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RANKED SET SAMPLING FROM LOCATION-SCALE FAMILIES OF SYMMETRIC DISTRIBUTIONS

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ABSTRACT

Statistical inference based on ranked set sampling has primarily been motivated by nonparametric problems. However, the sampling procedure can provide an improved estimator of the population mean when the population is partially known. In this article, we consider estimation of the population mean and variance for the location-scale families of distributions. We derive and compare different unbiased estimators of these parameters based on r independent replications of a ranked set sample of size n. Large sample properties, along with asymptotic relative efficiencies, help identify which estimators are best suited for different location-scale distributions.

Keywords: Asymptotic Relative Efficiency; Mean Squared Error; Order Statistics; Variance Estimators.

1 Introduction

In sampling situations where the units drawn from a population are difficult or expensive to quantify but can be easily ranked, the *ranked set sampling* procedure provides an efficient method to estimate the population mean and variance. Developed by McIntyre (1), the procedure involves randomly drawing a set of n units from the underlying population, where n is small enough so that the units can be ranked without judgment error. The n units are then ranked according to the characteristic of interest, and then smallest unit is quantified. A second set of n units is drawn, ranked and the unit ranked second smallest is quantified. This process is continued until a complete cycle is accomplished wherein, at the n^{th} stage, a set of n units is drawn, ranked and the unit ranked the largest is quantified. The sample obtained is called a (balanced) ranked set sample. The entire sequence can be repeated several times.

As the estimator of the population mean, one can show that the mean of a ranked set sample is unbiased and has smaller variance than the mean of a simple random sample of the same size. If the distribution function for underlying population is denoted by F with corresponding density function $f(x)$. From an independent sample $X_1, ..., X_n$ from F, the order is discerned and the smallest out of n is quantified as $X_{1:n}$. In another independent sample of n, the second smallest is quantified as $X_{2:n}$, and so on. Although n^2 units are identified, only n of them are actually measured. Each measurement is an independent order statistic, and we denote the set by $\{X_{(1:n)}, ..., X_{(n:n)}\}$. Suppose that the entire process is repeated r times. Then the RSS is denoted by $\{X_{(i:n)j}$: $i = 1, ..., n$; $j = 1, ..., r\}$. In general, we assume r is large compared to n.

The density function of $X_{i:n}$ is given by

$$f_{i:n}(x) = n\binom{n-1}{i-1} F^{i-1}(x)\bar{F}^{n-i}(x)f(x), \qquad -\infty < x < \infty, \quad i = 1, ..., n, \tag{1.1}$$

where $\bar{F} \equiv 1 - F$. The binomial expansion yields

$$f(x) = \frac{1}{n}\sum_{i=1}^{n} f_{i:n}(x). \tag{1.2}$$

Consistent with this notation, let $\mu = \int xf(x)dx$, $\sigma^2 = \int(x-\mu)^2 f(x)dx$, $\mu_{(i:n)} = \int xf_{i:n}(x)dx$, and $\sigma^2_{(i:n)} = \int(x-\mu_{(i:n)})^2 f_{i:n}(x)dx$. From (1.2), we have

$$\begin{aligned} \mu &= \frac{1}{n}\sum_{i=1}^{n}\mu_{(i:n)}, \\ \sigma^2 &= \frac{1}{n}\sum_{i=1}^{n}(\sigma^2_{(i:n)} + \mu_{(i:n)})^2 - \mu^2 \\ &= \frac{1}{n}\sum_{i=1}^{n}\sigma^2_{(i:n)} + \frac{1}{n}\sum_{i=1}^{n}(\mu_{(i:n)} - \mu)^2. \end{aligned} \tag{1.3}$$

Let $\bar{X}_{rss} = (nr)^{-1}\sum_{j=1}^{r}\sum_{i=1}^{n} X_{(i:n)j}$, and let $\bar{X}_{srs}$ denote the mean of a simple random sample (SRS) of the same size (nr). Both $\bar{X}_{rss}$ and $\bar{X}_{srs}$ are unbiased. Also, $Var(\bar{X}_{rss}) = \sum_{i=1}^{n}\sigma^2_{(i:n)}/(n^2 r)$, $Var(\bar{X}_{srs}) = \sigma^2/nr$, and from (1.3),

$$Var(\bar{X}_{srs}) = Var(\bar{X}_{rss}) + \frac{1}{rn^2}\sum_{i=1}^{n}(\mu_{(i:n)} - \mu)^2. \tag{1.4}$$

Thus $\bar{X}_{rss}$ is more efficient than $\bar{X}_{srs}$. The relative precision (RP) is given by

$$RP = \frac{Var(\bar{X}_{srs})}{Var(\bar{X}_{rss})} = \frac{1}{1 - \frac{1}{n}\sum_{i=1}^{n}\frac{(\mu_{(i:n)}-\mu)^2}{\sigma^2}}.$$

Note that RP does not depend on r, and satisfies $1 \leq RP \leq (n+1)/2$ with $RP = (n+1)/2$ if and only if F is a uniform distribution (Takahasi and Wakimoto, (2)).

Ranked set sampling has been used in finite population by Takahasi and Futatsuya (3) and Patil et al. ((4), (5)). Dell and Clutter (6) and David and Levine (7) studied “judgment errors” in ranking the sampling units, and Stokes (8) considered ranking the units employing a concomitant variable. See Kaur et al.(9) for a comprehensive review of the ranked set sampling. Kvam and Samaniego (10) derived the nonparametric maximum likelihood estimator of the distribution function F using an unbalanced RSS. Those results were extended to a Bayesian setup by Kvam and Tiwari (11).

Although the concept of ranked set sampling is a nonparametric one, it has applications in sampling problems where the underlying distribution is partially known. Sinha, et al. (12) considered normal and exponential distributions and derived the best linear unbiased estimators (BLUEs) of their parameters based on a ranked set sample. They provided solutions to several questions regarding selection and inclusion of different order statistics for ranked set samples.

In this paper, we more generally consider the location-scale families of distributions given by the density function

$$f(x) = \frac{1}{\sigma} f_0(\frac{x-\mu}{\sigma}), \qquad -\infty < x < \infty, \quad -\infty < \mu < \infty, \quad \sigma > 0, \tag{1.5}$$

where f_0 is a known density, symmetric about 0, and free of parameters μ and σ. In Section 2, the location parameter μ and the scale parameter σ are estimated using a balanced ranked set sample of size nr, where r represents the number of replications. We seek the BLUE of μ, based on a sensible optimality criteria. We also investigate unbiased estimators of σ. Large sample properties of these estimators, based on allowing r to tend to infinity, are derived in Section 3. Finally, we compare the asymptotic relative efficiency of the proposed estimator of μ with rival estimators in Section 4.

2 Estimation of the location and scale parameters

Consider the location-scale families of distributions in (1.5), where μ is the location parameter, and σ is the scale parameter. Examples of the density function f_0 include

i) The standard normal density: $f_0(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$,

ii) The double exponential density: $f_0(x) = \frac{1}{2} e^{-|x|}$, and

iii) The student t density with k degrees of freedom: $f_0(x) = \frac{\Gamma(\frac{k+1}{2})}{\sqrt{k\pi}\Gamma(\frac{k}{2})(1+x^2)^{\frac{k+1}{2}}}$.

For the normal distribution, μ and σ^2 are the mean and variance. For the double exponential distribution, μ is the mean and $2\sigma^2$ is the variance. For the student t distribution with k degrees of freedom, μ is the mean (provided $k > 1$) and $k\sigma^2/(k-2)$ is the variance (provided $k > 2$).

Let $\{X_{(1:n)j},...,X_{(n:n)j}\}, j = 1,...,r$ be a RSS of size nr, where r denotes the number of replications. Let $X \sim f$, and $Y \sim f_0$, then the relation $X = \mu + \sigma Y$ implies that $X_{(i:n)j} = \mu + \sigma Y_{(i:n)j}$, where $\{Y_{(1:n)j},...,Y_{(n:n)j}\}$, $j = 1,...,r$ is a RSS from f_0. When r=1, we shall denote the RSS from f simply by $\{X_{(1:n)},...,X_{(n:n)}\}$ and the RSS from f_0 by $\{Y_{(1:n)},...,Y_{(n:n)}\}$. Let $\nu_{(i:n)} = E(Y_{(i:n)})$ and $\eta^2_{(i:n)} = Var(Y_{(i:n)})$. Let F_0 be the cumulative distribution function corresponding to the density f_0. Then

$$\begin{aligned}\nu_{(i:n)} &= n\binom{n-1}{i-1}\int yF_0^{i-1}(y)\bar{F}_0^{n-i}(y)f_0(y)dy \\ &= \frac{1}{B(i,n+1-i)}\int_0^1 F_0^{-1}(u)u^{i-1}(1-u)^{n-i}du\end{aligned}$$

and

$$\eta^2_{(i:n)} = \frac{1}{B(i,n+1-i)}\int_0^1 F_0^{-2}(u)u^{i-1}(1-u)^{n-i}du - \nu^2_{(i:n)},$$

where $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$. Using Taylor's expansion

$$F_0^{-1}(u) \approx F_0^{-1}(\frac{i}{n+1}) + (u - \frac{i}{n+1})[f_0(F_0^{-1}(\frac{i}{n+1}))]^{-1},$$

we have

$$\nu_{(i:n)} \approx F_0{}^{-1}(\frac{i}{n+1}), \qquad \eta^2_{(i:n)} \approx \frac{i(n+1-i)}{(n+1)^2(n+2)}[f_0(F_0^{-1}(\frac{i}{n+1}))]^{-2}$$

and the other higher central moments of $Y_{(i:n)}$ can be approximated in the same manner. Also using (1.2) and the symmetry of f_0,

$$\sum_{i=1}^{n}\nu_{(i:n)} = 0, \qquad \sum_{i=1}^{n}\nu_{(i:n)}/\eta^2_{(i:n)} = 0. \tag{2.6}$$

Tables 1-5 lists the values of $\nu_{(i:n)}$ and $\eta^2_{(i:n)}$, for n=2 to 6 from the standard normal, the standard double exponential, and the Student's t distribution with k degrees of freedom with $k = 3, 4, 5$. A more detailed list of means and variances for order statistics from a $N(0,1)$ population are printed in Sinha, et al. (12) and Sarhan and Greenberg (13). Note that because the distributions are symmetric about zero, $\nu_{(i:n)} = 0$ for $i = (n+1)/2$ and odd values of n, and $\nu_{(i:n)} = -\nu_{(n-i+1:n)}$. Thus, only $\eta^2_{(i:n)}$ values for $i = 1,...,[n/2]$ and $\nu_{(i:n)}$ values for $i = 1,...,[(n+1)/2]$ are printed in these tables.

2.1 Least Squares Estimators

In view of $X_{(i:n)} = \mu + \sigma Y_{(i:n)}$, $i = 1,...,n$ we have

$$E(X_{(i:n)}) = \mu + \sigma\nu_{(i:n)}, \qquad Var(X_{(i:n)}) = \sigma^2\eta^2_{(i:n)}, \qquad i = 1,...,n.$$

Let $\beta = (\mu, \sigma)'$, and $\mathbf{D} = diag(\eta^2_{(1:n)}, ..., \eta^2_{(n:n)})$. Then

$$E(\mathbf{X}_n) = \mathbf{W}_n\beta, \qquad Var(\mathbf{X}_n) = \mathbf{\Sigma} = \sigma^2\mathbf{D}, \tag{2.7}$$

where $\mathbf{X}_n = (X_{(1:n)}, ..., X_{(n:n)})$ is a single RSS cycle, $\mathbf{W}_n = (\mathbf{1}_n, \nu_n)$, $\mathbf{1}_n$ is a n -column vector and $\nu'_n = (\nu_{(1:n)} \;....\; \nu_{(n:n)})$. The least-squares estimators of μ and σ are obtained by minimizing

$$\begin{aligned} Q_n(\beta) &= (\mathbf{X}_n - \mathbf{W}_n\beta)'(\sigma^2\mathbf{D})^{-1}(\mathbf{X}_n - \mathbf{W}_n\beta) \\ &= \sum_{i=1}^{n}(X_{(i:n)} - \mu - \sigma\nu_{(i:n)})^2/\sigma^2\eta^2_{(i:n)} \end{aligned}$$

with respect to β. By extending this least squares approach to r replicates of a RSS cycle, we obtain least squares estimates

$$\hat{\mu}_{ls} = \frac{\sum_{j=1}^{r}\sum_{i=1}^{n} X_{(i:n)j}/\eta^2_{(i:n)}}{r\sum_{i=1}^{n} 1/\eta^2_{(i:n)}}, \qquad \hat{\sigma}_{ls} = \frac{\sum_{j=1}^{r}\sum_{i=1}^{n}(X_{(i:n)j} - \hat{\mu}_{ls})^2/\eta^2_{(i:n)}}{\sum_{j=1}^{r}\sum_{i=1}^{n} X_{(i:n)j}\nu_{(i:n)}/\eta^2_{(i:n)}}. \tag{2.8}$$

For situations where both the mean vector and the variance matrix of $\mathbf{X}_n$ depend on σ^2, such as in (2.7), Godambe and Kale (14) discuss critical shortcomings of this least squares approach.

2.2 Estimating Functions Approach

Godambe and Kale ((14), Chapter 1) proposed an alternative approach for estimating the parameters, based on "estimating functions", which yield optimal estimators, in terms of mean squared error (MSE). For (2.7) the estimating functions are

$$\mathbf{W}'_n\mathbf{D}^{-1}(\mathbf{X}_n - \mathbf{W}_n\beta) = \mathbf{0}.$$

Based on r replicates of $\mathbf{X}_n$, the resulting in the estimators of μ and σ are given by

$$\begin{aligned} \hat{\mu}_{opt} &= \frac{\sum_{j=1}^{r}\sum_{i=1}^{n} X_{(i:n)j}/\eta^2_{(i:n)}}{r\sum_{i=1}^{n} 1/\eta^2_{(i:n)}}; \\ \hat{\sigma}_{opt} &= \frac{\sum_{j=1}^{r}\sum_{i=1}^{n} X_{(i:n)j}\nu_{(i:n)}/\eta^2_{(i:n)}}{r\sum_{i=1}^{n} \nu^2_{(i:n)}/\eta^2_{(i:n)}}, \end{aligned} \tag{2.9}$$

The estimates $\hat{\mu}_{opt}$ and $\hat{\sigma}_{opt}$ are optimal in the sense that both are linear unbiased estimators for μ and σ respectively, and both have minimum variance. Thus they are the best linear unbiased estimators (BLUEs). See Sinha, et al. (12) for another derivation of BLUEs of μ and σ for the particular case when f_0 is the standard normal distribution. It is interesting to note that the estimators $\hat{\mu}_{opt}$ and $\hat{\sigma}_{opt}$ can be obtained by minimizing $Q_n(\beta)$ ignoring the factor σ^2 in $\sigma^2\mathbf{D}$. As a serious drawback, both the estimators $\hat{\sigma}_{ls}$ and $\hat{\sigma}_{opt}$ can be negative, more frequently with small samples.

2.3 Stokes' Estimator for σ^2

For each of the three distributions mentioned earlier, the scale parameter is directly related to population variance, so we interchange σ and σ^2 in our discussion the scale parameter. For the case when F is not specified, Stokes (15) proposed an estimator for the population variance σ^2 based on a balanced RSS:

$$\hat{\sigma}_s^2 = \frac{1}{(nr-1)} \sum_{j=1}^{r} \sum_{i=1}^{n} (X_{(i:n)j} - \bar{X}_{rss})^2, \tag{2.10}$$

where $\bar{X}_{rss} = \frac{1}{nr} \sum_{j=1}^{r} \sum_{i=1}^{n} X_{(i:n)j}$. Stokes showed that

$$E(\hat{\sigma}_s^2) = \sigma^2 + \frac{1}{n(nr-1)} \sum_{i=1}^{n} (\mu_{(i:n)} - \mu)^2$$

and $E(\hat{\sigma}_s^2) \to 0$ as $r \to \infty$; that is, $\hat{\sigma}_s^2$ is an asymptotically unbiased estimator of σ^2. Let s^2 be the sample variance of a simple random sample of size nr. Stokes (15) obtained the mean squared error of $\hat{\sigma}_s^2$ and showed that $RP = Var(s^2)/MSE(\hat{\sigma}_s^2) \geq 1$.

2.4 Alternative Unbiased Estimators of σ^2

For the location-scale family of distributions, $\mu_{(i:n)} - \mu = \sigma\nu_{(i:n)}$, $i = 1, ..., n$ and $\sum_{i=1}^{n} \{\eta_{(i:n)}^2 + \nu_{(i:n)}^2\} = n$ so that

$$E(\hat{\sigma}_s^2) = \sigma^2 \{ \frac{1}{n} \sum_{i=1}^{n} \eta_{(i:n)}^2 + \frac{r}{nr-1} \sum_{i=1}^{n} \nu_{(i:n)}^2 \}.$$

Thus an unbiased estimator of σ^2 can be constructed from $\hat{\sigma}_s^2$:

$$\begin{aligned} \hat{\sigma}_{su}^2 &= \frac{\hat{\sigma}_s^2}{\{\frac{1}{n} \sum_{i=1}^{n} \eta_{(i:n)}^2 + \frac{r}{nr-1} \sum_{i=1}^{n} \nu_{(i:n)}^2\}} \\ &= \frac{\sum_{j=1}^{r} \sum_{i=1}^{n} (X_{(i:n)j} - \bar{X}_{rss})^2}{(nr-1) + \frac{1}{n} \sum_{i=1}^{n} \nu_{(i:n)}^2}. \end{aligned} \tag{2.11}$$

However, $\hat{\sigma}_s^2$ nor its unbiased alternative are necessarily optimal for estimating the population variance with location-scale distributions. Because $\hat{\mu}_{opt}$ is better for estimating μ than $\bar{X}_{rss}$, we propose an alternative unbiased estimator of σ^2 based on $\hat{\mu}_{opt}$:

$$\hat{\sigma}_u^2 = \frac{\sum_{j=1}^{r} \sum_{i=1}^{n} (X_{(i:n)j} - \hat{\mu}_{opt})^2 / \eta_{(i:n)}^2}{(nr-1) + r \sum_{i=1}^{n} \nu_{(i:n)}^2 / \eta_{(i:n)}^2}. \tag{2.12}$$

Yu, et al. (16) use this approach for the special case of variance estimation with the normal distribution. We can easily verify the unbiasedness of (2.12) as follows. First note that

$$
\begin{aligned}
& E[\{(nr-1)+r\sum_{i=1}^{n}\nu_{(i:n)}^2/\eta_{(i:n)}^2\}\hat{\sigma}_u^2] \\
= \; & \sigma^2 E[\sum_{j=1}^{r}\sum_{i=1}^{n} Y_{(i:n)j}^2/\eta_{(i:n)}^2 - (\sum_{j=1}^{r}\sum_{i=1}^{n} Y_{(i:n)j}/\eta_{(i:n)}^2)^2/r\sum_{i=1}^{n} 1/\eta_{(i:n)}^2]. \qquad (2.13)
\end{aligned}
$$

Now using $\nu_{(i:n)} = E(Y_{(i:n)})$ and $\eta_{(i:n)}^2 = Var(Y_{(i:n)})$,

$$
\begin{aligned}
E(\sum_{j=1}^{r}\sum_{i=1}^{n} Y_{(i:n)j}^2/\eta_{(i:n)}^2) &= r\sum_{i=1}^{n}(\eta_{(i:n)}^2+\nu_{(i:n)}^2)/\eta_{(i:n)}^2 \\
&= nr + r\sum_{i=1}^{n}\nu_{(i:n)}^2/\eta_{(i:n)}^2
\end{aligned}
$$

and

$$
E(\sum_{j=1}^{r}\sum_{i=1}^{n} Y_{(i:n)j}/\eta_{(i:n)}^2)^2/r\sum_{i=1}^{n} 1/\eta_{(i:n)}^2 = 1 + r^2(\sum_{i=1}^{n}\nu_{(i:n)}/\eta_{(i:n)}^2)^2 = 1,
$$

in view of (2.6). Substituting these expressions on the right hand side of (2.13), we have $E(\hat{\sigma}_u^2) = \sigma^2$.

2.5 Estimators based on Within-Replicate Variation

Other unbiased estimators of σ^2 can be formed based on balanced ranked set samples. Yu, et al. (16) discuss several approaches for estimating the variance of a normal distribution, and derived different variance estimators that improve on the ordinary Stokes (15) estimator. For example, some estimators exploit the known scale and location properties of the normal distribution by contrasting a given order statistic $X_{(i:n)}$ with its expected value $\mu + \sigma\nu_{(i:n)}$. Alternatively, one can contrast an order statistic from the j^{th} cycle with an estimate of the mean based on the j^{th} cycle, thus obtaining an estimator of σ^2 based on "within replicate" variability of the RSS sample. This is analogous to "within block" variability in a two-way analysis of variance. Define

$$
\hat{\sigma}_{w1}^2 = \frac{\sum_{j=1}^{r}\sum_{i=1}^{n}(X_{(i:n)j}-\bar{X}_{\cdot j})^2}{r(\frac{1}{n}\sum_{i=1}^{n}\nu_{(i:n)}^2 + n - 1)}, \qquad (2.14)
$$

$$
\hat{\sigma}_{w2}^2 = \frac{\sum_{j=1}^{r}\sum_{i=1}^{n}(X_{(i:n)j}-\hat{\mu}_{opt}^{(j)})^2/\eta_{(i:n)}^2}{r((n-1)+\sum_{i=1}^{n}\nu_{(i:n)}^2/\eta_{(i:n)}^2)}, \qquad (2.15)
$$

where $\bar{X}_{\cdot j} = \frac{1}{n}\sum_{i=1}^{n} X_{(i:n)j}$ and $\hat{\mu}_{opt}^{(j)} = [\sum_{i=1}^{n} X_{(i:n)j}/\eta_{(i:n)}^2]/\sum_{i=1}^{n} 1/\eta_{(i:n)}^2$. The "within replicate" estimators in (2.14) and (2.15) correspond to the unbiased estimators (2.11) and (2.12), respectively.

2.6 Comparisons of Variance Estimators

We have discussed six different estimators of σ or σ^2, not including the natural RSS variance, $\hat{\sigma}_s^2$ which Stokes showed was biased for σ^2. To further compare the estimators, we conducted several simulations to generate samples from location-scale families of distributions. Specifically, we computed the MSE for each estimator and compared the estimators using the relative efficiency with respect to the RSS variance; i.e., RMSE$(\hat{\sigma}_0^2)$ = MSE$(\hat{\sigma}_0^2)$/MSE$(\hat{\sigma}_s^2)$.

The results are listed in Tables 6-8. The normal Distribution is featured in Table 6, The double-exponential distribution is in Table 7, and The student-t distribution (with 3 degrees of freedom) is in Table 8. We focus on small or medium sized ranked set samples (n = 3,5,10) and various number of replicates (r = 2, 10, 20). For each case, 100,000 simulations are run.

¿From the results of Tables 6-8, we can see that $\hat{\sigma}_{su}$ (the unbalanced version of the Stokes estimator) is a relatively poor estimate of σ in a MSE sense. As $\hat{\sigma}_u$ appears superior to $\hat{\sigma}_{su}$, so too does $\hat{\sigma}_{w2}$ appear superior to $\hat{\sigma}_{w1}$. $\hat{\sigma}_u$ does not dominate $\hat{\sigma}_{w2}$ as it does the other estimators, but it outperforms $\hat{\sigma}_{w2}$ in most sampling scenarios in all three distributions.

3 Asymptotic Results

In this section, we focus on examining the properties of $\hat{\mu}_{opt}$ and $\hat{\sigma}_u$, the respective estimators of the location and scale parameter that performed best according to the criteria discussed in Section 2. The asymptotic normality of $\hat{\mu}_{opt}$ given in (2.9) and the consistency of the estimator $\hat{\sigma}_u^2$ given in (2.12) are established. First, we examine the asymptotic normality of $\hat{\mu}_{opt}$.

Throughout this section we assume that the sequence of random variables$\{X_{(i:n)j}\}$ and $\{Y_{(i:n)j}\}$ are defined on a common probability space. Using the relation $X_{(i:n)j} = \mu + \sigma Y_{(i:n)j}$; $i = 1, ..., n, j = 1, ..., r$ with $\nu_{(i:n)} = E(Y_{(i:n)})$ and $\eta^2_{(i:n)} = Var(Y_{(i:n)})$, we have

$$E(\hat{\mu}_{opt}) = \mu + \frac{\sigma}{r}\sum_{j=1}^{r}\bar{W}_{.j},$$

where

$$\bar{W}_{.j} = \frac{\sum_{j=1}^{r}\sum_{i=1}^{n} Y_{(i:n)j}/\eta^2_{(i:n)}}{\sum_{i=1}^{n} 1/\eta^2_{(i:n)}}, \qquad j = 1, ..., r.$$

Note that $\bar{W}_{.1}, ..., \bar{W}_{.r}$ are independent identically distributed (i.i.d.) with $E(\bar{W}_{.j}) = 0$ (see (2.6)) and $Var(\bar{W}_{.j}) = [\sum_{i=1}^{n} 1/\eta^2_{(i:n)}]^{-1}$, $j = 1, ..., r$. Similarly, we can write

$$\bar{X}_{rss} = \mu + \frac{\sigma}{r}\sum_{j=1}^{r}\bar{V}_{.j}$$

where $\bar{V}_{.j} = \sum_{i=1}^n Y_{(i:n)j}/n,\ j = 1, ..., r$ are i.i.d. with $E(\bar{V}_{.j}) = 0$ and $Var(\bar{V}_{.j}) = \sum_{i=1}^n \eta^2_{(i:n)}/n^2,\ j = 1, ..., r$. We have the following.

Theorem 3.1 *With a ranked set sample* $\{X_{(i:n)j}: i = 1, ..., n; j = 1, ..., r\}$, $\frac{\sqrt{r}(\hat{\mu}_{opt}-\mu)}{\sigma/[\sum_{i=1}^n 1/\eta^2_{(i:n)}]^{1/2}} \xrightarrow{d} N(0,1)$ *as* $r \to \infty$. *Also,* $\frac{\sqrt{r}(\bar{X}_{rss}-\mu)}{\sigma[\sum_{i=1}^n \eta^2_{(i:n)}/n^2]^{1/2}} \xrightarrow{d} N(0,1)$ *as* $r \to \infty$.

Let $\bar{W} = \sum_{j=1}^r \bar{W}_{.j}/r$, and $\bar{V} = \sum_{j=1}^r \bar{V}_{.j}/r$. We can rewrite (2.12) as

$$\hat{\sigma}^2_u = \sigma^2 \frac{\sum_{j=1}^r \sum_{i=1}^n Y^2_{(i:n)j}/\eta^2_{(i:n)} - r(\sum_{i=1}^n 1/\eta^2_{(i:n)})\bar{W}^2}{(nr-1) + r\sum_{i=1}^n \nu^2_{(i:n)}/\eta^2_{(i:n)}}. \tag{3.16}$$

Lemma 3.2 $\bar{W}^2 \xrightarrow{a.s.} 0$ *as* $r \to \infty$.

Proof. It follows from SLLN that $\bar{W} \xrightarrow{a.s.} 0$ as $r \to \infty$, and hence from Slutsky's theorem (cf. Sen and Singer, (17), Theorem 3.4.2) we have, $\bar{W}^2 \xrightarrow{a.s.} 0$ as $r \to \infty$. ⋄

Lemma 3.3 *Assume that* $E|Y_{(i:n)}|^{2+\delta} < \infty$ *for* $i = 1, ..., n$ *and for some* $0 < \delta < 1$. *Then* $\frac{\sum_{j=1}^r \sum_{i=1}^n Y^2_{(i:n)j}/\eta^2_{(i:n)}}{(nr-1)+r\sum_{i=1}^n \nu^2_{(i:n)}/\eta^2_{(i:n)}} \xrightarrow{P} 1$ *as* $r \to \infty$.

Proof. Note that $\sum_{i=1}^n E|\frac{Y_{(i:n)j}}{\eta_{(i:n)}}|^{2+\delta} < \infty$ for $j = 1, ..., r$ and

$$r^{-1-\delta/2} \sum_{j=1}^r \sum_{i=1}^n E|\frac{Y_{(i:n)j}}{\eta_{(i:n)}}|^{2+\delta} = c(r) \xrightarrow{a.s.} \quad \text{as} \quad r \to \infty.$$

Define $U_j = \sum_{i=1}^n Y^2_{(i:n)j}/\eta^2_{(i:n)}, j = 1, ..., r$. Then

$$\begin{aligned} E|U_j|^{1+\delta/2} &= E|\sum_{i=1}^n Y^2_{(i:n)j}/\eta^2_{(i:n)}|^{1+\delta/2} \\ &\leq \sum_{i=1}^n E|Y_{(i:n)j}/\eta_{(i:n)}|^{2+\delta} < \infty, \qquad j = 1, ..., r \end{aligned}$$

and

$$r^{-1-\delta/2} \sum_{j=1}^r E|U_j - E(U_j)|^{1+\delta/2} \leq r^{-1-\delta/2} \sum_{j=1}^r \sum_{i=1}^n E|Y_{(i:n)j}/\eta_{(i:n)}|^{2+\delta} = c(r) \xrightarrow{a.s.} 0 \quad \text{as} \quad \mathrm{r} \to \infty.$$

Hence from the Markov WLLN (cf. Sen and Singer, (17), Theorem 2.3.7),

$$\frac{1}{r}\sum_{j=1}^r U_j - E\frac{1}{r}\sum_{j=1}^r U_j = \frac{1}{r}\sum_{j=1}^r \sum_{i=1}^n \frac{Y^2_{(i:n)j}}{\eta^2_{(i:n)}} - (n + \sum_{i=1}^n \frac{\nu^2_{(i:n)}}{\eta^2_{(i:n)}}) \xrightarrow{P} 0 \quad \text{as} \quad \mathrm{r} \to \infty.$$

Thus from Slutsky's theorem,

$$\frac{\sum_{j=1}^r \sum_{i=1}^n Y_{(i:n)j}^2/\eta_{(i:n)}^2}{(nr-1)+r\sum_{i=1}^n \nu_{(i:n)}^2/\eta_{(i:n)}^2} = \frac{\frac{1}{r}\sum_{j=1}^r \sum_{i=1}^n Y_{(i:n)j}^2/\eta_{(i:n)}^2}{(n-\frac{1}{r})+\sum_{i=1}^n \nu_{(i:n)}^2/\eta_{(i:n)}^2} \xrightarrow{P} 1 \quad \text{as} \quad \text{r} \to \infty.\diamond$$

Similar to (3.16), we can also rewrite (2.11)

$$\hat{\sigma}_{su}^2 = \sigma^2 \frac{\sum_{j=1}^r \sum_{i=1}^n Y_{(i:n)j}^2 - nr\bar{V}^2}{\{(nr-1)+\frac{1}{n}\sum_{i=1}^n \nu_{(i:n)}^2\}}. \tag{3.17}$$

Along the lines of Lemma 3.2 we can show that $\bar{V}^2 \xrightarrow{a.s.} 0$ as $r \to \infty$. Also along the lines of Lemma 3.3,

$$\begin{aligned} & \frac{1}{r}\sum_{j=1}^r \{\frac{1}{n}\sum_{i=1}^n Y_{(i:n)j}^2\} - \frac{1}{n}\sum_{i=1}^n (\nu_{(i:n)j}^2 + \eta_{(i:n)}^2) \\ = \; & \frac{1}{r}\sum_{j=1}^r \{\frac{1}{n}\sum_{i=1}^n Y_{(i:n)j}^2\} - 1 \xrightarrow{P} 0 \quad \text{as} \quad \text{r} \to \infty. \end{aligned}$$

Hence from Slutsky's theorem

$$\frac{\frac{1}{nr}\sum_{j=1}^r \sum_{i=1}^n Y_{(i:n)j}^2}{(1-\frac{1}{nr})+\frac{1}{r}\sum_{i=1}^n \nu_{(i:n)}^2} \xrightarrow{P} 1 \quad \text{as} \quad \text{r} \to \infty.$$

Combining Lemma 3.2 and Lemma 3.3, we have

Theorem 3.4 *Under assumptions of Lemma 3.3, $\hat{\sigma}_u^2 \xrightarrow{P} \sigma^2$ as $r \to \infty$. Also, $\hat{\sigma}_{su}^2 \xrightarrow{P} \sigma^2$ as $r \to \infty$.*

We point out that the assumption of Lemma 3.3 is usually satisfied by assuming that the density function f_0 is known and symmetric about 0, hence it is not a strong assumption.

Theorem 3.5 $\frac{\sqrt{r}(\hat{\mu}_{opt}-\mu)}{\hat{\sigma}_u/[\sum_{i=1}^n 1/\eta_{(i:n)}^2]^{1/2}} \xrightarrow{d} N(0,\ 1)$ *as* $r \to \infty$. *Also,* $\frac{\sqrt{r}(\bar{X}_{rss}-\mu)}{\hat{\sigma}_{su}[\sum_{i=1}^n \eta_{(i:n)}^2/n^2]^{1/2}} \xrightarrow{d} N(0,\ 1)$ *as* $r \to \infty$.

4 Asymptotic Relative Efficiency

In this section, we compare the different estimators of the location parameter μ using asymptotic relative efficiency (ARE). The ARE of $\hat{\mu}_{opt}$, with respect to $\bar{X}_{rss}$, is given by

$$
\begin{aligned}
ARE(\hat{\mu}_{opt}; \bar{X}_{rss}) &= \frac{\frac{\sigma^2}{rn^2}\sum_{i=1}^{n}\eta_{(i:n)}^2}{\frac{\sigma^2}{r}[\sum_{i=1}^{n} 1/\eta_{(i:n)}^2]^{-1}} \\
&= \frac{\frac{1}{n}\sum_{i=1}^{n}\eta_{(i:n)}^2}{\frac{n}{\sum_{i=1}^{n} 1/\eta_{(i:n)}^2}} \geq 1. \qquad (4.18)
\end{aligned}
$$

The asymptotic relative efficiency is bounded below by one because $\frac{1}{n}\sum_{i=1}^{n}\eta_{(i:n)}^2$ is the arithmetic mean $(A.M.)$ of $\{\eta_{(i:n)}^2\}$ and $n/\sum_{i=1}^{n} 1/\eta_{(i:n)}^2$ is the harmonic mean $(H.M.)$ of $\{\eta_{(i:n)}^2\}$ and $A.M. \geq H.M.$ Also, if $\bar{X}_{srs}$ denotes the mean of a simple random sample of size nr from f, then

$$
\begin{aligned}
ARE(\bar{X}_{rss}; \bar{X}_{srs}) &= \frac{\frac{\sigma^2}{rn}}{\frac{\sigma^2}{rn^2}\sum_{i=1}^{n}\eta_{(i:n)}^2} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n}\eta_{(i:n)}^2} \\
&= \frac{1}{1-\frac{1}{n}\sum_{i=1}^{n}\nu_{(i:n)}^2} \geq 1. \qquad (4.19)
\end{aligned}
$$

Thus $\hat{\mu}_{opt}$ is more efficient than $\bar{X}_{rss}$, which in turn is more efficient than $\bar{X}_{srs}$. Finally,

$$
ARE(\hat{\mu}_{opt}; \bar{X}_{srs}) = \frac{\frac{\sigma^2}{rn}}{\frac{\sigma^2}{r}[\sum_{i=1}^{n} 1/\eta_{(i:n)}^2]^{-1}} = \frac{\sum_{i=1}^{n} 1/\eta_{(i:n)}^2}{n} \qquad (4.20)
$$

and $ARE(\hat{\mu}_{opt};\, \bar{X}) = (H.M.)^{-1} \geq (A.M.)^{-1} = ARE(\bar{X}_{rss}; \bar{X}) \geq 1$.

Although the ranking is uniform across choices for f_0, the relative dominance depends on the underlying distribution. The AREs for $\hat{\mu}_{opt}$ and $\bar{X}_{rss}$ (with respect to $\bar{X}_{srs}$) are plotted in Figures 1-3 for various ranked set sample sizes (n). The estimators appear close in ARE for the normal distribution (Figure 1), and the dominance of $ARE(\hat{\mu}_{opt}; \bar{X}_{srs}) \geq ARE(\bar{X}_{rss}; \bar{X}_{srs})$ is more apparent in Figure 2 ($f_0 \sim$ double exponential) and Figure 3 ($f_0 \sim$ student-t distribution with 3 degrees of freedom).

The efficiency gained by using $\hat{\mu}_{opt}$ carries over to the construction of statistical tests and confidence intervals. For testing the null hypothesis $H_0 : \mu = \mu_0$ versus the alternative hypothesis $H_1 : \mu \neq \mu_0$, for large r, we gain test power by using the test statistic

$$
Z = \frac{\sqrt{r}(\hat{\mu}_{opt} - \mu_0)}{\hat{\sigma}_u/[\sum_{i=1}^{n} 1/\eta_{(i:n)}^2]^{1/2}}
$$

and rejecting H_0 if $|Z| > Z_{\alpha/2}$, where $Z_{\alpha/2}$ is the $100(1-\alpha/2)$th percentile of the standard normal distribution. The gain in power, of course, is a direct function of the ARE.

5 Concluding Remarks

It is well known that ranked set sampling improves estimation of a population mean in situations where items are easily ranked but not easily measured. Although past research has emphasized nonparametric methods for RSS, applications exist where the underlying distribution is partially known. We have generalized results for the normal distribution by Sinha, et al. (12) and Yu, et al. (16) to inference for location-scale families of distributions.

There are various ways to derive estimators of the scale and location parameter in this case. We have used Godambe and Kale's (14) estimating functions approach to obtain the optimal estimates of μ and σ based on a RSS from the location-scale families of distributions. According the MSE criteria, the unbiased estimator for σ based on $\hat{\mu}_{opt}$ outperformed other unbiased estimators of the scale parameter. In Section 3, we focused further on two unbiased estimators of σ^2 and proved their consistency. In Section 4, we showed the optimal estimator for the location parameter, $\hat{\mu}_{opt}$, is asymptotically more efficient than its competing estimators $\bar{X}_{rss}$ and $\bar{X}$. Figures 1-3 show that the dominance of $\hat{\mu}_{opt}$ over $\bar{X}_{rss}$ and $\bar{X}_{srs}$ depends on the underlying scale-location family of distributions.

Table 1. $\nu_{(i:n)}$ and $\eta^2_{(i:n)}$, $i = 1, ..., n$ for the standard normal distribution.

n	$\nu_{(i:n)}$ for $i = 1, ..., [\frac{n}{2}]$	$\eta^2_{(i:n)}$ for $i = 1, ..., [\frac{n+1}{2}]$
2	(-0.5642)	(0.6817)
3	(-0.8463)	(0.5595,0.4487)
4	(-1.0294,-0.2970)	(0.4917,0.3605)
5	(-1.1630,-0.4950)	(0.4475,0.3115,0.2868)
6	(-1.2672,-0.6418,-0.2015)	(0.4159,0.2796,0.2462)

Table 2. $\nu_{(i:n)}$ and $\eta^2_{(i:n)}$, $i = 1, ..., n$ for the student-t distribution with 3 degrees of freedom.

n	$\nu_{(i:n)}$ for $i = 1, ..., [\frac{n}{2}]$	$\eta^2_{(i:n)}$ for $i = 1, ..., [\frac{n+1}{2}]$
2	(-0.7425)	(1.4031)
3	(-1.1137)	(1.3345,0.7132)
4	(-1.3631,-0.3657)	(1.3375,0.5795)
5	(-1.5515,-0.6094)	(1.3578,0.5464,0.4061)
6	(-1.7027,-0.7955,-0.2374)	(1.3840,0.5409,0.3498)

Table 3. $\nu_{(i:n)}$ and $\eta^2_{(i:n)}$, $i = 1, ..., n$ for the student-t distribution with 4 degrees of freedom.

n	$\nu_{(i:n)}$ for $i = 1, ..., [\frac{n}{2}]$	$\eta^2_{(i:n)}$ for $i = 1, ..., [\frac{n+1}{2}]$
2	(-0.7047)	(1.2110)
3	(-1.0571)	(1.1302,0.6275)
4	(-1.2942,-0.3459)	(1.1129,0.5078)
5	(-1.4736,-0.5764)	(1.1133,0.4671,0.3696)
6	(-1.6182,-0.7510,-0.2273)	(1.1206,0.4502,0.3180)

Table 4. $\nu_{(i:n)}$ and $\eta^2_{(i:n)}$, $i = 1, ..., n$ for the student-t distribution with 5 degrees of freedom.

n	$\nu_{(i:n)}$ for $i = 1, ..., [\frac{n}{2}]$	$\eta^2_{(i:n)}$ for $i = 1, ..., [\frac{n+1}{2}]$
2	(-0.6779)	(1.0912)
3	(-1.0168)	(1.0006,0.5829)
4	(-1.2442,-0.3348)	(0.9705,0.4708)
5	(-1.4158,-0.5579)	(0.9593,0.4267,0.3502)
6	(-1.5537,-0.7261,-0.2216)	(0.9561,0.4047,0.3010)

Table 5. $\nu_{(i:n)}$ and $\eta^2_{(i:n)}$, $i = 1, ..., n$ for the double exponential distribution.

n	$\nu_{(i:n)}$ for $i = 1, ..., [\frac{n}{2}]$	$\eta^2_{(i:n)}$ for $i = 1, ..., [\frac{n+1}{2}]$
2	(-0.7495)	(1.4327)
3	(-1.1243)	(1.4083,0.6385)
4	(-1.3844,-0.3438)	(1.4334,0.5207)
5	(-1.5873,-0.5729)	(1.4604,0.5025,0.3512)
6	(-1.7548,-0.7500,-0.2188)	(1.4826,0.5080,0.3033)

Table 6. Relative Mean Squared Error (RMSE)with respect to $\bar{x}_{rss}$ for estimators of the scale parameter for the normal distribution.

n	r	$RMSE(\hat{\sigma}^2_{su})$	$RMSE(\hat{\sigma}^2_{u})$	$RMSE(\hat{\sigma}^2_{w1})$	$RMSE(\hat{\sigma}^2_{w2})$
3	2	1.302	0.791	0.922	0.902
3	5	1.098	0.898	1.124	1.100
3	10	1.046	0.932	1.197	1.171
3	20	1.022	0.949	1.227	1.201
5	2	1.251	0.785	0.880	0.821
5	5	1.089	0.867	0.997	0.932
5	10	1.042	0.896	1.035	0.966
5	20	1.021	0.906	1.057	0.985
10	2	1.179	0.769	0.905	0.781
10	5	1.038	0.821	0.973	0.841
10	10	1.033	0.838	0.997	0.861
10	20	1.017	0.845	1.007	0.869

Table 7. Relative Mean Squared Error (RMSE)with respect to $\bar{x}_{rss}$ for estimators of the scale parameter for the double exponential distribution.

n	r	$RMSE(\hat{\sigma}^2_{su})$	$RMSE(\hat{\sigma}^2_{u})$	$RMSE(\hat{\sigma}^2_{w1})$	$RMSE(\hat{\sigma}^2_{w2})$
3	2	1.529	0.164	0.583	0.176
3	5	1.182	0.133	0.616	0.109
3	10	1.094	0.096	0.601	0.108
3	20	1.049	0.062	0.576	0.201
5	2	1.407	0.116	0.631	0.120
5	5	1.155	0.082	0.656	0.087
5	10	1.080	0.054	0.653	0.057
5	20	1.041	0.032	0.648	0.035
10	2	1.258	0.059	0.722	0.060
10	5	1.104	0.034	0.750	0.035
10	10	1.053	0.020	0.758	0.020
10	20	1.027	0.011	0.762	0.012

Table 8. Relative Mean Squared Error (RMSE)with respect to $\bar{x}_{rss}$ for estimators of the scale parameter for the student-t distribution with ν=3 degrees of freedom.

n	r	$RMSE(\hat{\sigma}^2_{su})$	$RMSE(\hat{\sigma}^2_{u})$	$RMSE(\hat{\sigma}^2_{w1})$	$RMSE(\hat{\sigma}^2_{w2})$
3	2	1.444	0.194	0.520	0.185
3	5	1.132	0.203	0.522	0.186
3	10	1.062	0.200	0.517	0.180
3	20	1.029	0.209	0.505	0.185
5	2	1.311	0.133	0.625	0.135
5	5	1.102	0.134	0.626	0.136
5	10	1.047	0.137	0.616	0.139
5	20	1.023	0.133	0.620	0.135
10	2	1.165	0.059	0.756	0.059
10	5	1.062	0.055	0.758	0.057
10	10	1.030	0.054	0.756	0.057
10	20	1.024	0.053	0.758	0.057

Figure 1: Asymptotic Relative Efficiencies (4.19) graphed as dotted line, and (4.20) graphed as solid line, based on the normal(0,1)distribution.

Figure 2: Asymptotic Relative Efficiencies (4.19) graphed as dotted line, and (4.20) graphed as solid line, based on the double exponential distribution.

Figure 3: Asymptotic Relative Efficiencies (4.19) graphed as dotted line, and (4.20) graphed as solid line, based on the student-t distribution.

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