1994

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This is a pre-publication author manuscript of the final, published article.

AN INVARIANT SUBSPACE PROBLEM FOR $p = 1$ BERGMAN SPACES ON SLIT DOMAINS

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Abstract. In this paper, we characterize the z-invariant subspaces that lie between the Bergman spaces $A^1(G)$ and $A^1(G\backslash K)$, where G is a bounded region in the complex plane and K is a compact subset of a simple arc of class C^1.

1. Introduction

For a bounded region $U \subset \mathbb{C}$, we define the *Bergman space* $A^1(U)$ as the space of analytic functions f on U with $\int_U |f| dA < \infty$ (Here dA is Lebesgue measure on $\mathbb{C}$) and the operator S on $A^1(U)$ by $(Sf)(z) = zf(z)$. Characterizing the subspaces $\mathcal{M}$ of $A^1(U)$ for which $S\mathcal{M} \subset \mathcal{M}$ is a difficult and unsolved problem which has received considerable attention over the past 40 years. In this paper, we give a complete characterization of the S-invariant subspaces $\mathcal{M}$ with

$$A^1(G) \subset \mathcal{M} \subset A^1(G\backslash K). \tag{1.1}$$

Here G is a bounded region in $\mathbb{C}$ and K is a compact subset of a simple arc of class C^1. This paper will be a continuation of an L^p version of this problem [5] to the largest of the Bergman spaces $p = 1$. Different techniques will be used here since the papers mentioned above use duality and the reflexivity of L^p, a luxury not afforded us in the non-reflexive setting of L^1. Our main theorem is:

Theorem 1.1. *For $\mathcal{M}$ of type (1.1) and S-invariant, there is a closed $F \subset K$ with $\mathcal{M} = A^1(G\backslash F)$.*

The author would like to thank Prof. Peter Jones and Prof. Dmitry Khavinson for a helpful conversation.

2. Preliminaries

Before proceeding, we point out that some of the techniques used here are somewhat standard and fall under the general name of 'Havin's lemma'. We refer the reader to [3] and [8], Ch. 4, section 2, for further details. For the sake of completeness, we will outline these results here.

For a bounded region U in the complex plane $\mathbb{C}$, we identify the dual of $L^1(U) = L^1(U, dA)$ with $L^\infty(U) = L^\infty(U, dA)$ by the bi-linear pairing

$$< f, g >= \int_U fg dA, \quad f \in L^1(U),\ g \in L^\infty(U).$$

For a linear manifold X in $L^1(U)$ we let $X^\perp$ be the annihilator of X and note that $X^\perp$ is weak-star closed in $L^\infty(U)$. For a linear manifold Y in $L^\infty(U)$, we let ${}^\perp Y$ be the pre-annihilator of Y and note that ${}^\perp Y$ is norm closed in $L^1(U)$. We also note that by the Hahn-Banach theorem ${}^\perp(X^\perp)$ is the norm closure of X in $L^1(U)$ and $({}^\perp Y)^\perp$ is the weak-star closure of Y in $L^\infty(U)$.

Lemma 2.1. *$A^1(U)^\perp$ is the weak-star closure of $\overline{\partial} C_0^\infty(U)$, where $\overline{\partial} = \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$.*

Proof. By Weyl's lemma [2], p. 172, ${}^\perp(\overline{\partial} C_0^\infty(U)) = A^1(U)$, hence

$$\left({}^\perp\ (\overline{\partial} C_0^\infty(U))\ \right)^\perp = A^1(U)^\perp. \tag{2.1}$$

By Hahn-Banach, the left-hand side of (2.1) is the weak-star closure of $\overline{\partial} C_0^\infty(U)$. □

Remark: We will show, in Proposition 3.1, that in fact $\overline{\partial} C_0^\infty(U)$ is weak-star sequentially dense in $A^1(U)^\perp$, a technicality that will be important later in the paper.

We now relate $A^1(U)^\perp$ with a certain type of Sobolev space on U via $\overline{\partial}$. Let $\mathcal{W} = \mathcal{W}(\mathbb{C})$ be the Banach space of $f \in L^\infty = L^\infty(\mathbb{C}, dA)$ such that $\overline{\partial} f$ (in the sense of distributions) belongs to L^∞. We norm $\mathcal{W}$ by

$$\|f\|_{\mathcal{W}} = \|f\|_\infty + \|\overline{\partial} f\|_\infty.$$

Remark: We pause here for a moment to mention that $\mathcal{W}$ contains, but is not equal to $W^{1,\infty}(\mathbb{C})$, the Sobolev space of $f \in L^\infty$ whose first partial derivatives (in the sense of distributions) also belong to L^∞. In fact, if $f \in \mathcal{W}$, then the first partial derivatives belong to BMO but are not always bounded (see [9] and [10], p. 164, and [4]).

For $g \in L^\infty$ with compact support, define the *Cauchy transform* Tg by

$$(Tg)(w) = -\frac{1}{\pi} \int \frac{g(z)}{z - w} dA(z) \tag{2.2}$$

and note that Tg is continuous on $\mathbb{C}$ ([11], p. 40), analytic off of the support of g, $(Tg)(\infty) = 0$, and $\phi = T(\overline{\partial}\phi)$ for all $\phi \in C_0^\infty$ ([2], p. 170).

Lemma 2.2. *Every $f \in \mathcal{W}$ has a continuous representative.*

Proof. Let $f \in \mathcal{W}$ and $\phi \in C_0^\infty$. Note that $\phi f \in \mathcal{W}$ and, by distribution theory [2], p. 174 - 175, $\phi f = T(\overline{\partial}(\phi f))$ a.e. (dA). Since $T(\overline{\partial}(\phi f))$ is continuous, we can conclude that f has a continuous representative. □

Assuming now, and for the rest of the paper that all functions in $\mathcal{W}$ are continuous, we let $\mathcal{W}_0(U)$ be the subspace of functions in $\mathcal{W}$ which vanish off of U. ($\mathcal{W}_0(U)$ is not the same as the closure of $C_0^\infty(U)$ in the $\mathcal{W}$ norm.) For $f \in \mathcal{W}_0(U)$, one sees from Lemma 2.2 that $f = T(\overline{\partial} f)$, and thus for $w \in U$

$$|f(w)| \leq \frac{1}{\pi} \int_U \frac{|\overline{\partial} f(z)|}{|z - w|} dA(z) \leq C_U \|\overline{\partial} f\|_\infty. \tag{2.3}$$

Here C_U is a positive constant depending only on the region U. Hence an equivalent norm on $\mathcal{W}_0(U)$ is

$$\|f\|_{\mathcal{W}_0} = \|\overline{\partial} f\|_\infty.$$

If $f \in \mathcal{W}_0(U)$ and $w \notin U$, then

$$0 = f(w) = -\frac{1}{\pi} \int_U \frac{\overline{\partial} f(z)}{z - w} dA(z).$$

But since $A^1(U)$ is the closed linear span of $\{(z-w)^{-1} : w \notin U\}$ [1], then $\overline{\partial} f \in A^1(U)^\perp$ and moreover $\overline{\partial} : \mathcal{W}_0(U) \to A^1(U)^\perp$ is an isometry.

Proposition 2.3. *$\overline{\partial} : \mathcal{W}_0(U) \to A^1(U)^\perp$ is invertible.*

Proof. Since $\overline{\partial}$ is an isometry, it suffices to show that $\overline{\partial}$ is onto. To this end, let $g \in A^1(U)^\perp$. By distribution theory [2], p. 174, $\overline{\partial}(Tg) = g \in L^\infty$, so $Tg \in \mathcal{W}$. Since $g \in A^1(U)^\perp$, then by (2.2), $(Tg)(w) = 0$ for all $w \notin U$. Hence $Tg \in \mathcal{W}_0(U)$. □

Proposition 2.4. *$\mathcal{W}_0(U)$ can be equivalently re-normed to make it a Banach algebra.*

Proof. If $f = T(\overline{\partial} f)$ and $g = T(\overline{\partial} g)$ both belong to $\mathcal{W}_0(U)$, then one has ([2], p. 178, Lemma 3.11) $fg = T(f\overline{\partial} g + g\overline{\partial} f)$, hence

$$\overline{\partial}(fg) = f\overline{\partial} g + g\overline{\partial} f, \tag{2.4}$$

from which we obtain $\|\overline{\partial}(fg)\|_\infty \le \|f\overline{\partial} g\|_\infty + \|g\overline{\partial} f\|_\infty$. Using (2.3) will yield $\|\overline{\partial}(fg)\|_\infty \le 2C_U\|\overline{\partial} f\|_\infty \|\overline{\partial} g\|_\infty$. We conclude from this that $\mathcal{W}_0(U)$ can be equivalently re-normed to make it a Banach algebra. □

3. Invariant subspaces

Define the operator R on $A^1(U)^\perp$ by $(Rg)(z) = zg(z)$ and M on the Sobolev space $\mathcal{W}_0(U)$ by $(Mh)(z) = zh(z)$ and notice that R and M are well defined and continuous. For $f \in \mathcal{W}_0(U)$, observe that

$$\overline{\partial}(zf) = z\overline{\partial} f,$$

and thus $\overline{\partial} M = R\overline{\partial}$.

If $\mathcal{M}$ is invariant with

$$A^1(G) \subset \mathcal{M} \subset A^1(G\backslash K), \tag{3.1}$$

we can take annihilators to get

$$A^1(G\backslash K)^\perp \subset \mathcal{M}^\perp \subset A^1(G)^\perp \tag{3.2}$$

with $R\mathcal{M}^\perp \subset \mathcal{M}^\perp$. Taking $T = \overline{\partial}^{-1}$ (Proposition 2.3) of both sides of (3.2) will yield

$$\mathcal{W}_0(G\backslash K) \subset T\mathcal{M}^\perp \subset \mathcal{W}_0(G)$$

and using $\overline{\partial} M = R\overline{\partial}$, we get that $T\mathcal{M}^\perp$ is z-invariant. We will eventually show that $T\mathcal{M}^\perp$ is an ideal of the Banach algebra $\mathcal{W}_0(G)$ and that $T\mathcal{M}^\perp = \mathcal{W}_0(G\backslash Z_\mathcal{M})$, and hence

$$\mathcal{M} = A^1(G\backslash Z_\mathcal{M}),$$

where

$$Z_\mathcal{M} = \{z \in K : (Tg)(z) = 0 \ \ \forall g \in \mathcal{M}^\perp\}, \tag{3.3}$$

but first we need a few preliminary lemmas.

In Lemma 2.1, we saw that $\overline{\partial} C_0^\infty(G)$ is weak-star dense in $A^1(G)^\perp$. This next result gives us slightly more.

Proposition 3.1. *$\overline{\partial} C_0^\infty(G)$ is weak-star sequentially dense in $A^1(G)^\perp$.*

The proof of Proposition 3.1 will depend on the following lemma which uses a certain "mollifier" of Ahlfors [1].

Lemma 3.2. *Let $h \in \mathcal{W}_0(G)$. Then there is a sequence $h_n \in \mathcal{W}_0(G)$ with supp$(h_n) \subset G$ and $\overline{\partial} h_n \to \overline{\partial} h$ weak-star.*

Proof. Since $h \equiv 0$ off of G, then one can show [11], p. 40, that for all z and w in $\mathbb{C}$

$$|h(z) - h(w)| \leq C|z-w||\log|z-w||.$$

Thus if $d(z)$ equals the minimum of e^{-2} and $\text{dist}(z, \partial G)$, then

$$|h(z)| \leq Cd(z)|\log d(z)|. \tag{3.4}$$

We now construct the "Ahlfors mollifier" w_n as follows [1]: Let $j(t)$ be an infinitely differentiable function on $\mathbb{R}$ with $0 \leq j \leq 1$, $j(t) = 0$ for all $t \leq 1$, and $j(t) = 1$ for all $t > 2$. For $n \in \mathbb{N}$ and $z \in G$ let

$$w_n(z) = j\left(\frac{n}{\log\log d(z)}\right) \tag{3.5}$$

and notice that $w_n \equiv 0$ near ∂G. Thus define w_n on $\mathbb{C}$ be defining $w_n \equiv 0$ off G.

Since $d(z)$ is Lipschitz continuous with constant 1 and $j'(t) = 0$ outside $1 < t < 2$, one can check [1] that

$$|\overline{\partial} w_n(z)| \leq \frac{C}{n}\frac{1}{d(z)|\log d(z)|} \quad \forall z \in G. \tag{3.6}$$

Hence $w_n \in \mathcal{W}_0(G)$ and so, by Proposition 2.4, $h_n \equiv w_n h$ also belongs to $\mathcal{W}_0(G)$ with $\text{supp}(h_n) \subset G$.

We now show that $\overline{\partial} h_n \to \overline{\partial} h$ weak-star. If $f \in L^1(G)$, then by (2.4)

$$|\int_G f(\overline{\partial} h_n - \overline{\partial} h)dA| \leq |\int_G f\overline{\partial} h(w_n - 1)dA| + \int_G |f||h||\overline{\partial} w_n|dA. \tag{3.7}$$

By (3.4) and (3.6), we get

$$\int_G |f||h||\overline{\partial} w_n|dA \leq \frac{C}{n}\int_G |f|dA$$

which goes to zero as $n \to \infty$. The first integral in (3.7) goes to zero since $w_n \to 1$ pointwise and $w_n \leq 1$. Thus $\overline{\partial} h_n \to \overline{\partial} h$ weak-star. ☐

Proof of Proposition 3.1

Let $h_n = w_n h$ be as in Lemma 3.2. For $n, k \in \mathbb{N}$ let φ_k be a mollifier [2], p. 171, and define

$$h_{n,k}(w) = \int \varphi_k(w - z)h_n(z)dA(z).$$

Notice that $h_{n,k} \in C_0^\infty(G)$ (since h_n has compact support in G) and $h_{n,k} \to h_n$ uniformly as $k \to \infty$. By a change of variables and Fubini's theorem, one checks that

$$\overline{\partial} h_{n,k}(w) = -\int \varphi_k(z)\overline{\partial} h_n(w - z)dA(z).$$

Since $\overline{\partial} h_n \to \overline{\partial} h$ weak-star, then

$$\sup_n \|\overline{\partial} h_n\|_\infty = M < \infty$$

and so $|\overline{\partial} h_{n,k}(w)| \leq M \int \varphi_k(z)dA(z) = M$. Hence

$$\sup_{n,k} \|\overline{\partial} h_{n,k}\|_\infty \leq M < \infty.$$

Choose $k(n)$ so that $\|h_{n,k(n)} - h_n\|_\infty \le 1/n$ and let $H_n = h_{n,k(n)}$. We shall conclude by showing that $\overline{\partial} H_n \to \overline{\partial} h$ weak-star. Let $f \in L^1(G)$ and choose a sequence $\phi_j \in C_0^\infty(G)$ with $\phi_j \to f$ in L^1. Then

$$\begin{aligned} \Big| \int f(\overline{\partial} H_n - \overline{\partial} h) dA \Big| &\le \Big| \int (f - \phi_j)(\overline{\partial} H_n - \overline{\partial} h) dA \Big| + \Big| \int \phi_j (\overline{\partial} H_n - \overline{\partial} h_n) dA \Big| \\ &+ \Big| \int \phi_j (\overline{\partial} h_n - \overline{\partial} h) dA \Big| \\ &\le M \|f - \phi_j\|_{L^1} + \int |\overline{\partial} \phi_j| \frac{1}{n} dA + \Big| \int \phi_j (\overline{\partial} h_n - \overline{\partial} h) dA \Big| . \end{aligned}$$

For $\varepsilon > 0$ given, choose j' such that

$$\|f - \phi_{j'}\|_{L^1} \le \varepsilon.$$

Hence

$$\Big| \int f(\overline{\partial} H_n - \overline{\partial} h) dA \Big| \le M\varepsilon + \frac{1}{n} \int |\overline{\partial} \phi_{j'}| dA + \Big| \int \phi_{j'} (\overline{\partial} h_n - \overline{\partial} h) dA \Big| .$$

Now use Lemma 3.2 and let $n \to \infty$ to get the desired conclusion. Λ

Lemma 3.3. *If $f, g \in \mathcal{W}_0(G)$ and ϕ_n in $C_0^\infty(G)$ with $\overline{\partial} \phi_n \to \overline{\partial} g$ weak-star, then $\overline{\partial}(f \phi_n) \to \overline{\partial}(fg)$ weak-star.*

Proof. Let $h \in L^1(G)$. By (2.4) we have

$$\int_G \overline{\partial}(f\phi_n - fg) h dA = \int_G h f \overline{\partial}(\phi_n - g) dA + \int_G h(\phi_n - g) \overline{\partial} f dA. \tag{3.8}$$

The first integral on the right-hand side of (3.8) goes to zero since $hf \in L^1(G)$ and $\overline{\partial} \phi_n \to \overline{\partial} g$ weak-star. For the second integral, we first notice that by (2.3) $\|\phi_n\|_\infty \le C \|\overline{\partial} \phi_n\|_\infty$ and that $\|\overline{\partial} \phi_n\|_\infty$ is uniformly bounded in n (since $\overline{\partial} \phi_n$ is weak-star convergent). Since $(z - w)^{-1} \in L^1(G)$ for all $w \in G$, and $\overline{\partial} \phi_n \to \overline{\partial} g$ weak-star, then

$$\phi_n(w) = -\pi^{-1} \int \overline{\partial} \phi_n(z)(z - w)^{-1} dA \to -\pi \int \overline{\partial} g(z)(z - w)^{-1} dA = g(w) \quad \forall w \in G.$$

Now apply the dominated convergence theorem to get the second integral on the right-hand side of (3.8) goes to zero as $n \to \infty$. $\square$

This next lemma can be found in [5] and uses very strongly that K lies on γ a simple compact C^1 arc.

Lemma 3.4. *Let $\psi \in C^1(\mathbb{C})$ and $\varepsilon > 0$ be given. Then there is a polynomial $p(z)$ and a $\Psi \in C^1(\mathbb{C})$ with*

(i) $\Psi = \psi$ *on* γ
(ii) $\|p - \Psi\|_\infty < \varepsilon$
(iii) $\|\overline{\partial}(p - \Psi)\|_\infty < \varepsilon$.

As a consequence of this lemma, we have that $T\mathcal{M}^\perp$ is not only z-invariant but invariant under multiplication by any $C_0^\infty(G)$ function.

Corollary 3.5. *If $\psi \in C_0^\infty(G)$, then $\psi(T\mathcal{M}^\perp) \subset T\mathcal{M}^\perp$.*

Proof. Let $\psi \in C_0^\infty(G)$ and $\varepsilon > 0$ be given and Ψ and p be as in Lemma 3.4. If $f \in T\mathcal{M}^\perp$, then $\Psi f \in \mathcal{W}_0(G)$ and $\Psi f - \psi f = 0$ on K, so $\Psi f - \psi f \in \mathcal{W}_0(G \backslash K) \subset T\mathcal{M}^\perp$. Hence

$$\text{dist}(\psi f, T\mathcal{M}^\perp) = \text{dist}(\Psi f, T\mathcal{M}^\perp) \le \|\overline{\partial}(pf - \Psi f)\|_\infty \le C\varepsilon \|\overline{\partial} f\|_\infty \qquad \square$$

This immediately yields the following:

Proposition 3.6. *$T\mathcal{M}^{\perp}$ is an ideal of $\mathcal{W}_0(G)$.*

Proof. Let $f \in T\mathcal{M}^{\perp}$ and $g \in \mathcal{W}_0(G)$ and notice that $fg \in \mathcal{W}_0(G)$. Employing the weak-star sequential density of $\overline{\partial}C_0^{\infty}(G)$ in $A^1(G)^{\perp}$, Proposition 3.1, we can find a sequence $\phi_n \in C_0^{\infty}(G)$ with $\overline{\partial}\phi_n \to \overline{\partial}g$ weak-star. By Corollary 3.5, $\phi_n f \in T\mathcal{M}^{\perp}$ and by Lemma 3.3, $\overline{\partial}(\phi_n f) \to \overline{\partial}(fg)$ weak-star. Since $\mathcal{M}^{\perp}$ is weak-star closed, then $\overline{\partial}(fg) \in \mathcal{M}^{\perp}$, hence $fg \in T\mathcal{M}^{\perp}$. □

Proof of Theorem 1.1

Let $Z_{\mathcal{M}}$ be as in (3.3). Since $\mathcal{W}_0(G\backslash K) \subset T\mathcal{M}^{\perp}$, then $T\mathcal{M}^{\perp} \subset \mathcal{W}_0(G\backslash Z_{\mathcal{M}})$ and so $\mathcal{M}^{\perp} \subset A^1(G\backslash Z_{\mathcal{M}})^{\perp}$. To prove $A^1(G\backslash Z_{\mathcal{M}})^{\perp} \subset \mathcal{M}^{\perp}$, we apply Lemma 3.2 to see that it suffices to show that $\overline{\partial}\phi \in \mathcal{M}^{\perp}$ for all $\phi \in \mathcal{W}_0(G\backslash Z_{\mathcal{M}})$ with support in $G\backslash Z_{\mathcal{M}}$. For this, we use an argument of Sarason [7], p. 41, Lemma 1, along with the fact that $T\mathcal{M}^{\perp}$ is an ideal to find a $g \in T\mathcal{M}^{\perp}$ with $g \equiv 1$ on the support of ϕ. Thus, since $T\mathcal{M}^{\perp}$ is an ideal, $\phi = g\phi \in T\mathcal{M}^{\perp}$ and hence $\overline{\partial}\phi \in \mathcal{M}^{\perp}$. Λ

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