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Logics and the Sorites Paradox

Devin A. Neubrander

Abstract: Renderings of the ancient Sorites paradox in classical first-order logic, Lukasiewicz's three-valued first-order logic, and a Pavelka-style derivation system for Lukasiewicz's fuzzy first-order logic are derived. It can be seen that only in the last logic mentioned is the conclusion of the Sorites paradox false while the premises are true thus resolving the paradox.

Introduction

Modern binary symbolic logic, also known simply as classical logic, has its origin in the work of Aristotle. His collected writings on the subject of logic are found in the tome Oranon. According to Kosko, though George Boole and Gottlob Frege helped lay the ground work for what now constitutes binary symbolic logic, little has changed since Aristotle first described its principles well over 2,000 years ago [4].

The Principle of Bivalence and the closely related Law of the Excluded Middle form part of the foundation of classical logic. The Principle of Bivalence states that every declarative sentence is either true or false. The Law of the Excluded Middle says that P or $\sim P$ is a tautology. In classical logic, nothing is both not true and not false – nothing is vague or fuzzy.

Vagueness constitutes a major part of human experience. Consider a man thirty-five years old. Is he old or not old? An eighty-five year old would say that such a man is not old. A teenager might say that, no, this man is in fact old. Classical logic assigns a truth value of either true or false to every declarative sentence - in this case, "A thirty-five year old man is old." Yet this need not hold in real life. Only a proper subset of all declarative sentences (some members of which are "All dogs are not cats," and "Every time a match is struck, it catches fire") is either only true or only false.

The need for a logic that deals with vagueness existed from antiquity. This need became apparent with the realization that classical logic cannot satisfactorily resolve certain anciently proposed paradoxes. One of the most famous and most confounding paradoxes: the Sorites paradox.

Eubulides of Miletus originally proposed the Sorites paradox. A contemporary of Aristotle, his main contribution to logic arises from his paradoxes [5]. The Greek Zeno and even Buddha proposed similar puzzles prior to Eubulides [4]. The word sorites in Greek simply means 'heap'. So the Sorites Paradox literally means the paradox of the heap.

Sorites Paradox [2]

1) A grain of sand does not constitute a heap.
2) Adding a grain of sand to a non-heap does not make it a heap.
3) Therefore no amount of sand constitutes a heap

Many formulations of this same paradox exist. The Bald Man puzzle is another common rendering of the paradox. This puzzle says that a man with one hair on his head is bald, and a man with two hairs on his head is bald. Therefore a man with many hairs on his head is bald [6]. Bergmann gives the general form of a Sorites paradox.

Sorites Form [1]

> Premise 1: x is T (where x is something of which T is clearly true).
> Premise 2: Some type of small change to a thing that is T results in something that is also T (called the Principal of Charity premise).

The Sorites paradox poses a problem, as will be seen in section one, since in classical logic, although preposterous, the paradox is valid, indicating of a weakness in binary logic. But if classical logic cannot give a result that can be accepted from the premises of the Sorites paradox, what kind of logic can?

This question became significant in the 20th century. Mathematicians, from the time of Aristotle until the 1920's, considered Binary logic the only valid or reasonable logic. Then, in the 1920's, the Polish logician, Jan Lukasiewicz, developed multi-valued and fuzzy logic systems [4]. In 1965 that fuzzy logic began to take flight in the mathematical community. On this year Lotfi Zadeh presented a paper on fuzzy sets. He both originated fuzzy set theory and contributed largely to it for over thirty years [3]. Other highly important contributors to the field include Joseph Goguen, who generalized Zadeh's concept of fuzzy sets, relating them to algebra, and Jan Pavelka, who created a complete and consistent axiomatic system for propositional fuzzy logic with graded rules of inference [1]. Pavelka's work made possible the resolution of the Sorites paradox.

In his text, An Introduction to Many-Valued and Fuzzy Logics, Merrie Bergmann explains many facets of modern logics and analyzes a specific rendering of the Sorites paradox under these logics' differing derivation systems. He does not, however, fully show the harmonization of the Sorites paradox using a fuzzy logic. It is the goal of this work to do so.

Classical First Order Logic

Simple sentences such as "John is a teenager" are symbolized by uppercase roman letters along with logical connectives $\sim$, $\wedge$, $\vee$, $\rightarrow$, and $\longleftrightarrow$ make up the language of classical propositional logic. The simple sentences are called atomic formulas. The rules for forming formulas in classical propositional logic are as follows.

Rules [1]

> 1) Every uppercase roman letter is a formula.
> 2) If P is a formula, the $\sim P$ is as well.
> 3) If P and Q are formulas, so are $(P \wedge Q)$, $(P \vee Q)$, $(P \rightarrow Q)$, and $(P \longleftrightarrow Q)$.

Formulas made using one or more connectives are called compound formulas. The truth-values of compound formulas are a function of the truth-values of their constituent atomic formulas. In classical logic, true and false are the only truth values. Truth tables describe truth value operations.

Truth Tables [1]

P	$\wedge$	Q	P	$\vee$	Q	P	$\rightarrow$	Q	P	$\leftrightarrow$	Q
T	T	T	T	T	T	T	T	T	T	T	T
T	F	F	T	T	F	T	F	F	T	F	F
F	F	T	F	T	T	F	T	T	F	F	T
F	F	F	F	F	F	F	T	F	F	T	F

P	$\sim P$
T	F

A truth-value assignment is an assignment of truth values to all atomic formulas considered. Tautologies are formulas that are true on all truth-value assignments, and contradictions are formulas that are false on all truth-value assignments. Two formulas are equivalent if they have the same truth-value on each truth-value assignment. A Set T of formulas entails P if whenever all formulas in T are true, P is true. An argument consists of premises (formulas) and a formula called the conclusion. An argument is valid if its premises entail its conclusion. An example of traditional argument form follows.

Example 1 [1]

$$\frac{\begin{array}{c} J \rightarrow C \\ J \end{array}}{C}$$

$J \rightarrow C$ and J are premises and C is the conclusion.

An axiomatic derivation system in logic is a set of formulas or axioms along with a set of rules which can be used to derive new formulas from previous ones. The most basic system (in binary logic) is classical logic axiomatic system (CLA). First, an axiom schema must be defined. An axiom schema stands for all formulas (instances) that have the overall form shown by the schema. The plural of axiom schema is axiom schemata. Every instance of an axiom schema expresses an axiom. CLA contains three axiom schemata –

$CL1.$ $P \rightarrow (Q \rightarrow P)$
$CL2.$ $(P \rightarrow (Q \rightarrow R)) \rightarrow ((P \rightarrow Q) \rightarrow (P \rightarrow R))$
$CL3.$ $(\neg P \rightarrow \neg Q) \rightarrow (Q \rightarrow P)$

– and the single inference rule Modus Ponens (MP): from P and $P \rightarrow Q$, infer Q [1]. An example instance of $CL1$: $(A \wedge B) \rightarrow ((P \vee Q) \rightarrow (A \wedge B))$. In this instance, $(A \wedge B)$ has been substituted for P and $(P \vee Q)$ has been substituted for Q. Each axiom schemata is a tautology. This is why each schemata expresses a set of axioms.

A derivation is a sequence of formulas. The sequence begins with assumption formulas. These are followed by formulas which are either instances of an axiom schema, or formulas that can be derived from earlier formulas in the sequence using the derivation rule MP. The following is an example. It is the derivation of $A \rightarrow B$ from the assumption $\neg A$.

Derivation [1]

1	$\sim A$	Assumption
2	$\sim A \rightarrow (\sim B \rightarrow \sim A)$	$CL1$ instance
3	$\sim B \rightarrow \sim A$	MP
4	$(\sim B \rightarrow \sim A) \rightarrow (A \rightarrow B)$	$CL3$ instance
5	$A \rightarrow B$	MP

A theorem is defined to be any formula which can be derived without the use of assumptions. A derivation that does not use assumptions is called a proof [1].

Classical first-order logic is also called predicate logic. It uses the same connectives found in propositional logic, but it analyzes simple sentences by breaking them down into terms and predicates. Uppercase roman letters are used to symbolize predicates, lowercase letters a through t are used as constants, and lowercase letters u through z are variables. Constants and variables are called terms. The predicate applies to the terms. Terms are subjects. The arity of a predicate is the number of terms it applies to. There are two standard quantifiers – the universal and existential – in predicate logic. Variables (never constants) are used along with quantifiers to mark quantified positions relative to predicates. In addition to the three rules for forming formulas in propositional logic, there are two more in first-order logic:

1. Every predicate of arity n followed by n terms is a formula (atomic).
2. If P is a formula, so are $(\forall x) P$ and $(\exists x) P$.

All formulas other than ones formed in accordance with clause 1 are compound formulas.

One symbolization of the statement, "John runs" is Rj. John is a single specific person (or subject) necessitating the placement of the constant (j) after the predicate (R). One

symbolization of the statement, "John dates Sally," is Djs. A formula Nab means that someone or something (a) has something to do with or to (N) someone or something (b). The formula $(\forall x)\,Px$ can mean "Every man wears pants." In this case the predicate P stands for wears. The formula $(\exists y)\,Qyx$ can mean "There exists a female teenager who does not own a cell phone." In this case the domain of y is the set of all female teenagers. x stands for any member of the set of all cell phones, and the predicate Q stands for "does not own." Classical first-order logic symbolizations of some sentences follow.

Sentence Symbolizations

> If Sara is a cell phone owner, then all cell phone owners call her.
> $Psx \rightarrow (\forall x)Cxs$
> Everyone who lives in America gets fat.
> $(\forall x)\,((Px \wedge Lxa) \rightarrow Fx)$
> Everyone loved everyone who rock climbed.
> $(\forall x)\,(Px \rightarrow (\forall y)\,((Py \wedge Ry) \rightarrow Lxy))$

Sentences often necessitate restructuring prior to symbolization. Restructuring the sentence, "Everyone loved everyone who rock climbed," as "For every x, x being a person implied that for every y, y being a person and y being someone who rock climbed implied that x loved y" makes possible the creation of a symbolic expression defined equivalent to it.

Before discussing an axiomatic derivation system for first-order logic, the characteristics of a good derivation system are discussed. Why must a derivation system for predicate logic differ from propositional logic's system? Why is CLA formed the way it that it is? In answering these questions, the terms sound and complete are defined. A system is usually considered good only if it is both sound and complete. Soundness means that all theorems are tautologies and whenever a formula P is derivable from a set Γ of formulas, then P is entailed by the set Γ. A system is complete if every tautology of the given logic is a theorem in the system and whenever a set Γ of formulas entails a formula P, P is also derivable from Γ within the system. CLA is complete and sound for classical propositional logic. Adding to CLA two axiom schemata and one rule of inference produces a sound and complete axiomatic system for predicate logic. Bergmann calls this system $CL\forall A$ [1]. The two added axiom schemata are:

$CL\forall 1.$ $(\forall x)\,(P \rightarrow Q) \rightarrow (P \rightarrow (\forall x)\,Q)$ where P is a formula in which x does not occur free

$CL\forall 2.$ $(\forall x)\,P \rightarrow P\,(a/x)$ where a is any individual constant and the expression $P\,(a/x)$ means: expression $P\,(a/x)$ means: the result of substituting the constant a for the variable x wherever x occurs free in P.

$P\,(a/x)$ is called the substitution instance of P. The added rule is Universal Generalization (UG): From $P\,(a/x)$ infer $(\forall x)\,P$ where x is any individual variable, provided that no assumption contains the constant a and that P itself does not contain the constant a. An example of an assumption that contains the constant a follows. Let a stand for my brother and Pa symbolize the sentence, "My brother is buff." The conclusion that everyone is buff, $(\forall x)\,Px$, from the fact that my brother is, Pa, is erroneous. An inference of $(\forall x)\,Hxa$ from the formula Haa exemplifies a violation of the second part of the rule for correct usage of UG. Haa means that everything stands in relation H to itself. $(\forall x)\,Hxa$ means that everything stands in relation H to something designated as a. In this case, let $P = Hxa$. Thus P contains a. Intuitively then, the UG inference is incorrect.

An axiomatic derivation system does not contain all the axioms (tautologies) and rules that hold in the system's logic. But, if complete, it provides what is necessary to derive all the tautologies and rules of the system. In addition to $CL\forall A$, only two derived rules and three derived axioms are needed for our Sorites argument derivation in classical first-order logic.

Derived Axioms [1]

$CL\forall D1.$ $P \rightarrow P$
$CL\forall D2.$ $P \rightarrow{\sim}{\sim} P$
$CL\forall D3.$ $(P \rightarrow Q) \rightarrow (\sim Q \rightarrow \sim P)$

Derived Rules [1]

Universal Instantiation (UI).   From $(\forall x)\,P$ infer $P\,(a/x)$
(a direct result of $CL\forall 2$).

Transposition $(TRAN)$.   From $P \;\rightarrow\; (Q \rightarrow R)$ infer $Q \;\rightarrow\; (P \rightarrow R)$.

Sorites Height Argument in CL∀A

John is $6'7''$ tall. Don, Juan, and George are his siblings. John's height exceeds Don's by $\frac{1}{8}''$, Don's height exceeds Juan's by $\frac{1}{8}''$, and Juan's height exceeds George's by $\frac{1}{8}''$. Their heights exceed $6'6''$ and are universally regarded as tall. Suppose John has one hundred and ninety-two brothers, all but the youngest having exactly one brother $\frac{1}{8}''$ shorter than himself. Are all the brothers tall? Some simple (yet tedious) arithmetic shows the shortest brother's height - $4'7''$. Almost always considered short for a man, the height $4'7''$ seems tall in the Sorites argument.

Sorites Height Argument

$6'7''$ is tall.
Any height that is $\frac{1}{8}''$ less than a tall height is tall.
_____
$4'7''$ is tall.

This argument excludes an implicit premise: the height $6'7''$ becomes the height $4'7''$ due to repeated $\frac{1}{8}''$ reductions. The symbolized argument (stating the implicit premise explicitly) using the language of classical first-order logic follows.

$$Ts_1$$
$$(\forall x)\,(\forall y)\,((Tx \wedge Eyx) \rightarrow Ty)$$
$$Es_2s_1$$
$$Es_3s_2$$
$$Es_4s_3$$
$$\vdots$$
$$Es_{193}s_{192}$$
_____
$$Ts_{193}$$

In this symbolization, T means "is tall." E means "is $\frac{1}{8}''$ less than."

$$\{s_i \mid i \in I,\; 1 \le i \le 193\}$$

is the set of heights $6'7''$ down to $4'7''$.

$$(\forall x)\,(\forall y)\,((Tx \wedge Eyx) \rightarrow Ty)$$

is the Principal of Charity premise. With the premises of the Sorites height argument written in this form, the conclusion, Ts_{193}, is derivable in $CL\forall A$.

Table 1: Truth Tables [1]

P	∧	Q	P	∨	Q	P	→	Q	P	↔	Q	P	~P
T	T	T	T	T	T	T	T	T	T	T	T		
T	F	F	T	T	F	T	F	F	T	F	F		
F	F	T	F	T	T	F	T	T	F	F	T	P	~P
F	F	F	F	F	F	F	T	F	F	T	F	T	F
N	N	N	N	N	N	N	T	N	N	T	N	F	T
N	N	T	N	T	T	N	T	T	N	N	T	N	N
N	F	F	N	N	F	N	N	F	N	N	F		
T	N	N	T	T	N	T	N	N	T	N	N		
F	F	N	F	N	N	F	T	N	F	N	N		

Sorites Derivation $(CL\forall A)$ [1]

1	Ts_1	Assumption
2	$(\forall x)(\forall y)((Tx \wedge Eyx) \rightarrow Ty)$	Assumption
3	Es_2s_1	Assumption
4	Es_3s_2	Assumption
	$\vdots$	
194	$Es_{193}s_{192}$	Assumption
195	$(\forall y)(\sim(Ts_1 \rightarrow\sim Eys_1) \rightarrow Ty)$	UI
196	$\sim(Ts_1 \rightarrow\sim Es_2s_1) \rightarrow Ts_2$	UI
197	$(Ts_1 \rightarrow\sim Es_2s_1) \rightarrow (Ts_1 \rightarrow\sim Es_2s_1)$	$CL\forall D1$
198	$Ts_1 \rightarrow ((Ts_1 \rightarrow\sim Es_2s_1) \rightarrow\sim Es_2s_1)$	$TRAN$
199	$(Ts_1 \rightarrow\sim Es_2s_1) \rightarrow\sim Es_2s_1$	MP
200	$((Ts_1 \rightarrow\sim Es_2s_1) \rightarrow\sim Es_2s_1) \rightarrow$ $(\sim\sim Es_2s_1 \rightarrow\sim(Ts_1 \rightarrow\sim Es_2s_1))$	$CL\forall D3$
201	$Es_2s_1 \rightarrow\sim(Ts_1 \rightarrow\sim Es_2s_1)$	MP
202	$Es_2s_1 \rightarrow\sim\sim Es_2s_1$	$CL\forall D2$
203	$\sim\sim Es_2s_1$	MP
204	$\sim(Ts_1 \rightarrow\sim Es_2s_1)$	MP
205	Ts_2	MP
206 − 215	Exactly like steps $195 - 204$ except for the substitution of s_2 for s_1 and s_3 for s_2	
216	Ts_3	
$\vdots$	Repeating $195 - 205$ with appropriate	
$\vdots$	substitutions we end with	
2090	Ts_{193}	

Thus the Sorites height argument premises entail the idea that $4'7''$ is tall in classical first-order logic. The Sorite paradox remains unharmonized.

Polish logician Jan Lukasiewicz's created Lukasiewicz's three-valued first-order logic (abbreviated L_3). The term three-valued comes from the fact that the Principle of Bivalence is dropped and sentences are allowed to have one of three different truth values: true (T), false (F), and neutral (N) in this logic. Thus, this logic addresses vagueness. The connectives in it are all the same as those in classical logic, but the truth tables describing their meaning are expanded to incorporate the extra truth value N.

Table 1 shows that tautologies (such as $P \to P$) exist in L_3. Also, whenever a connective combines formulas with classical truth-values (true and false), the compound formula which results has the same truth-value that it does in classical logic. For this reason, L_3 is a normal system. L_3 is also uniform. This means that whenever the truth-value of a formula formed with its connective is uniquely determined by the truth-value of one of its constituent formulas in classical logic, then the truth-value of the formula formed with its connective is also uniquely so determined in L_3. For example, a false conjunct guarantees the falsehood of a conjunction in classical logic; this is also the case in L_3. Thus conjunction is uniform in L_3. Similarly, the other connectives are uniform as well.

The Normality Lemma clarifies something expected at this point. It says that because L_3 is normal, a classical truth-value assignment behaves exactly as it does in classical logic - every true formula on that assignment in L_3 is also true on that assignment in classical logic, and every false formula on that assignment in L_3 is also false on that assignment in classical logic.

With the concept of more than two truth-values comes the notion of a formula that is either never false or never true but not necessarily either always true or always false. Such formulas are called quasi-tautologies and quasi-contradictions. A quasi-tautology in L_3 is a formula which always has truth-values T or N. A quasi-contradiction in L_3 is a formula which only has truth-values F and N. Also, a concept of degree-entailment arises. To define it, the three truth-values are ranked as $T \geq N \geq F$ - a commonly accepted standard. Now a set of formulas Γ degree-entails a formula P if $P's$ value can never be less than the least value of the formulas in Γ. For example, if all the formulas in Γ have the values T or N, then P must have either the value T or N.

Just as an axiom schemata and a few rules of inference defined an axiomatic derivation system which was both sound and complete for classical first-order logic ($CL\forall A$), there is also a similar axiom schemata (given below) that defines a sound and complete axiomatic derivation system for Lukasiewicz's first-order three-valued logic. Bergmann calls it $L_3\forall A$ [1].

$L_3\forall 1.$	$P \to (Q \to P)$
$L_3\forall 2.$	$(P \to Q) \to ((Q \to R) \to (P \to R))$
$L_3\forall 3.$	$(\sim P \to \sim Q) \to (Q \to P)$
$L_3\forall 4.$	$((P \to \sim P) \to P) \to P$
$L_3\forall 5.$	$(\forall x)(P \to Q) \to (P \to (\forall x)Q)$ where P is a formula in which x does not occur free
$L_3\forall 6.$	$(\forall x)P \to P(a/x)$ where a is any individual constant and the expression $P(a/x)$ means: the result of substituting the constant a for the variable x wherever x occurs free in P.

The derivation rules are MP and UG (unaltered).

A derivation of the Sorites argument requires the use of an additional derived inference rule, Conjunction Introduction (CI): from P and Q, infer $P \wedge Q$. Figure 1 shows the derivation of Sorites height argument in $L_3\forall A$. This proves the validity the Sorites height argument in $L_3\forall A$.

Polish logician Jan Lukasiewicz's created Lukasiewicz's three-valued first-order logic (abbreviated L_3). The term three-valued comes from the fact that the Principle of Bivalence is dropped and sentences are allowed to have one of three different truth values: true (T), false (F), and neutral (N) in this logic. Thus, this logic addresses vagueness. The connectives in it are all the same as those in classical logic, but the truth tables describing their meaning are expanded to incorporate the extra truth value N.

Table 1 shows that tautologies (such as $P \rightarrow P$) exist in L_3. Also, whenever a connective combines formulas with classical truth-values (true and false), the compound formula which results has the same truth-value that it does in classical logic. For this reason, L_3 is a normal system. L_3 is also uniform. This means that whenever the truth-value of a formula formed with its connective is uniquely determined by the truth-value of one of its constituent formulas in classical logic, then the truth-value of the formula formed with its connective is also uniquely so determined in L_3. For example, a false conjunct guarantees the falsehood of a conjunction in classical logic; this is also the case in L_3. Thus conjunction is uniform in L_3. Similarly, the other connectives are uniform as well.

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Just as an axiom schemata and a few rules of inference defined an axiomatic derivation system which was both sound and complete for classical first-order logic $(CL\forall A)$, there is also a similar axiom schemata (given below) that defines a sound and complete axiomatic derivation system for Lukasiewicz's first-order three-valued logic. Bergmann calls it $L_3\forall A$ [1].

$L_3\forall 1.$ $\quad P \rightarrow (Q \rightarrow P)$
$L_3\forall 2.$ $\quad (P \rightarrow Q) \rightarrow ((Q \rightarrow R) \rightarrow (P \rightarrow R))$
$L_3\forall 3.$ $\quad (\sim P \rightarrow \sim Q) \rightarrow (Q \rightarrow P)$
$L_3\forall 4.$ $\quad ((P \rightarrow \sim P) \rightarrow P) \rightarrow P$
$L_3\forall 5.$ $\quad (\forall x)(P \rightarrow Q) \rightarrow (P \rightarrow (\forall x) Q)$ where
$\quad\quad\quad\quad$ P is a formula in which x does not occur free
$L_3\forall 6.$ $\quad (\forall x) P \rightarrow P(a/x)$ where a is any individual
$\quad\quad\quad\quad$ constant and the expression $P(a/x)$ means:
$\quad\quad\quad\quad$ the result of substituting the constant a for the
$\quad\quad\quad\quad$ variable x wherever x occurs free in P.

The derivation rules are MP and UG (unaltered).

A derivation of the Sorites argument requires the use of an additional derived inference rule, Conjunction Introduction (CI): from P and Q, infer $P \wedge Q$. Figure 1 shows the derivation of Sorites height argument in $L_3\forall A$. This proves the validity the Sorites height argument in $L_3\forall A$.

In three-valued logic, Ts_n evaluates to true for n such that $s_n \in \left[5'11'', 6'7''\right]$, Ts_n evaluates to neutral for n such that $s_n \in [5'3'', 5'11'']$, and Ts_n evaluates to false for n such that $s_n \in \left[4'7'', 5'3''\right]$. Sets of heights must be defined as tall, neither tall nor not tall, and not tall. There are clear cut-off points between the sets of heights which are considered tall, neither tall nor not tall, and not tall. This seems intuitively incorrect. In everyday life, no clear distinction exists between someone who is, say, only $5'10\frac{7}{8}''$ tall and someone who is $5'11''$ tall. A smooth continuum of heights exists among people. A need exists for a logic that reflects this. Infinitely many degrees of tallness accurately describe height. Three-valued logic deals with only three degrees of tallness.

To what degree is someone tall? Let the degree to which some height may be considered a part of a set be indicated with values between 0 and 1 inclusive. The set important to this discussion is the set of tall heights. The height $4'7''$ is a member of this set to degree 0. The height $6'7''$ is a member to degree 1. The height $5'3''$ is a member to degree 0.333 since it lies one-third of the way from $4'7''$ to $6'7''$.

This describes a fuzzy set, which is: a set defined by a function that assigns to each entity in its domain a value between 0 and 1 inclusive, representing the entity's degree of

Figure 1: Sorites Derivation [$(L_3 \forall A)$] [1]

1	Ts_1	*Assumption*
2	$(\forall x)(\forall y)((Tx \wedge Eyx) \to Ty)$	*Assumption*
3	Es_2s_1	*Assumption*
4	Es_3s_2	*Assumption*
	$\vdots$	
194	$Es_{193}s_{192}$	*Assumption*
195	$(\forall x)(\forall y)((Tx \wedge Eyx) \to Ty) \to$ $(\forall y)((Ts_1 \wedge Eys_1) \to Ty)$	$L_3\forall 6.$
196	$(\forall y)((Ts_1 \wedge Eys_1) \to Ty)$	MP
197	$(\forall y)((Ts_1 \wedge Eys_1) \to Ty) \to$ $(\forall y)((Ts_1 \wedge Es_2s_1) \to Ts_2)$	$L_3\forall 6.$
198	$(Ts_1 \wedge Es_2s_1) \to Ts_2$	MP
199	$Ts_1 \wedge Es_2s_1$	CI
200	Ts_2	MP
$201-205$	Exactly like steps $195-200$ except for the substitution of s_2 for s_1 and s_3 for s_2	
206	Ts_3	
$\vdots$	Repeating $195-200$ with appropriate	
$\vdots$	substitutions we end with	
1346	Ts_{193}	

membership in the set. In our example, the degree of membership of a height in the set of tall heights corresponds to the degree of truth of the statement that the height is tall. A logic in which sentences may have an infinite number of degrees of truth (values between 0 and 1) is an infinite-valued logic. When degrees of truth are assigned based on fuzzy sets in a logical system, the system is called a fuzzy logic.

Lukasiewicz's fuzzy propositional logic ($Fuzzy_L$ or F_L) provides a foundation for the rest of this work. Let V be a function which assigns fuzzy truth-values between 0 and 1 inclusive to atomic formulas of the language. Thus if P is a formula in the language, $V(P) \in [0,1]$. Instead of truth tables, the following expressions describe the connectives.

Truth Formulas [1]

$$V\left(\sim P\right) = 1 - V\left(P\right)$$
$$V\left(P \wedge Q\right) = \min\left(V\left(P\right), V\left(Q\right)\right)$$
$$V\left(P \vee Q\right) = \max\left(V\left(P\right), V\left(Q\right)\right)$$
$$V\left(P \rightarrow Q\right) = \min\left(1, 1 - V\left(P\right) + V\left(Q\right)\right)$$
$$V\left(P \leftrightarrow Q\right) = \min\left(1, 1 - V\left(P\right) + V\left(Q\right), 1 - V\left(Q\right) + V\left(P\right)\right)$$

A tautology represents a formula which always has the value 1 and a contradiction is a formula which always has the value 0. A set of formulas Γ entail P if P has the value 1 whenever all the formulas in Γ have the value 1. A formula is an n- tautology if and only if n is the greatest lower bound of all the truth-values the formula can have. A formula is an n- contradiction if n is the least upper bound of the set of truth-values that the formula can have. Also, a set of formulas Γ degree-entails a formula P if on every fuzzy truth-value assignment, the value of P is greater than or equal to the greatest lower bound of the values of the members of Γ on that assignment. An argument is degree-valid if the premises degree-entail the conclusion.

The following axiom schemata along with the inference rule, MP, make up the fuzzy

Lukasiewicz axiomatic system. It is called $F_L A$ [1].

$$F_L 1. \quad P \to (Q \to P)$$
$$F_L 2. \quad (P \to Q) \to ((Q \to R) \to (P \to R))$$
$$F_L 3. \quad (\sim P \to \sim Q) \to (Q \to P)$$
$$F_L 4. \quad ((P \to Q) \to Q) \to ((Q \to P) \to P)$$

A Pavelka-Style Derivation System for Fuzzy$_{L\forall}$

The Pavelka-style axiomatic derivation system gives the added expressive power needed to do derivations to establish not only tautologousness and validity but also n-tautologousness and degree-validity. Not an entirely new system, $F_L A$ simply adds to F_L just as classical first-order logic adds to classical propositional logic giving it more expressive power.

Augmenting the language for F_L in the Pavelka-style system are special atomic formulas m, n, and p which denote rational truth-values m, n, and p in the unit interval. When an expression involves one of these formulas, the symbole $\to$ is not a conditional symbol. Given a truth value p, the formula $p \to Q$ means Q has at least the value p and the formula $Q \to p$ means Q has at most the value p. The following three "truth tables" give examples of this.

Example 2

v	P	$v \to P$
$\frac{1}{2}$	1	1
1	$\frac{1}{2}$	0
0	0	1

Example 3

t	Q	$t \to Q$
$\frac{1}{4}$	$\frac{1}{8}$	0
1	1	1
0	$(\forall x)\, \{x : x \in \mathbb{Q} \land x \in [0,1]\}$	1

Example 4

m	R	$R \to m$
$\frac{1}{1000}$	$\frac{1}{1001}$	1
0	$\frac{1}{2}$	0
1	$(\forall x)\, \{x : x \in \mathbb{Q} \land x \in [0,1]\}$	1

$p \leftrightarrow Q$ means Q has exactly the value p. The pair $[Q, p]$ where Q is any formula and p is any rational truth value in the unit interval is called a graded formula. The value p in the graded formula indicates that Q has at least the value p. Graded formulas make up every derivation in a Pavelka-style system.

Graded formulas make possible the creation of a system powerful enough to satisfactorily deal with the troublesome Sorites paradox (height scenario). These formulas, called the Pavelka-style axiomatic derivation system for first-order Lukasiewicz's fuzzy logic ($F_L \forall PA$) [1], follow.

$F_L \forall P1.$ $[P \rightarrow (Q \rightarrow P), 1]$

$F_L \forall P2.$ $[(P \rightarrow Q) \rightarrow ((Q \rightarrow R) \rightarrow (P \rightarrow R)), 1]$

$F_L \forall P3.$ $[(\sim P \rightarrow \sim Q) \rightarrow (Q \rightarrow P), 1]$

$F_L \forall P4.$ $[((P \rightarrow Q) \rightarrow Q) \rightarrow ((Q \rightarrow P) \rightarrow P), 1]$

$F_L \forall P5.$ All graded formulas $[(m \rightarrow n) \rightarrow p, 1]$
where $p = \min(1, 1 - m + n)$

$F_L \forall P6.$ All graded formulas $[p \rightarrow (m \rightarrow n), 1]$
where $p = \min(1, 1 - m + n)$

$F_L \forall P7.$ All graded formulas $[\sim m \rightarrow p, 1]$ where $p = 1 - m$

$F_L \forall P8.$ All graded formulas $[p \rightarrow \sim m, 1]$ where $p = 1 - m$

$F_L \forall P9.$ $[m, m]$ for any rational value m in the unit interval

$F_L \forall P10.$ $[(\forall x)(P \rightarrow Q) \rightarrow (P \rightarrow (\forall x) Q), 1]$ where P is a
formula in which x does not occur free

$F_L \forall P11.$ $[(\forall x) P \rightarrow P(a/x), 1]$ where a is any individual
constant and the expression $P(a/x)$ means:
the result of substituting the constant a for the
variable x wherever x occurs free in P.

The common rules of inference are altered to deal with graded formulas and one new rule, Truth-value Constant Introduction (TCI), is added. These rules follow.

MP From $[P, m]$ and $[P \rightarrow Q, n]$, infer $[Q, p]$,
where $p = \max(0, m + n - 1)$

TCI From $[P, m]$ infer $[m \rightarrow P, 1]$, where m
is the atomic formula that denotes the value m

UG From $[P(a/x), m]$ infer $[(\forall x) P, m]$ where x is
any individual variable, provided that no
assumption contains the constant a and that P
does not contain the constant a.

In addition to these rules, the $F_L \forall PA$ Sorites derivation requires one derived rule, Weak Conjunction Introduction (WCI). This rule says that from $[P, m]$ and $[Q, n]$ infer $[P \wedge Q, p]$ where $p = \min(m, n)$.

Derivation of the Sorites Heights Argument in $F_L \forall PA$

Using a Pavelka-style axiomatic derivation system to express the premises of the Sorites height argument results in the ability to express the precise truth of the Principal of Charity premise using a graded formula. The truth of this premise evaluates to not exactly 1, but to a little less, because lessening a height by $\frac{1}{8}''$ results in a height that is less tall instead of just "tall." In deriving the Sorites height conclusion (and it's associated truth value) in $F_L \forall PA$, two derivations are necessary. It must first be shown that

$$[Ts_1, 1]$$
$$\left[(\forall x)\,(\forall y)\,((Tx \wedge Eyx) \to Ty),\, \tfrac{191}{192} \right]$$
$$[Es_2 s_1, 1]$$
$$[Es_3 s_2, 1]$$
$$\cdots$$
$$\underline{[Es_{193} s_{192}, 1]}$$
$$[Ts_{193}, 0]$$

and then that

$$[Ts_1 \to 1, 1]$$
$$\left[(\forall x)\,(\forall y)\,((Tx \wedge Eyx) \to Ty) \to \tfrac{191}{192},\, 1 \right]$$
$$[Es_2 s_1 \to 1, 1]$$
$$[Es_3 s_2 \to 1, 1]$$
$$\vdots$$
$$\underline{[Es_{193} s_{192} \to 1, 1]}$$
$$[Ts_{193} \to 0, 1]$$

If these derivations are done, then the assumptions

$$Ts_1 \leftrightarrow 1$$
$$(\forall x)(\forall y)((Tx \wedge Eyx) \rightarrow Ty) \leftrightarrow \tfrac{191}{192}$$
$$Es_2s_1 \leftrightarrow 1$$
$$Es_3s_2 \leftrightarrow 1$$
$$\vdots$$
$$Es_{193}s_{192} \leftrightarrow 1$$

entail the desired conclusion, $Ts_{193} \leftrightarrow 0$.

Sorites Derivation Part I ($F_L \forall PA$) [1]

1	$[Ts_1, 1]$	*Assumption*
2	$\left[(\forall x)(\forall y)((Tx \wedge Eyx) \rightarrow Ty), \tfrac{191}{192}\right]$	*Assumption*
3	$[Es_2s_1, 1]$	*Assumption*
4	$[Es_3s_2, 1]$	*Assumption*
	...	
194	$[Es_{193}s_{192}, 1]$	*Assumption*
195	$[(\forall x)(\forall y)((Tx \wedge Eyx) \rightarrow Ty) \rightarrow$ $(\forall y)((Ts_1 \wedge Eys_1) \rightarrow Ty), 1]$	$F_L \forall P9$
196	$\left[(\forall y)((Ts_1 \wedge Eys_1) \rightarrow Ty), \tfrac{191}{192}\right]$	MP
197	$[(\forall y)((Ts_1 \wedge Eys_1) \rightarrow Ty) \rightarrow$ $((Ts_1 \wedge Es_2s_1) \rightarrow Ts_2), 1]$	$F_L \forall PA$
198	$\left[(Ts_1 \wedge Es_2s_1) \rightarrow Ts_2, \tfrac{191}{192}\right]$	MP
199	$[Ts_1 \wedge Es_2s_1, 1]$	WCI
200	$\left[Ts_2, \tfrac{191}{192}\right]$	MP
$201 - 205$	Exactly like steps $195 - 200$ except for the substitution of s_2 for s_1, s_3 for s_2, and $\tfrac{190}{192}$ for $\tfrac{191}{192}$	
206	$\left[Ts_3, \tfrac{190}{192}\right]$	
$\vdots$	Repeating $195 - 200$ with appropriate	
$\vdots$	substitutions we end with	
1346	$[Ts_{193}, 0]$	

Sorites Derivation Part II $(F_L \forall PA)$ [1]

1	$[Ts_1 \to 1, 1]$	*Assumption*
2	$[(\forall x)(\forall y)((Tx \wedge Eyx) \to Ty) \to \frac{191}{192}, 1]$	*Assumption*
3	$[Es_2 s_1 \to 1, 1]$	*Assumption*
4	$[Es_3 s_2 \to 1, 1]$	*Assumption*
	...	
194	$[Es_{193} s_{192} \to 1, 1]$	*Assumption*
195	$[((\forall x)(\forall y)((Tx \wedge Eyx) \to Ty) \to \frac{191}{192})$ $\to (\forall y)((Ts_1 \wedge Eys_1) \to Ty) \to \frac{191}{192}, 1]$	$F_L \forall P9$
196	$[(\forall y)((Ts_1 \wedge Eys_1) \to Ty) \to \frac{191}{192}, 1]$	MP
197	$[((\forall y)((Ts_1 \wedge Eys_1) \to Ty) \to \frac{191}{192})$ $\to ((Ts_1 \wedge Es_2 s_1) \to Ts_2) \to \frac{191}{192}, 1]$	$F_L \forall PA$
198	$[((Ts_1 \wedge Es_2 s_1) \to Ts_2) \to \frac{191}{192}, 1]$	MP
199	$[(Ts_1 \wedge Es_2 s_1) \to 1, 1]$ since (letting $P = Ts_1$ and $Q = Es_2 s_1$) $V(P \wedge Q) = \min(V(P), V(Q))$ and $V(P) \leq 1$ and $V(Q) \leq 1$ by assumption.	
200	Let $P = (Ts_1 \wedge Es_2 s_1)$ and $Q = Ts_2$. Now $V(P \to Q) = \min(1, 1 - V(P) + V(Q))$. $V(P \to Q) \leq \frac{191}{192}$ by step 198. Thus $V(P \to Q) = 1 - V(P) + V(Q)$. By step 199, $V(P) \leq 1$. Thus $1 - \frac{191}{192} + V(Q) \leq V(P) \leq 1$. This implies $1 \geq \frac{1}{192} + V(Q)$ which means $V(Q) \leq \frac{191}{192}$. Therefore, $[Ts_2 \to \frac{191}{192}, 1]$	
$201 - 205$	Exactly like steps $195 - 200$ except for the substitution of s_2 for s_1, s_3 for s_2, and $\frac{190}{192}$ for $\frac{191}{192}$	
206	$[Ts_3 \to \frac{190}{192}, 1]$	
$\vdots$	Repeating $195 - 200$ with appropriate substitutions we end with	
1346	$[Ts_{193} \to 0, 1]$	

Conclusion

The Sorites height premises entail a paradoxical conclusion under classical first-order logic and Lukasiewicz's three-valued first-order logic. This paradoxical conclusion mainly arises from the inability of these logical systems to capture the exact truth-value of the Principal of Charity premise. One sees that no problematic idea is entailed by the Sorites premises in Pavelka-style Lukasiewicz's fuzzy first-order logic.

Jan Lukasiewicz's logic systems, among others, deal with the Sorites paradox. Three other three-valued logics exist - Kleen's Strong logic (K_3^S), Bochvar's Internal logic (B_3^I), and Bochvar's External logic (B_3^E) [1]. One can create Infinite truth-value generalizations of these logics; however, tautologies and contradictions exist only in B_3^E and thus only in B_3^E can an axiomatic derivation system be constructed. This paper excludes any study of such a (fuzzy) derivation system and it's application to the Sorites paradox.

Although $Fuzzy_{L\forall}$ is the only fuzzy logic system for which a construction in a Pavelka-style graded derivation system exists, other fuzzy logic systems exist in the literature under which the Sorites paradox can be analyzed [1]. Two other logics developed in the twentieth century include Godel Fuzzy logic and Product Fuzzy logic. A sound and complete axiomatic derivation system exists For Godel Fuzzy logic. An analysis of the Sorites paradox under Godel Fuzzy first-order logic remains an important extension of this work.

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