1 and \mathcal{D}_2 . For the DDPs (IDPs) \mathcal{D}_1 and \mathcal{D}_2 of the multiplicative form,

(i) If $X_1 \leq_{st} X_2$, then $\mathcal{D}_1 \leq_{st} \mathcal{D}_2$.

(ii) If $X_1 \leq_{lr} X_2$, then $\mathcal{D}_1 \leq_{lr} \mathcal{D}_2$.

(iii) If $\mathcal{D}_1 \leq_{lr} \mathcal{D}_2$, then $\mathcal{D}_1 \leq_{st} \mathcal{D}_2$.

Proof: For (i),

$$\begin{aligned} Pr[X_1 \leq x] \geq Pr[X_2 \leq x] &\Leftrightarrow Pr[X_1 \cdot \eta(t) \leq \mathcal{D}_f] \geq Pr[X_2 \cdot \eta(t) \leq \mathcal{D}_f] \quad \text{for } \eta(t) > 0 \\ &\Leftrightarrow Pr[\mathcal{D}_1 \leq \mathcal{D}_f] \geq Pr[\mathcal{D}_2 \leq \mathcal{D}_f] \\ &= F_{MD,1} \geq F_{MD,2} \quad (\bar{F}_{MI,1} \geq \bar{F}_{MI,2}), \end{aligned}$$

which implies that $\mathcal{D}_1 \leq_{st} (\geq_{st}) \mathcal{D}_2$.

For (ii), using the DDPs \mathcal{D}_1 and \mathcal{D}_2 , a partial ordering of X_1 and X_2 can be translated into a partial ordering on an increasing function of t , $\mathcal{D}_f/\eta(t) \geq 0$, where $\mathcal{D}_f \geq 0$ is a constant and $t \geq 0$. Therefore,

$$\begin{aligned} \left\{ \frac{G_{X_1}(x)}{G_{X_2}(x)}, \frac{g_{X_1}(x)}{g_{X_2}(x)} \right\} \swarrow x &\Leftrightarrow \left\{ \frac{G_{X_1}(\mathcal{D}_f/\eta(t))}{G_{X_2}(\mathcal{D}_f/\eta(t))}, \frac{g_{X_1}(\mathcal{D}_f/\eta(t))}{g_{X_2}(\mathcal{D}_f/\eta(t))} \right\} \swarrow t \\ &\Leftrightarrow \left\{ \frac{F_{MD,1}(t)}{F_{MD,2}(t)}, \frac{f_{MD,1}(t)}{f_{MD,2}(t)} \right\} \swarrow t. \end{aligned}$$

Similarly for the IDPs \mathcal{D}_1 and \mathcal{D}_2 , a partial ordering of X_1 and X_2 can be reversely translated into a partial ordering on a decreasing function of t , $\mathcal{D}_f/\eta(t) \geq 0$, therefore

$$\begin{aligned} \left\{ \frac{G_{X_1}(x)}{G_{X_2}(x)}, \frac{g_{X_1}(x)}{g_{X_2}(x)} \right\} \swarrow x &\Leftrightarrow \left\{ \frac{G_{X_1}(\mathcal{D}_f/\eta(t))}{G_{X_2}(\mathcal{D}_f/\eta(t))}, \frac{g_{X_1}(\mathcal{D}_f/\eta(t))}{g_{X_2}(\mathcal{D}_f/\eta(t))} \right\} \nearrow t \\ &\Leftrightarrow \left\{ \frac{F_{MI,1}(t)}{F_{MI,2}(t)}, \frac{f_{MI,1}(t)}{f_{MI,2}(t)} \right\} \nearrow t. \end{aligned}$$

See Keilson and Sumita [11] for detailed proof of (iii). ■

It can be easily shown that the partial ordering of random effects X_1 and X_2 is preserved with the DDPs \mathcal{D}_1 and \mathcal{D}_2 in the additive model. Therefore, Theorem 5.1 can be identically applied to the DDPs (IDPs) \mathcal{D}_1 and \mathcal{D}_2 in the additive model if only a failure is defined for $\mathcal{D}_f \geq (\leq)\eta(t)$.

6 Reliability Ordering of a degradation lifetime distribution

Partial orderings of life distributions in terms of their aging properties were introduced by Kochar and Wiens [12]. We will employ some of those definitions, summarized below, to investigate how the stochastic orderings affect comparisons in the lifetime distributions. The orderings are based on the previously defined stochastic orders IFR, IFRA and another ordering called “New Better than Used” (NBU). A distribution F is NBU iff

$$\bar{F}(x)\bar{F}(y) \geq \bar{F}(x+y)$$

for all values of x and y . Note that this ordering is less restrictive than the IFRA ordering.

Definition 6.1 F_1 is **more IFR** than F_2 , written $F_1 <_{IFR} F_2$ if $F_2^{-1}(F_1(t))$ is a convex function in t on the support of F_1 . This is also equivalent to **convex ordering** denoted by $F_1 <_c F_2$. If the failure rates exist, an equivalent formulation is

$$\frac{r_1(F_1^{-1}(z))}{r_2(F_2^{-1}(z))} \text{ is nondecreasing in } z \in [0, 1].$$

Definition 6.2 F_1 is **more IFRA** than F_2 , written $F_1 <_{IFRA} F_2$ if $F_2^{-1}(F_1(t))$ is star-shaped. This is also equivalent to **star-ordering** denoted by $F_1 <_* F_2$.

Definition 6.3 F_1 is **more NBU** than F_2 , written $F_1 <_{NBU} F_2$ if $F_2^{-1}(F_1(t))$ is superadditive. This is also equivalent to **superadditive ordering** denoted by $F_1 <_{SU} F_2$. F_1 is said to be superadditive with respect to F_2 if

$$F_2^{-1}(F_1(x+y)) \geq F_2^{-1}(F_1(x)) + F_2^{-1}(F_1(y))$$

for all x and y in the support of F_1 .

Each of the above orderings are scale invariant, and has the property that for a standard exponential distribution $F_2(t) = 1 - e^{-t}$, if F_1 has aging property κ , then $F_1 <_{\kappa} F_2$ for $\kappa \in \{IFR, IFRA, NBU\}$. It also follows that

$$F_1 <_{IFR(c)} F_2 \Rightarrow F_1 <_{IFRA(*)} F_2 \Rightarrow F_1 <_{NBU(SU)} F_2.$$

Theorem 6.1 *Let G_{X_1} and G_{X_2} be distribution functions of random effects X_1 and X_2 for corresponding degradation paths \mathcal{D}_1 and \mathcal{D}_2 in a multiplicative degradation model, then*

(i) *If $G_{X_1} <_{IFR(IFRA)} G_{X_2}$, then $F_{MD,1} <_{IFR(IFRA)} F_{MD,2}$ if $\eta(t)$ is a decreasing function of t .*

(ii) *If $G_{X_1} <_{NBU} G_{X_2}$, then $F_{MD,1} <_{NBU} F_{MD,2}$ if $\eta(t)$ is a convex increasing function of t .*

Proof: (i) For the distribution functions of the DDPs \mathcal{D}_1 and \mathcal{D}_2 , $F_{MD,i}(t) = G_{X_i}(\mathcal{D}_f/\eta(t))$, $i = 1, 2$, let $Z(t) = \mathcal{D}_f/\eta(t)$. Then

$$\begin{aligned} F_{MD,2}^{-1}(F_{MD,1}(t)) &= \{G_{X_2}(Z(t))\}^{-1} [G_{X_1}(Z(t))] \\ &= Z^{-1} \left\{ G_{X_2}^{-1}[G_{X_1}(Z(t))] \right\}. \end{aligned}$$

Since $G_{X_2}^{-1}(G_{X_1}(x))$ is a convex (star-shaped) function, $G_{X_2}^{-1}(G_{X_1}(Z(t)))$ is also a convex (star-shaped) function with respect to t , $t \geq 0$. For any monotonic increasing function $Z(t)$, an inverse function $Z^{-1}(t)$ is also increasing. Consequently, an increasing function of a convex (star-shaped) function, $Z^{-1}\{G_{X_2}^{-1}[G_{X_1}(Z(t))]\}$ is also a convex (star-shaped) function, which proves that $F_{MD,1} <_{IFR(IFRA)} F_{MD,2}$.

(ii) Since $\eta(t)$ is convex, $Z(t) = \mathcal{D}_f/\eta(t)$ is concave, hence $Z(t_1+t_2) \geq Z(t_1)+Z(t_2)$ for arbitrary values of $t_1 \geq 0$ and $t_2 \geq 0$. Therefore,

$$\begin{aligned} Z^{-1} \left\{ G_{X_2}^{-1}[G_{X_1}(Z(t_1+t_2))] \right\} &\geq Z^{-1} \left\{ G_{X_2}^{-1}[G_{X_1}(Z(t_1) + Z(t_2))] \right\} \\ &\geq Z^{-1} \left\{ G_{X_2}^{-1}[G_{X_1}(Z(t_1))] \right\} + Z^{-1} \left\{ G_{X_2}^{-1}[G_{X_1}(Z(t_2))] \right\}, \end{aligned}$$

which proves $F_{MD,1} <_{NBU} F_{MD,2}$. ■

7 Conclusion

In modeling material fatigue and degradation, the relationship between the randomness in degradation and randomness in the resulting lifetime distribution is strong and direct, albeit hard to discern. By making tacit assumptions about the degradation distribution, the resulting implications to the lifetime distribution may surprise the experimenter or contradict the assumptions about the failure characteristics in the study.

This research reveals surprising restrictions on the properties of the degradation path when assumptions are made on the degradation distribution. Interestingly, some degradation models, under distributional assumption of random parameters in the model, directly imply lifetime distributions having bathtub shaped failure rate.

The comparisons in this article are based on basic degradation functions that are applied to show the relationships between degradation and lifetime with minimal complexity. Certainly, many practical applications would use degradation functions extended beyond the elementary ones listed here. The results bolster the need for further attention to this implied relationship of interest; few efforts have been pursued toward this end.

While degradation modeling is growing beyond the purview of manufacturing and materials testing (*e.g.*, logistics-performance degradation in supply-chain networks [20]), reliability prediction through degradation modeling will be further emphasized as a supporting tool in lifetime data analysis.

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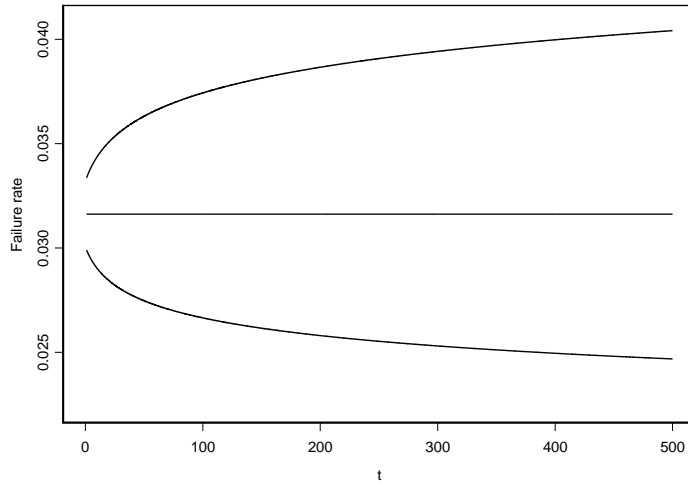


Figure 1: Failure rate plot of the DDP in a multiplicative model with random effect X distributed Weibull

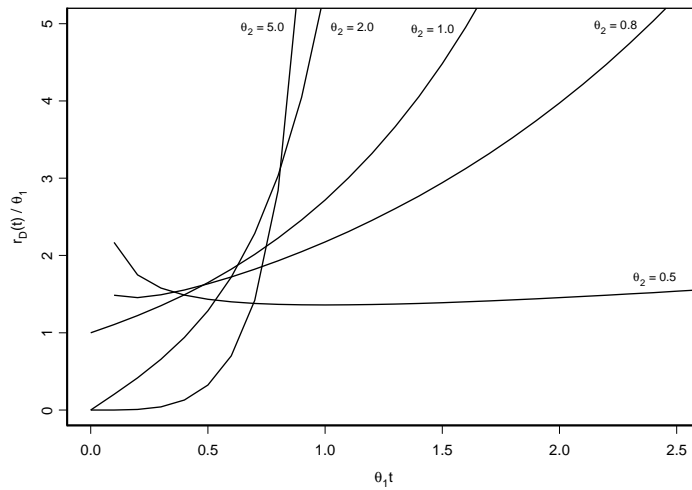


Figure 2: Failure rate plot of exponential power distribution

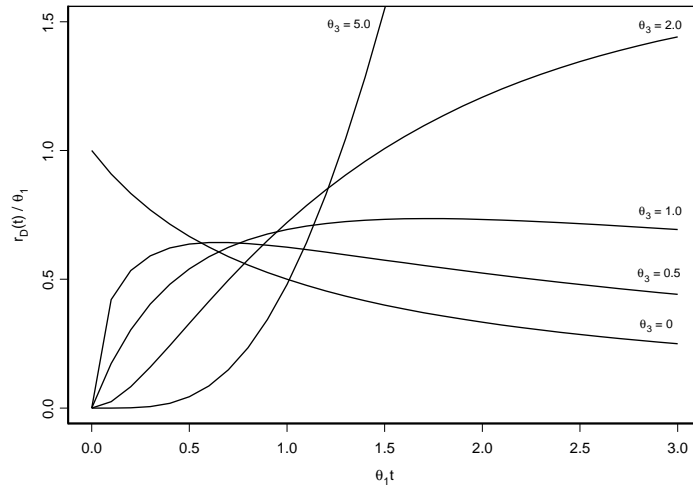


Figure 3: Failure rate plot of 2-parameter distribution II

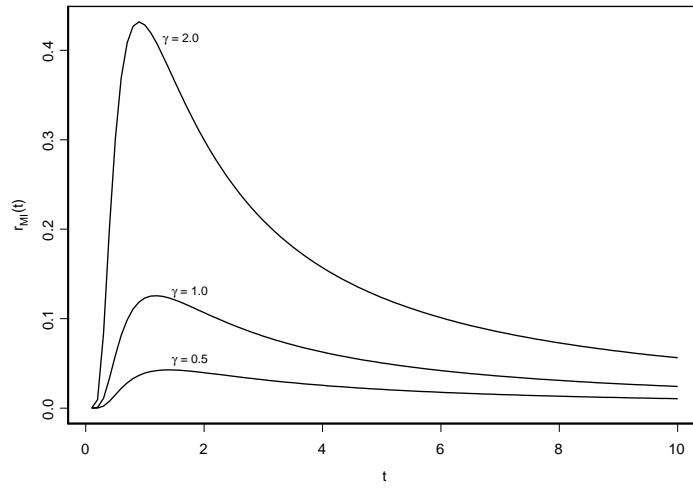


Figure 4: Failure rate plot of functional gamma distribution with $\lambda = 1$

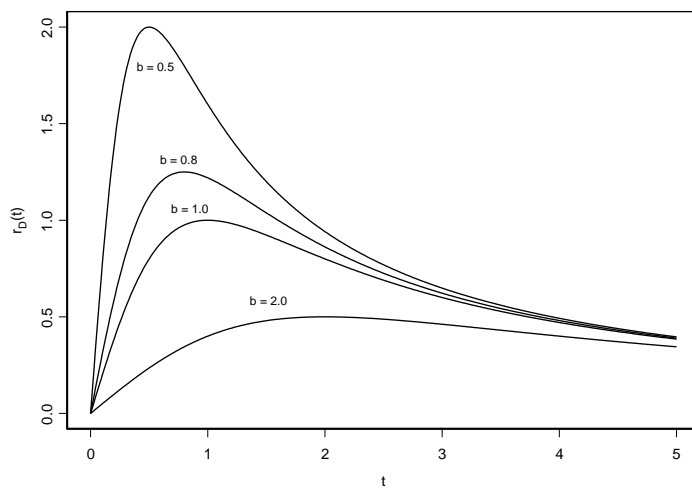


Figure 5: Failure rate plot of $bF_{2,2}^{1/2}$ distribution

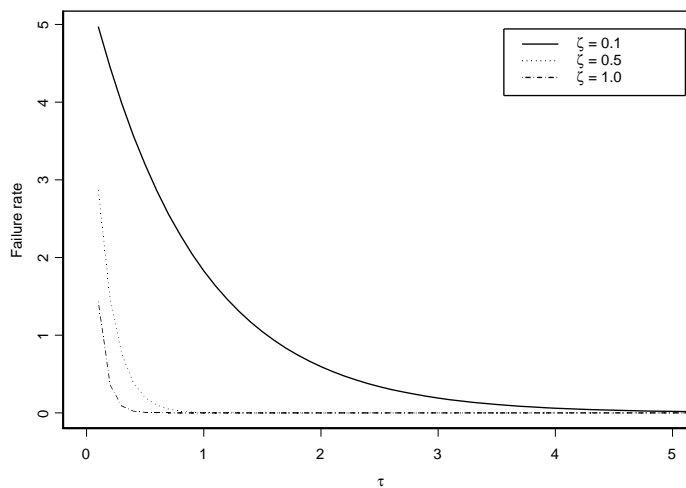


Figure 6: Failure rate plot of a crack growth rate model with the Paris-Erdogan relationship