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Gauss' hypergeometric equation

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GAUSS'S HYPERGEOMETRIC EQUATION

by

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CHAPTER I

HISTORICAL INTRODUCTION

As early as the seventeenth century the English mathematician, John Wallis (1616-1703), used the term "hypergeometric" to describe a series which he was studying. This series, $\sum (a)(a+b)(a+2b)\cdots(a+\overline{n-1}\,b)$ is quite different from the usual geometric series, hence the term, "hyper" (=above) plus "geometric" was used to signify that the series was of greater complexity than the geometric series. Wallis did not consider his series a power series or a function of x.[1]

In 1769 this series received a remarkable development at the hands of Leonhard Euler who, following the example of Wallis, applied the word "hypergeometric" to it. He observed that the series is dependent upon the integration of a linear partial differtial equation of the second order. In his work the series is treated from three distinct standpoints: (i) as a power series, (ii) as an integral of a certain linear equation of the second order, (iii) and as a definite integral.[2]

The adjective, "hypergeometric", was first used in the modern sense by Ernst Eduard Kummer (1810-1866) when he christened the series, $1+\frac{\alpha\beta}{1\cdot\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}+\cdots+\frac{\alpha(\alpha+1)\cdots(\alpha+n-2)\beta(\beta+1)\cdots(\beta+n-2)}{1\cdot 2\cdots(n-1)\gamma(\gamma+1)\cdots(\gamma+n-2)}x^{n-1}+\cdots\cdots$, the hypergeometric series.[3] Kummer did a great deal of work upon the differential equation whose solution is expressed in hypergeometric series. The twenty-four integrals were found and published in his memoir en-

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1. Cajori: History of Mathematics, p. 185   2. Ibidem, p. 238

3. Whittaker and Watson: Modern Analysis, p. 281

titled "Über die Hypergeometrische Reihe" which may be found in Volume XV of Crelle's Journal.[1]

There were no rigorous investigations of the series, however, until Karl Friedrich Gauss (1777-1855) turned his attention to the matter. He established a criterion for convergence which settles the question of convergence in every case which the series is supposed to cover. This remarkable work is the first important and strictly rigorous investigation of infinite series. Gauss gave an exhaustive treatment of the subject and showed that, for special values of its letters , the hypergeometric series represents almost every function known at that time. Because of its strangeness of treatment and unusual rigor Gauss' paper received little attention from the mathematicians of the time.[2] This memoir entitled, "Disquisitiones generales circa seriem infinitam, $1+\frac{\alpha\beta}{1\cdot\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}x^2+\ldots$" is included in the third volume of the Collected Works of Gauss. In the same volume one may find a different treatment of the subject in the paper,"Determinatio serei nostrae per aequationem differentialem secundi ordinis."

The general theory when no restriction is imposed on the variable has been worked out by J. Tannery of Paris who used L.Fuch's method of linear partial differential equations. Tannery by this method obtained the twenty-four integrals previously found by Kummer.

Georg Friedrich Bernhard Riemann (1826-1866) has applied his conception that a function of a single variable is defined by the nature and position of its singularities to the linear partial differential equation which is satisfied by the hypergeometric series.[1]

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1. Cajori: History of Mathematics, p. 383

2. Ibidem, p. 373

He remarked in 1857 that functions expressed by Gauss' hypergeometric series, $F(\alpha,\beta,\gamma,x)$, which satisfy a homogeneous linear partial differential of the second order with rational coefficiants could be utilized in the solution of any linear differential equation.[1]

The study of the subject was later taken up by Edouard Goursat (1858-), professor of mathematical analysis in the University of Paris. His results, obtained by developmant of a method due originally to Jacobi, agree with those of Kummer and Schwarz.[2] These are embodied in his paper, "Sur l'equation differentiele qui admet pour integral le serie hypergeometrique".

The subject also attracted the attention of Schwarz who publishthe results of his work upon it in "Über diejenigen Falle in welchen die Gaussische hypergeometrische Reihe eine algebraische Function ihres vierten Elementes darstellt." Later he embodied further work upon the series in a second paper, "Über einige Abbildungsaufgauben". The former memoir may be found in Crelle's Journal, Volume LXX: the latter, in Volume XXV.[3]

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1. Schlesinger: Entwickelung d. Theorie d. linearen Differentialgleichungen (1865)
2. Cajori: History of Mathematics, p.385
3. Forsyth: Differential Equations, p. 211

CHAPTER II

THE SOLUTION OF THE HYPERGEOMETRIC EQUATION BY THE METHOD OF FROBENIUS

1. SOLUTION IN ASCENDING POWERS OF x. The hypergeometric equation,

$$\frac{d^2y}{dx^2} + \frac{\gamma-(\alpha+\beta+1)x}{x(1-x)}\frac{dy}{dx} - \frac{\alpha\beta}{x(1-x)}y = 0 \tag{2.1}$$

for convenience in employment of integration in series may be written thus:

$$x^2y'' - xy'' + \alpha xy' + \beta xy' + xy' - \gamma y' + \alpha\beta y = 0$$

Let $y = x^m$ The left hand member becomes

$$-m(m+\gamma-1)x^{m-1} + (m+\alpha)(m+\beta)x^m \tag{2.2}$$

For a trial series let us take

$$y = c_0x^m + c_1x^{m+1} + c_2x^{m+2} + \cdots + c_rx^{m+r} + \cdots$$

$$y' = c_0mx^{m-1} + c_1(m+1)x^m + c_2(m+2)x^{m+1} + \cdots + c_r(m+r)x^{m+r-1} + \cdots$$

$$y'' = c_0m(m-1)x^{m-2} + c_1(m+1)(m)x^{m-1} + c_2(m+2)(m+1)x^m + \cdots + c_r(m+r)(m+r-1)x^{m+r-2} + \cdots$$

Substituting these values in the differential equation, (2.1), we have

$x^2y'' = c_0m(m-1)$	$x^m + c_1(m+1)m$	$x^{m+1} + \cdots + c_r(m+r)(m+r-1)$	$x^{m+r} + \cdots$
$-xy'' = -c_1(m+1)m$	$-c_2(m+2)(m+1)$	$+ \cdots - c_{r+1}(m+r+1)(m+r+2)$	
$\alpha xy' = \alpha c_0 m$	$+\alpha c_1(m+1)$	$+ \cdots + \alpha c_r(m+r)$	
$\beta xy' = \beta c_0 m$	$+\beta c_1(m+1)$	$+ \cdots + \beta c_r(m+r)$	
$xy' = c_0 m$	$+c_1(m+1)$	$+ \cdots + c_r(m+r)$	
$-\gamma y' = -\gamma c_1(m+1)$	$-\gamma c_2(m+2)$	$+ \cdots - \gamma c_{r+1}(m+r-1)$	
$\alpha\beta y = \alpha\beta c_0$	$+\alpha\beta c_1$	$+ \cdots + \alpha\beta c_r$	

Since each power of x must vanish we may write

$$c_1\left[(m+1)m + \gamma(m+1)\right] - c_0\left[m^2 - m + \alpha m + \beta m + m + \alpha\beta\right] = 0$$

or $$c_1 = \frac{(m+\alpha)(m+\beta)}{(m+1)(m+\gamma)}\,c_0$$

Similarly

$$c_2 = \frac{(m+\alpha+1)(m+\beta+1)}{(m+2)(m+\gamma+1)}\,c_1$$

$$= \frac{(m+\alpha+1)(m+\beta+1)}{(m+2)(m+\gamma+1)}\cdot\frac{(m+\alpha)(m+\beta)}{(m+1)(m+\gamma)}\,c_0$$

$$c_r = \frac{(m+\alpha+r-1)(m+\beta+r-1)}{(m+r)(m+\gamma+r-1)}\,c_{r-1}$$

From the indicial equation, $-m(m+\gamma+1)=0$, we have:

$$m = 0 \quad , \quad m = 1-\gamma$$

For $m = 0$:

$$c_0 = c_0$$

$$c_1 = \frac{\alpha\beta}{1\cdot\gamma}\,c_0$$

$$c_2 = \frac{(\alpha+1)(\beta+1)}{2\cdot(1+\gamma)}\,c_1 = \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}\,c_0$$

$$c_r = \frac{\alpha(\alpha+1)\cdots(\alpha+r-1)\beta(\beta+1)\cdots(\beta+r-1)}{1\cdot 2\cdot 3\cdots r\cdot\gamma(\gamma+1)\cdots(\gamma+r-1)}$$

We now have as one integral of (2.1),

$$y_1 = c_0\left(1 + \frac{\alpha\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}x^2 + \cdots\cdots\right.$$

$$\left.\cdots + \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-2)\beta(\beta+1)(\beta+2)\cdots\cdots(\beta+n-2)}{1\cdot 2\cdot 3\cdots(n-1)\gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-2)}x^{n-1} + \cdots\right)$$

The right member of the foregoing with the constant multiplier, $c_0 = 1$, is the hypergeometric series: it is usually represented by the symbol, $F(\alpha, \beta, \gamma, x)$.

For $m = 1-\gamma$, we have

$$c_0 = c_0$$

$$c_1 = \frac{(1-\gamma+\alpha)(1-\gamma+\beta)}{1\cdot(2-\gamma)}\, c_0$$

$$c_2 = \frac{(\alpha-\gamma+1)(\alpha-\gamma+2)(\beta-\gamma+1)(\beta-\gamma+2)}{1\cdot 2\ \cdot(2-\gamma)(2-\gamma+1)}\, c_0$$

$$c_r = \frac{(\alpha-\gamma+r)(\beta-\gamma+r)}{(1-\gamma+r)\, r}\, c_{r-1}$$

Hence a second particular integral, provided $1-\gamma$ is not equal to a positive integer or zero, is

$$y_2 = F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, x)$$

If $1-\gamma$ is not a positive integer or zero, the general solution of (2.1) is

$$y = Ay_1 + By_2$$

If α or β is a negative integer, y_1 reduces to a polynomial.

The following facts should be noted.

(1) If $1-\gamma = -g$ where g is a positive integer or zero, y_1 is still an integral but y_2 will be of form

$$y_2 = y_1 \log x + \text{a power series in } x$$

provided that α or $\beta \neq 1, 2, 3 \cdots, g$. If α or $\beta = 1, 2, 3, \cdots, g$ then y_1 and y_2 remain integrals but y_2 reduces to a polynomial if the integer to which α or β is equal exceeds 1. In case α or $\beta = 1$, a vanishing factor appears in the numerator and denominator of each coefficient beginning with c_g. The result obtained by ignoring these factors (i.e., by treating their quotient as 1) gives an available form for y_2.

(ii) If $1-\gamma = h$, where h is a positive integer, y_2 remains an integral but y_1 will be of the form

$$y_1 = y_2 \log x + \text{a power series in } x$$

provided α or $\beta \neq 0, -1, -2, \cdots -(h-1)$. In this case y_1 remains an integral and reduces to a polynomial if the integer to which α or β is equal exceeds the value of γ. Otherwise beginning with $c_{1-\gamma}$ a vanishing factor appears in the numerator and denominator of every coefficient. An available form of y_1 results by treating the quotient of these factors as 1.

(iii) The integrals do not converge for $|x| > 1$.

2. SOLUTION IN DESCENDING POWERS OF x. If in (2.2), $(m+\alpha)(m+\beta)=0$ be chosen as our indicial equation our trial series will be in descending powers of x. Let the trial series be

$$y = c_0 x^m + c_{-1} x^{m-1} + c_{-2} x^{m-2} + \cdots + c_{-(r-1)} x^{m-r+1} + c_{-r} x^{m-r} + \cdots$$

$$y' = c_0 m x^{m-1} + c_{-1}(m-1) x^{m-2} + c_{-2}(m-2) x^{m-3} + \cdots c_{-r}(m-r) x^{m-r-1} + \cdots$$

$$y'' = c_0 m(m-1) x^{m-2} + c_{-1}(m-1)(m-2) x^{m-3} + \cdots + c_{-r}(m-r)(m-r-1) x^{m-r-2} + \cdots$$

Substituting the foregoing in (2.1), we get

$$\begin{array}{r|l|l|l|l}
x^2y'' = c_0 m(m-1) & x^m + c_1(m-1)(m-2) & x^{m-1} + c_{-2}(m-2)(m-3) & x^{m-2} + \cdots + c_{-r}(m-r)(m-r-1) & x^{m-r} + \cdots \\
-xy'' \qquad\qquad & - c_0 (m)(m-1) & - c_{-1}(m-1)(m-2) & + \cdots c_{-(r-1)}(m-r+1)(m-r) & \\
\alpha x y' \quad \alpha c_0 m & + \alpha c_{-1}(m-1) & + \alpha c_{-2}(m-2) & + \alpha c_{-r}(m-r) & \\
\beta x y' \quad \beta c_0 m & + \beta c_{-1}(m-1) & + \beta c_{-2}(m-2) & + \beta c_{-r}(m-r) & \\
-\gamma y' \qquad\qquad & - \gamma c_0 m & - \gamma c_{-1}(m-1) & - \gamma c_{-(r-1)}(m-r+1) & \\
\alpha\beta y \quad \alpha\beta c_0 & + \alpha\beta c_{-1} & + \alpha\beta c_{-2} & + \alpha\beta c_{-r} & \\
xy' \quad c_0 m & + c_1(m-1) & + c_{-2}(m-2) & + c_{-r}(m-r) &
\end{array}$$

Hence

$$c_0 = c_0$$

$$c_{-1} = -\frac{m(m+\gamma-1)}{(m-1)(m+\alpha+\beta-1)+\alpha\beta}$$

$$c_{-2} = \frac{m(m-1)(m+\gamma-1)(m+\gamma-2)}{[(m-1)(m-2)][(m+\alpha+\beta-1)+\alpha\beta][(m+\alpha+\beta-2)+\alpha\beta]}\, c_0$$

$$c_{-r} = \frac{(m-r+1)(m+\gamma-r)}{(m-r)(m+\alpha+\beta-r)+\alpha\beta}\, c_{-(r-1)}$$

For $m=-\alpha$:

$$c_0 = c_0$$

$$c_{-1} = \frac{\alpha(1+\alpha-\gamma)}{1+\alpha-\beta}\, c_0$$

$$c_{-2} = \frac{\alpha(\alpha+1)(1+\alpha-\gamma)(1+\alpha-\gamma+1)}{1\cdot 2(1+\alpha-\beta)(1+\alpha-\beta+1)}\, c_0$$

$$c_{-r} = \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+r-1)(1+\alpha-\gamma)(1+\alpha-\gamma+1)\cdots(1+\alpha-\gamma+r-1)}{1\cdot 2\cdot 3\cdots r\,(1+\alpha-\beta)(1+\alpha-\beta+1)(1+\alpha-\beta+2)\cdots(1+\alpha-\beta+r-1)}\, c_0$$

We have now for an integral,

$$\bar{y}_1 = x^{-\alpha}\left(1+\frac{\alpha(\alpha+1-\gamma)}{1\cdot(1+\alpha-\beta)}x^{-1}+\frac{\alpha(\alpha+1)(\alpha+1-\gamma)(\alpha+1-\gamma+1)}{1\cdot 2\cdot(1+\alpha-\beta)(1+\alpha-\beta+1)}x^{-2}+\cdots\right) \quad \text{or}$$

$$\bar{y}_1 = x^{-\alpha} F\left(\alpha,\ 1+\alpha-\gamma,\ 1+\alpha-\beta,\ \frac{1}{x}\right)$$

Thus we have found one particular integral of (2.1).

For $m=-\beta$, symmetry enables us to at once write

$$\bar{y}_2 = x^{-\beta} F\left(\beta,\ 1+\beta-\gamma,\ 1+\beta-\alpha,\ \frac{1}{x}\right)$$

The general solution in descending powers of x is therefore

$$y = A\bar{y}_1 + B\bar{y}_2 \quad .$$

y_1 and y_2 converge for $|x| > 1$.

CHAPTER III

THE SERIES

1. INTRODUCTION. The series, $1+\frac{\alpha\beta}{1\cdot\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}x^2+\cdots\cdots$ is called the hypergeometric series and is denoted by the symbol, $F(\alpha,\beta,\gamma,x)$ as previously pointed out. It is a function of the four quantities α, β, γ, and x. Trroughout this discussion α, β, and γ will be treated as constant quantities: x, as a variable quantity. A glance at the function will show that it is symmetrie with respect to α, and β : hence an interchange of these two elements does not alter the value of it. Further, on writing out the nth term,

$$\frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-2)\beta(\beta+1)(\beta+2)\cdots(\beta+n-2)}{1\cdot2\cdot3\cdots\cdots(n-1)\gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-2)}x^{n-1}$$

it is evident that γ must never be a negative integer for if this were the case the $(2+|\gamma|)$th and all terms thereafter are either indefinitely great or indeterminate. If α is a negative integer, the $(2+|\alpha|)$th and all succeeding terms vanish, reducing the series to a polynomial. Similarly if β is a negative integer, the $(2+|\beta|)$th and all terms thereafter vanish and we have a polynomial. If γ is a negative integer whose absolute value is greater than α, and α is a negative integer, the vanishing terms following the $(1-|\alpha|)$th term may mislead one into thinking that the series reduces to a polynomial. In reality the series will be of the form,

a polynomial + vanishing terms + indeterminate terms[1]

In the remaining cases the series proceeds to infinity.

In case $F(\alpha,\beta,\gamma,x)$ is finite, it represents a rational algebraic function: otherwise it represents, in most cases, a transcendental function. The hypergeometric series, for special values of its letters frequently represents well known functions. The expression of functions in hypergeometric series will be dealt with in Chapter VII.

2. CONVERGENCE OF THE HYPERGEOMETRIC SERIES, $F(\alpha,\beta,\gamma,x)$

$$F(\alpha,\beta,\gamma,x) = 1 + \frac{\alpha\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)(\beta)(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}x^2 + \cdots + \frac{\alpha(\alpha+1)\cdots(\alpha+n-2)(\beta)(\beta+1)\cdots(\beta+n-2)}{1\cdot 2\cdot 3\cdots(n-1)\gamma(\gamma+1)\cdots(\gamma+n-2)}x^{n-1} + \cdots$$

We shall first employ the ratio test.

$$\frac{U_{n+1}}{U_n} = \frac{(\alpha+n-1)(\beta+n-1)}{n(\gamma+n-1)}$$

$$\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\frac{\left(\frac{\alpha}{n}+1-\frac{1}{n}\right)\left(\frac{\beta}{n}+1-\frac{1}{n}\right)}{\frac{\gamma}{n}+1-\frac{1}{n}}\right|\cdot|x| = |x|$$

Therefore the series converges absolutely for $|x|<1$ and diverges for $|x|>1$.

The ratio test fails for $x=1$. Let us use Gauss' Test.

$$\frac{U_{n+2}}{U_{n+1}} = \frac{(\alpha+n)(\beta+n)}{(n+1)(\gamma+n)} = \frac{n^2+n(\alpha+\beta)+\alpha\beta}{n^2+n(1+\gamma)+\gamma}$$

By Gauss' Test the series converges only when

$$(1+\gamma)-(\alpha+\beta) > 1$$

or when $\alpha+\beta-\gamma<0$ and then absolutely.

The question of convergence for $x=-1$ remains to be settled. The following theorem will be used.

" If $\frac{U_n}{U_{n+1}}$ can be expressed in the form

$$\frac{U_n}{U_{n+1}} = 1 + \frac{\mu}{n} + O\left(\frac{1}{n^p}\right),\quad p>1$$

the series $\sum(-1)^{n-1}u_n$ is convergent if $\mu > 0$.[1,2]

For the $F(\alpha,\beta,\gamma,x)$ we have

$$\frac{u_n}{u_{n+1}} = \frac{n(\gamma+n-1)}{(\alpha+n-1)(\beta+n-1)}$$

For convenience $\frac{u_{n+1}}{u_{n+2}}$ will be used.

$$\frac{u_{n+1}}{u_{n+2}} = \frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)} = \frac{n^2+n(1+\gamma)+\gamma}{n^2+n(\alpha+\beta)+\alpha\beta}$$

$$= 1 + \frac{1+\gamma-\alpha-\beta}{n} + \frac{\gamma-\alpha\beta-(\alpha+\beta)(1+\gamma-\alpha-\beta)}{n^2+n(\alpha+\beta+\alpha\beta)} - \frac{\alpha\beta(1+\gamma-\alpha-\beta)}{n(n^2+\overline{\alpha+\beta}\,n+\alpha\beta)}$$

It is now necessary to show that

$$\left[\frac{\gamma-\alpha\beta-(\alpha+\beta)(1+\gamma-\alpha-\beta)}{n^2+(\alpha+\beta)n+\alpha\beta} - \frac{\alpha\beta(1+\gamma-\alpha-\beta)}{n(n^2+\overline{\alpha+\beta}\,n+\alpha\beta}\right] = O\left(\frac{1}{n^2}\right) \qquad \text{or}$$

$$n^2\left[\frac{\gamma-\alpha\beta-(\alpha+\beta)(1+\gamma-\alpha-\beta)}{n^2+(\alpha+\beta)n+\alpha\beta} - \frac{\alpha\beta(1+\gamma-\alpha-\beta)}{n(n^2+\overline{\alpha+\beta}\,n+\alpha\beta}\right] < K$$

for all values of $n > n_0$ where K is a constant. A rearrangment of the left member gives

$$\left|\frac{\gamma-\alpha\beta-(\alpha+\beta)(1+\gamma-\alpha-\beta)}{1+\frac{\alpha+\beta}{n}+\frac{\alpha\beta}{n^2}}\right| < \left|\gamma-\alpha\beta-(\alpha+\beta)(1+\gamma-\alpha-\beta)\right| = C \qquad (3.1)$$

for all values of n.

We now proceed to show that the second term of the left member can be made less than some constant for a sufficiently large value of

1. The theorem is from Bromwich's "Theory of Series," p.56. A proof is given in Note I of the Appendix.
2.

n, say n_1, and all values of n thereafter, that is

$$\left|\frac{n\alpha\beta(-\gamma-1+\alpha+\beta)}{n^2+(\alpha+\beta)n+\alpha\beta}\right| < \left|\frac{\alpha\beta(-\gamma-1+\alpha+\beta)}{1+\alpha+\beta+\alpha\beta}\right| = M \tag{3.2}$$

which may be written

$$\left|\frac{n}{n^2+(\alpha+\beta)n+\alpha\beta}\right| < \frac{1}{1+\alpha\beta+\alpha+\beta}$$

for all values of n after n_n. The truth of this inequality is obvius since the denominator is of higher degree than the numerator in n. These two inequalities, (3.1) and (3.2) enable us to write

$$n^2\left|\frac{\gamma-\alpha\beta-(\alpha+\beta)(1+\gamma-\alpha-\beta)}{n^2+(\alpha+\beta)n+\alpha\beta}\right| + \left|\frac{\alpha\beta(-1-\gamma+\alpha+\beta)}{n(n^2+\overline{\alpha+\beta}\,n+\alpha\beta)}\right| < C+M = K$$

for all values of $n > n_n$. Thus we have shown that $\frac{U_{n+1}}{U_{n+2}}$ may be expressed in the form,

$$\frac{U_{n+1}}{U_{n+2}} = 1 + \frac{1+\gamma-\alpha-\beta}{n} + O\left(\frac{1}{n^2}\right)$$

Therefore $F(\alpha,\beta,\gamma,x)$ converges for $x=-1$, if $1+\gamma-\alpha-\beta>0$ or $\alpha+\beta-\gamma-1<0$. The question of convergence has now been settled for every case. The results of this article may be summarized as follows:

The hypergeometric series, $F(\alpha,\beta,\gamma,x)$, converges absolutely when $|x|<1$ and diverges when $|x|>1$. When $x=1$, $F(\alpha,\beta,\gamma,x)$ converges only when $\alpha+\beta-\gamma<0$ and then absolutely. When $x=-1$, $F(\alpha,\beta,\gamma,x)$ converges only when $\alpha+\beta-\gamma-1<0$, and absolutely if $\alpha+\beta-\gamma<0$.

3. THE DIFFERENTIATION OF $F(\alpha,\beta,\gamma,x)$. The differentiation of the hypergeometric series term by term is valid for all values of the variable within the interval of convergence since all power series are uniformly convergent in the interval of convergence.

For the first derivative we have

$$\frac{d}{dx}\left(F(\alpha,\beta,\gamma,x)\right) = \frac{\alpha\beta}{1\cdot\gamma} + \frac{2\cdot\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}x + \cdots\cdots$$

$$\cdots + \frac{n\,\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{1\cdot 2\cdot 3\cdots(n-1)\,n\cdot\gamma(\gamma+1)\cdots(\gamma+n-1)}x^{n-1} + \cdots$$

$$= \frac{\alpha\beta}{\gamma}F(\alpha+1,\ \beta+1,\ \gamma+1,\ x) \tag{3.3}$$

The second derivative is

$$\frac{d^2}{dx^2}\left[F(\alpha,\beta,\gamma,x)\right] = \frac{\alpha\beta}{\gamma}\left[\frac{(\alpha+1)(\beta+1)}{1\cdot(\gamma+1)} + \frac{2(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}{1\cdot 2\cdot(\gamma+1)(\gamma+2)}x + \cdots\right.$$

$$\left.\cdots + \frac{n(\alpha+1)(\alpha+2)\cdots(\alpha+n)(\beta+1)(\beta+2)\cdots(\beta+n)}{1\cdot 2\cdots n\cdot(\gamma+1)(\gamma+2)\cdots(\gamma+n)}x^{n-1} + \cdots\right]$$

$$= \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}F(\alpha+2,\beta+2,\gamma+2,x) \tag{3.4}$$

and the third

$$\frac{d^3}{dx^3}\left[F(\alpha,\beta,\gamma,x)\right] = \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}F(\alpha+3,\beta+3,\gamma+3,x) \tag{3.5}$$

Equations (3.3), (3.4), (3.5), lead to the assumption

$$\frac{d^r}{dx^r}\left[F(\alpha,\beta,\gamma,x)\right] = \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+r-1)\beta(\beta+1)(\beta+2)\cdots(\beta+r-1)}{\gamma(\gamma+1)(\gamma+2)\cdots(\gamma+r-1)}F(\alpha+r,\beta+r,\gamma+r,x) \tag{3.6}$$

To establish this we shall show that it holds true for $r = r+1$

$$\frac{d^{r+1}}{dx^{r+1}}F(\alpha,\beta,\gamma,x) = \frac{d}{dx}\left\{\frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+r-1)\beta(\beta+1)(\beta+2)\cdots(\beta+r-1)}{\gamma(\gamma+1)(\gamma+2)\cdots(\gamma+r-1)}\right.\left[1 + \right.$$

$$\left.\frac{(\alpha+r)(\beta+r)}{1\cdot(\gamma+r)}x + \frac{(\alpha+r)(\alpha+r+1)(\beta+r)(\beta+r+1)}{1\cdot 2\cdot(\gamma+r)(\gamma+r+1)}x^2 + \cdots\cdots\right]$$

$$\frac{d^{r+1}}{dx^{r+1}} F(\alpha,\beta,\gamma,x) = \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+r-1)\beta(\beta+1)(\beta+2)\cdots(\beta+r-1)}{\gamma(\gamma+1)(\gamma+2)\cdots(\gamma+r-1)} \Big\{ \frac{(\alpha+r)(\beta+r)}{\gamma+r}$$

$$+ \frac{2(\alpha+r)(\alpha+r+1)(\beta+r)(\beta+r+1)}{1\cdot 2\cdot(\gamma+r)(\gamma+r+1)} x + \cdots$$

$$\cdots + \frac{n(\alpha+r)(\alpha+r+1)\cdots(\alpha+r+n-1)(\beta+r)(\beta+r+1)\cdots(\beta+r+n-1)}{1\cdot 2\cdot 3\cdots n\cdot(\gamma+r)(\gamma+r+1)\cdots(\gamma+r+n-1)} x^{n-1} + \cdots \Big\}$$

$$= \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+\overline{r+1}-1)(\beta)(\beta+1)(\beta+2)\cdots(\beta+\overline{r+1}-1)}{\gamma(\gamma+1)(\gamma+2)\cdots(\gamma+\overline{r+1}-1)} F(\alpha+r+1,\beta+r+1,\gamma+r+1,x)$$

which establishes (3.6).

4. THE INTEGRATION OF $F(\alpha,\beta,\gamma,x)$. Since the $F(\alpha,\beta,\gamma,x)$ is a power series it is uniformly convergent in the interval of convergence. Therefore integration term by term is valid for all values of x within this interval. Integrating once we have

$$\int F(\alpha,\beta,\gamma,x) = \Big(x + \frac{\alpha\beta}{1\cdot 2\cdot\gamma} x^2 + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot 3\cdot\gamma(\gamma+1)} x^3 + \cdots$$

$$\cdots + \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-2)\beta(\beta+1)(\beta+2)\cdots(\beta+n-2) x^n + \cdots\Big) + c_1$$

$$= x\Big(1 + \frac{\alpha\beta}{1\cdot 2\cdot\gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot 3\cdot\gamma(\gamma+1)} x^2 + \cdots$$

$$\cdots + \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-2)\beta(\beta+1)(\beta+2)\cdots(\beta+n-2)}{1\cdot 2\cdot 3\cdots(n-1)n\cdot\gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-2)} x^{n-1} + \cdots\Big) + c_1$$

Let $S(\alpha,\beta,\delta,\gamma,x)$ denote the function,

$$\frac{1}{\delta} + \frac{\alpha\beta}{\delta(\delta+1)} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\delta(\delta+1)(\delta+2)\gamma(\gamma+1)} x^2 + \cdots + \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-2)\beta(\beta+1)(\beta+2)\cdots(\beta+n-2)}{\delta(\delta+1)(\delta+2)\cdots(\delta+n-1)\gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-2)} x^{n-1} + \cdots$$

Then $\int F(\alpha,\beta,\gamma,x) = x\, S(\alpha,\beta,1,\gamma,x) + c_1$

$$\iint F(dx)^2 = \Big(\frac{x^2}{2} + \frac{\alpha\beta}{3!\,\gamma} x^3 + \cdots + \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-2)\beta(\beta+1)(\beta+2)\cdots(\beta+n-2)}{(n+1)!\,\gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-2)} x^{n+1} + \cdots\Big)$$

$$+ c_1 x + c_2$$

$$= x^2 S(\alpha,\beta,2,\gamma,x) + c_1 x + c_2$$

The following may be easily verified

$$\iiint F(dx)^3 = \frac{x^3}{2!} S(\alpha, \beta, 3, \gamma, x) + c_1 x^2 + c_2 x + c_3$$

$$\iiiint F(dx)^4 = \frac{x^4}{3!} S(\alpha, \beta, 4, \gamma, x) + c_1 x^3 + c_2 x^2 + c_3 x + c_4$$

$$\iiiiint F(dx)^5 = \frac{x^5}{4!} S(\alpha, \beta, 5, \gamma, x) + c_1 x^4 + c_2 x^3 + c_3 x^2 + c_4 x + c_5$$

This leads to the following for the general case:

$$\iiint \cdots \int F(dx)^K = \frac{x^K}{(K-1)!} S(\alpha, \beta, K, \gamma, x) + c_1 x^K + c_2 x^{K-1} + \cdots c_{K-1} x + c_K \qquad (3.7)$$

We establish this by showing that (3.7) holds true for $K = K+1$.

$$\iint \cdots \int F(dx)^{K+1} = \iint \left\{ \frac{x^K}{(K-1)!} S(\alpha, \beta, K, \gamma, x) + c_1 x^K + c_2 x^{K-1} + \cdots + c_{K-1} x + c_K \right\} dx$$

$$= \iint \left\{ \frac{x^K}{(K-1)!} \left[\frac{1}{K} + \frac{\alpha \beta x}{K(K+1)\gamma} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{K(K+1)(K+2)\gamma(\gamma+1)} x^2 + \cdots \right. \right.$$

$$\left. \cdots + \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-2)\beta(\beta+1)(\beta+2)\cdots(\beta+n-2)}{K(K+1)(K+2)\cdots(K+n-1)\gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-2)} x^{n-1} + \cdots \right]$$

$$\left. + c_1 x^K + c_2 x^{K-1} + \cdots c_{K-1} x + c_K \right\} dx$$

$$= \frac{1}{(K-1)!} \left[\frac{x^{K+1}}{K(K+1)} + \frac{\alpha \beta x^{K+2}}{K(K+1)(K+2)\gamma} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{K(K+1)(K+2)(K+3)\gamma(\gamma+1)} x^{K+3} + \cdots \right.$$

$$\left. \cdots + \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-2)\beta(\beta+1)(\beta+2)\cdots(\beta+n-2)}{K(K+1)(K+2)\cdots(K+n)\cdot\gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-2)} x^{K+n} + \cdots \right]$$

$$+ c_1 x^{K+1} + c_2 x^K + c_3 x^{K-1} + \cdots + c_K x + c_{K+1}$$

$$= \frac{x^{K+1}}{K!} S(\alpha, \beta, K+1, \gamma, x) + c_1 x^{K+1} + c_2 x^K + \cdots$$

$$\cdots + c_K x + c_{K+1}$$

This establishes (3.7). Equation (3.7) is true for all positive integral values of n.

5. CONTIGUOUS FUNCTIONS. When one or more of the constant elements α, β, and γ of $F(\alpha,\beta,\gamma,x)$ are increased or diminished by one, the functions which arise are known as contiguous functions of $F(\alpha,\beta,\gamma,x)$. For convenience in writing $F(\alpha,\beta,\gamma,x)$ will hereafter be denoted by $F(\alpha,\beta,\gamma,x)$. $F(\alpha,\beta,\gamma)$ when such notation is desirable.

The primary function, $F(\alpha,\beta,\gamma)$ furnishes six contiguous functions:

$$F(\alpha+1,\ \beta,\ \gamma)$$
$$F(\alpha+1,\ \beta+1,\ \gamma)$$
$$F(\alpha+1,\ \beta+1,\ \gamma+1)$$
$$F(\alpha,\ \beta+1,\ \gamma)$$
$$F(\alpha,\ \beta+1,\ \gamma+1)$$
$$F(\alpha,\ \beta,\ \gamma+1)$$

Between any two of these and the primary function itself exists a relation expressed by a very simple linear equation. Since the number of such pairs of contiguous functions is $\frac{6.5}{1.2}$, there are 15 such linear relations. To derive these we proceed as follows. Let

$$\frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)\beta(\beta+1)(\beta+2)\cdots(\beta+n-2)}{1\cdot2\cdot3\cdots n\cdot\gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-1)} = M$$

The coefficient of X^n in $F(\alpha,\beta,\gamma)$ is

$$\frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)\beta(\beta+1)(\beta+2)\cdots(\beta+n-1)}{n!\ \gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-1)} = \alpha(\beta+n-1)M \tag{3.8}$$

The coefficient of X^n in $F(\alpha,\beta-1,\gamma)$ is

$$\frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)\beta(\beta+1)(\beta+2)\cdots(\beta+n-2)}{n!\cdot\gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-1)} = \alpha(\beta-1)M \tag{3.9}$$

Similarly for the coefficients of x^n in the following series we have

$$F(\alpha+1, \beta, \gamma) \quad , \quad (\alpha+n)(\beta+n-1)M \tag{3.10}$$

$$F(\alpha, \beta, \gamma-1) \quad , \quad \frac{\alpha(\beta+n-1)(\gamma+n-1)}{\gamma-1} M \tag{3.11}$$

$$F(\alpha+1, \beta, \gamma) \quad , \quad n(\gamma+n-1) M \tag{3.12}$$

It follows from (3.8),(3.10), and (3.12) that the coefficients of x^n in

$$(\gamma-\alpha-1)F(\alpha,\beta,\gamma) + \alpha F(\alpha+1,\beta\ \gamma) + (1-\gamma)F(\alpha,\beta,\gamma-1) = 0$$

Since this is true for every n we may write

$$(\gamma-\alpha-1)F(\alpha,\beta,\gamma) + \alpha F(\alpha+1,\beta,\gamma) + (1-\gamma)F(\alpha,\beta,\gamma-1) = 0 \tag{3.13}$$

From (3.9), (3.12), (3.8) weobtain by a similar process:

$$(\gamma-\alpha-\beta)F(\alpha,\beta,\gamma) + \alpha(1-x)F(\alpha+1,\beta,\gamma) + (\beta-\gamma)F(\alpha,\beta-1,\gamma,x) = 0 \tag{3.14}$$

Permuting α with β in (3.13) we get

$$(\gamma-\beta-1)F(\alpha,\beta,\gamma) + \beta F(\alpha,\beta+1,\gamma) + (1-\gamma)F(\alpha,\beta,\gamma-1) = 0 \tag{3.15}$$

Eliminating $F(\alpha,\beta,\gamma-1)$ from (3,13) and (3.15),

$$(\beta-\alpha)F(\alpha,\beta,\gamma) + \alpha F(\alpha+1,\beta,\gamma) - \beta F(\alpha,\beta+1,\gamma) = 0 \tag{3.16}$$

Permuting α with β in (3.14) ,

$$(\gamma-\alpha-\beta)F(\alpha,\beta,\gamma) + \beta(1-x)F(\alpha,\beta+1,\gamma) - (\gamma-\alpha)F(\alpha-1,\beta,\gamma) = 0 \tag{3.17}$$

Eliminating $F(\alpha,\beta+1,\gamma)$ from (3.15) and (3.17),

$$[\alpha-1-(\gamma-\beta-1)x]F(\alpha,\beta,\gamma)+(\gamma-\alpha)F(\alpha-1,\beta,\gamma)-(\gamma-1)(1-x)F(\alpha,\beta,\gamma-1)=0 \tag{3.18}$$

By permutation of α with β in (3.18),

$$[\beta-1-(\gamma-\alpha-1)x]F(\alpha,\beta,\gamma)+(\gamma-\beta)F(\beta-1,\alpha,\gamma)-(\gamma-1)(1-x)F(\alpha,\beta,\gamma-1)=0 \tag{3.19}$$

Combining (3.18) and 3.19),

$$(\beta-\alpha)(1-x)F(\alpha,\beta,\gamma)-(\gamma-\alpha)F(\alpha-1,\beta,\gamma)+(\gamma-\beta)F(\alpha,\beta-1,\gamma)=0 \tag{3.20}$$

Using (3.16) and (3.17),

$$[\gamma-2\alpha-(\beta-\alpha)x]F(\alpha,\beta,\gamma)+\alpha(1-x)F(\alpha+1,\beta,\gamma)-(\gamma-\alpha)F(\alpha-1,\beta,\gamma)=0 \tag{3.21}$$

A permutation of α with β gives

$$[\gamma-2\beta-(\alpha-\beta)x]F(\alpha,\beta,\gamma)+\beta(1-x)F(\beta+1,\alpha,\gamma)-(\gamma-\beta)F(\beta-1,\alpha,\gamma)=0 \tag{3.22}$$

Replace α by $\alpha-1$ and γ by $\gamma+1$ in (3.13). The result is

$$(\gamma-\alpha+1)F(\alpha-1,\beta,\gamma+1)+(\alpha-1)F(\alpha,\beta,\gamma+1)-\gamma F(\alpha-1,\beta,\gamma)=0 \tag{3.23$'$}$$

In (3.18) replace γ by $\gamma+1$. We then have

$$[\alpha-1-(\gamma-\beta)x]F(\alpha,\beta,\gamma+1)+(\gamma+1-\alpha)F(\alpha-1,\beta,\gamma+1)-\gamma(1-x)F(\alpha,\beta,\gamma)=0 \tag{3.24$'$}$$

When (3.24) is subtracted from (3.23), $F(\alpha-1,\beta,\gamma+1)$ is eliminated and the result is

$$\gamma(1-x)F(\alpha,\beta,\gamma)-\gamma F(\alpha-1,\beta,\gamma)+(\gamma-\beta)xF(\alpha,\beta,\gamma+1)=0 \tag{325}$$

1. This is not one of the 15 linear relations between $F(\alpha,\beta,\gamma)$ and any two of its contiguous functions.

By permutation of α with β, we obtain from (3.25)

$$\gamma(1-x)F(\alpha,\beta,\gamma) - \gamma F(\alpha,\beta-1,\gamma) + (\gamma-\alpha)x F(\alpha,\beta,\gamma+1) = 0 \tag{3.26}$$

A combination of (3.21) and (3.25) gives

$$\gamma\,[\alpha-(\gamma-\beta)x]\,F(\alpha,\beta,\gamma) - \alpha\gamma(1-x)F(\alpha+1,\beta,\gamma) + (\gamma-\alpha)(\gamma-\beta)\,x\,F(\alpha,\beta,\gamma+1) = 0 \tag{3.27}$$

From permutation of α with β in (3.27) results

$$\gamma\,[\beta-\overline{\gamma-\alpha}\,x]\,F(\alpha,\beta,\gamma) - \beta\gamma(1-x)\,F(\alpha,\beta+1,\gamma) + (\gamma-\beta)(\gamma-\alpha)x\,F(\alpha,\beta,\gamma+1) = 0 \tag{3.28}$$

A combination of (3.25) and (3.27) gives

$$\begin{aligned}\gamma\,[\gamma-1-(2\gamma-\alpha-\beta-1)x]\,F(\alpha,\beta,\gamma) + (\gamma-\alpha)(\gamma-\beta)\,x\,F(\alpha,\beta,\gamma+1)&\\ -\gamma(\gamma-1)(1-x)\,F(\alpha,\beta,\gamma-1) &= 0\end{aligned} \tag{3.29}$$

6. LINEAR RELATIONS BETWEEN $F(\alpha,\beta,\gamma)$, $F(\alpha+1,\beta+1,\gamma+1,x)$ AND $F(\alpha+2,\beta+2,\gamma+2)$

The following notation will be used

$$\begin{aligned}
F &= F(\alpha,\beta,\gamma,x) = F(\alpha,\beta,\gamma)\\
F' &= F(\alpha+1,\beta,\gamma,x) = F(\alpha+1,\beta,\gamma)\\
F'' &= F(\alpha+1,\beta+1,\gamma,x) = F(\alpha+1,\beta+1,\gamma)\\
F''' &= F(\alpha+1,\beta+1,\gamma+1,x) = F(\alpha+1,\beta+1,\gamma+1)\\
F^{iv} &= F(\alpha+2,\beta+1,\gamma+1,x) = F(\alpha+2,\beta+1,\gamma+1)\\
F^{v} &= F(\alpha+2,\beta+2,\gamma+1,x) = F(\alpha+2,\beta+2,\gamma+1)\\
F^{vi} &= F(\alpha+2,\beta+2,\gamma+2,x) = F(\alpha+2,\beta+2,\gamma+2)
\end{aligned}$$

Setting $\alpha=\alpha+1$ in (3.17) we have

$$(\gamma-\alpha-\beta-1)F'-(\gamma-\alpha-1)F+\beta(1-x)F''=0 \tag{3.30}$$

Setting $\alpha=\alpha+1$, $\beta=\beta+1$ in (3.26),

$$\gamma(1-x)F''-\gamma F'+(\gamma-\alpha-1)xF''=0 \tag{3.31}$$

Setting $\alpha=\alpha+1$, $\beta=\beta+1$, $\gamma=\gamma+1$, in(3.13),

$$(\gamma-\alpha-1)F'''+(\alpha+1)F^{IV}-\gamma F''=0 \tag{3.32}$$

Placing $\alpha=\alpha+2$, $\beta=\beta+2$, $\gamma=\gamma+1$ in(3.26),

$$(\gamma+1)(1-x)F^{V}-(\gamma+1)F^{IV}+(\gamma-\alpha-1)xF^{VI}=0 \tag{3.33}$$

Placing $\alpha=\alpha+2$, $\beta=\beta+2$, $\gamma=\gamma+1$ in(3.17),

$$(\gamma-\alpha-\beta-2)F^{IV}-(\gamma-\alpha-1)F'''+(\beta+1)(1-x)F^{V}=0 \tag{3.34}$$

Eliminating F' from (3.30) and (3.31), we obtain

$$\gamma F-\gamma(1-x)F''-(\gamma-\alpha-\beta-1)xF'''=0 \tag{3.35}$$

Eliminating F'' from (3.35) and (3.32),

$$\gamma F-(\gamma-\alpha-1-\beta x)F'''-(\alpha+1)(1-x)F^{IV}=0 \tag{3.36}$$

Eliminating F^{IV} from (3.34) and (3.33),

$$(\gamma+1)F'''-(\gamma+1)F^{IV}+(\beta+1)xF^{VI}=0 \tag{3.37}$$

Eliminating F^{IV} from (3.37) and (3.36),

$$\gamma(\gamma+1)F-(\gamma+1)(\gamma-\overline{\alpha+\beta+1}\,x)F'''-(\alpha+1)(\beta+1)x(1-x)F^{VI}=0 \tag{3.38}$$ [1]

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1. From this equation Gauss obtained the differential equation satisfied by the hypergeometric series. See Appendix, Note IV.

The following type of relations are easily obtained.

$$F(\alpha,\beta,\gamma) - F(\alpha,\beta,\gamma-1) = (1-1) + \left(\frac{\alpha\beta}{1\cdot\gamma} - \frac{\alpha\beta}{1\cdot(\gamma-1)}\right)x + \left(\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)} - \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2(\gamma-1)\gamma}\right)x^2 + \cdots$$

$$= -\frac{\alpha\beta}{\gamma(\gamma-1)} - \frac{\alpha\beta}{\gamma(\gamma-1)}\left(\frac{(\alpha+1)(\beta+1)}{1\cdot(\gamma+1)}\right)x - \frac{\alpha\beta}{\gamma(\gamma-1)}\left(\frac{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}{1\cdot 2\cdot(\gamma+1)(\gamma+2)}\right)x^2 + \cdots$$

$$= -\frac{\alpha\beta}{\gamma(\gamma-1)} F(\alpha+1,\beta+1,\gamma+1) = -\frac{\alpha\beta}{\gamma(\gamma-1)} F'''$$

Similarly one can show that

$$F(\alpha,\beta+1,\gamma) - F(\alpha,\beta,\gamma) = \frac{\alpha x}{\gamma} F(\alpha+1,\beta+1,\gamma+1) = \frac{\alpha\beta}{\gamma} F'''$$

$$F(\alpha+1,\beta,\gamma) - F(\alpha,\beta,\gamma) = \frac{\beta x}{\gamma} F(\alpha+1,\beta+1,\gamma+1) = \frac{\beta}{\gamma} x F'''$$

$$F(\alpha,\beta+1,\gamma+1) - F(\alpha,\beta,\gamma) = \frac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)} x F(\alpha+1,\beta+1,\gamma+2) \tag{3.39}$$

$$F(\alpha+1,\beta,\gamma+1) - F(\alpha,\beta,\gamma) = \frac{\beta(\gamma-\alpha)}{\gamma(\gamma+1)} F(\alpha+1,\beta+1,\gamma+2)$$

$$F(\alpha-1,\beta+1,\gamma) - F(\alpha,\beta,\gamma) = \frac{\alpha-\beta-1}{\gamma} x F(\alpha,\beta+1,\gamma+1)$$

$$F(\alpha+1,\beta-1,\gamma) - F(\alpha,\beta,\gamma) = \frac{\beta-\alpha-1}{\gamma} x F(\alpha+1,\beta,\gamma+1)$$

$$F(\alpha,\beta+1,\gamma) - F(\alpha+1,\beta,\gamma) = \frac{\alpha-\beta}{\gamma} x F(\alpha+1,\beta+1,\gamma+1)$$

$$= \frac{\alpha-\beta}{\gamma} x F'''$$

There are many other like relations between such functions. According to Gauss the number of relations between $F(\alpha,\beta,\gamma)$, $F(\alpha+\lambda, \beta+\mu, \gamma+\nu)$, and $F(\alpha+\lambda', \beta+\mu', \gamma+\nu')$, where λ, μ, ν, λ', μ', and ν' are either 0, $+1$, or -1, is as large a sum as 325.[1]

7. CONTINUED FRACTIONS. Let

$$\mathcal{G}(\alpha,\beta,\gamma,x) = \frac{F(\alpha,\beta+1,\gamma+1,x)}{F(\alpha,\beta,\gamma,x)}$$

Then

$$\mathcal{G}(\beta,\alpha,\gamma,x) = \frac{F(\beta,\alpha+1,\gamma+1,x)}{F(\alpha,\beta,\gamma,x)} = \frac{F(\alpha+1,\beta,\gamma+1,x)}{F(\alpha,\beta,\gamma,x)}$$

Divide Equation (3.39) by $F(\alpha,\beta+1,\gamma+1,x)$ thus:

$$\frac{F(\alpha,\beta+1,\gamma+1)}{F(\alpha,\beta+1,\gamma+1)} = \frac{F(\alpha,\beta,\gamma) + \frac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)} x \, F(\alpha+1,\beta+1,\gamma+2)}{F(\alpha,\beta+1,\gamma+1)}$$

or

$$1 - \frac{1}{\mathcal{G}(\alpha,\beta,\gamma,x)} = \frac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)} x \; \mathcal{G}(\beta+1,\alpha,\gamma+1,x)$$

Solving for $\mathcal{G}(\alpha,\beta,\gamma,x)$,

$$\mathcal{G}(\alpha,\beta,\gamma,x) = \frac{1}{1 - \frac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)} x \; \mathcal{G}(\beta+1,\alpha,\gamma+1,x)}$$

Then for $\mathcal{G}(\beta+1,\alpha,\gamma+1,x)$ we will have

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1. Gauss: Werke, t. III, p. 133

$$\mathcal{G}(\beta+1, \alpha, \gamma+1, x) = \cfrac{1}{1 - \cfrac{(\beta+1)(\gamma+1-\alpha)}{(\gamma+1)(\gamma+2)} \times \mathcal{G}(\alpha+1, \beta+1, \gamma+2, x)}$$

Similarly

$$\mathcal{G}(\alpha+1, \beta+1, \gamma+2, x) = \cfrac{1}{1 - \cfrac{(\alpha+1)(\gamma+2-\beta-1)}{(\gamma+2)(\gamma+3)} \times \mathcal{G}(\beta+2, \alpha+1, \gamma+3, x)}$$

$$\mathcal{G}(\beta+2, \alpha+1, \gamma+3, x) = \cfrac{1}{1 - \cfrac{(\beta+2)(\gamma+3-\alpha-1)}{(\gamma+3)(\gamma+4)} \times \mathcal{G}(\alpha+2, \beta+2, \gamma+4, x)}$$

The result is the following continued fraction for $\mathcal{G}(\alpha,\beta,\gamma,x)$

$$\mathcal{G}(\alpha,\beta,\gamma,x) = \frac{F(\alpha,\beta+1,\gamma+1,x)}{F(\alpha,\beta,\gamma,x)} = \cfrac{1}{1 - \cfrac{C_0 x}{1 - \cfrac{K_0 x}{1 - \cfrac{C_1 x}{1 - \cfrac{K_1 x}{1 - \cfrac{C_2 x}{1 - \cfrac{K_2 x}{1 - \cfrac{C_3 x}{1 - \text{etc}}}}}}}}$$

where
$$C_0 = \frac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)} \qquad K_0 = \frac{(\beta+1)(\gamma+1-\alpha)}{(\gamma+1)(\gamma+2)}$$

$$C_1 = \frac{(\alpha+1)(\gamma+1-\beta)}{(\gamma+2)(\gamma+3)} \qquad K_1 = \frac{(\beta+2)(\gamma+2-\alpha)}{(\gamma+3)(\gamma+4)}$$

$$C_2 = \frac{(\alpha+2)(\gamma+2-\beta)}{(\gamma+4)(\gamma+5)} \qquad K_2 = \frac{(\beta+3)(\gamma+3-\alpha)}{(\gamma+5)(\gamma+6)}$$

. . . .

$$C_n = \frac{(\alpha+n)(\gamma+n-\beta)}{(\gamma+2n)(\gamma+2n+1)} \qquad K_n = \frac{(\beta+n+1)(\gamma+n+1-\alpha)}{(\gamma+2n+1)(\gamma+2n+2)}$$

The law of progression of this fraction is evident. If in the fraction α, β, $\gamma-\alpha$, or $\gamma-\beta$, is a negative integer, the fraction will terminate: Otherwise it is continued indefinitely.

The most interesting application of (3.40) is found when β and γ are set equal to 0 and $\gamma-1$, respectively. Then

$$g(\alpha,\beta,\gamma,x) = \frac{F(\alpha,\beta+1,\gamma+1,x)}{F(\alpha,\beta,\gamma,x)} = \frac{F(\alpha,1,\gamma,x)}{1}$$

$$F(\alpha,1,\gamma,x) = 1 + \frac{\alpha}{\gamma}x + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}x^3 + \cdots$$

$$= \cfrac{1}{1 - \cfrac{C_0 x}{1 - \cfrac{K_0 x}{1 - \cfrac{C_1 x}{1 - \cfrac{K_1 x}{1 - \cfrac{C_2 x}{1 - \text{etc.}}}}}}}$$

where

$$c_0 = \frac{\alpha}{\gamma} \qquad K_0 = \frac{\gamma-\alpha}{\gamma(\gamma+1)}$$

$$c_1 = \frac{\gamma(\alpha+1)}{(\gamma+1)(\gamma+2)} \qquad K_1 = \frac{2(\gamma+1-\alpha)}{(\gamma+2)(\gamma+3)}$$

$$c_2 = \frac{(\alpha+2)(\gamma+1)}{(\gamma+3)(\gamma+4)} \qquad K_2 = \frac{3(\gamma+2-\alpha)}{(\gamma+4)(\gamma+5)}$$

$$\cdots\cdots\cdots \qquad \cdots\cdots\cdots$$

$$c_n = \frac{(\alpha+n)(\gamma-1+n)}{(\gamma-1+2n)(\gamma+2n)} \qquad K_n = \frac{(n+1)(\gamma+n-\alpha)}{(\gamma+2n)(\gamma+2n+1)}$$

But $(1+x)^n = F(-n,\beta,\beta,-x)$ by Equation (7.2)

Hence

$$(1+x)^n = \cfrac{1}{1-\cfrac{nx}{1+\cfrac{\frac{n+1}{2}x}{1-\cfrac{\frac{n-1}{2.3}x}{1+\cfrac{\frac{2(n+2)}{3.4}x}{1-\cfrac{\frac{2(n-2)}{4.5}x}{1+\text{etc.}}}}}}}$$

Similarly, since $e^t = \lim\limits_{K\to\infty} F\left(1, K, 1, \frac{t}{K}\right)$ by Equation (7.8)

we obtain

$$e^t = \cfrac{1}{1-\cfrac{t}{1+\cfrac{\frac{1}{3}t}{1-\cfrac{\frac{1}{6}t}{1+\cfrac{\frac{1}{6}t}{1-\cfrac{\frac{1}{10}t}{1+\cfrac{\frac{1}{10}t}{1-\text{etc.}}}}}}}}$$

Other functions may be expressed in the form of continued fractions in this way. Thus , by use of Equation $(A.11)$, $t = \sin t \cos t\, F(1, 1, \tfrac{3}{2}, \sin^2 t)$, we obtain

$$t = \cfrac{\sin t \cos t}{1 - \cfrac{\frac{1\cdot 2}{1\cdot 3}\sin^2 t}{1 - \cfrac{\frac{1\cdot 2}{3\cdot 5}\sin^2 t}{1 - \cfrac{\frac{3\cdot 4}{5\cdot 7}\sin^2 t}{1 - \cfrac{\frac{3\cdot 4}{7\cdot 9}\sin^2 t}{1 - \cfrac{\frac{5\cdot 6}{9\cdot 11}\sin^2 t}{1 - \cfrac{\frac{5\cdot 6}{11\cdot 13}\sin^2 t}{1 - \cfrac{\frac{7\cdot 8}{13\cdot 15}\sin^2 t}{1 - \cfrac{\frac{7\cdot 8}{15\cdot 17}\sin^2 t}{1 - \cfrac{\frac{9\cdot 10}{17\cdot 19}\sin^2 t}{1 - \cfrac{\frac{9\cdot 10}{19\cdot 21}\sin^2 t}{1 - \text{etc.}}}}}}}}}}}$$

CHAPTER IV

THE Π FUNCTION OF GAUSS

1. DEFINITION AND FUNDAMENTAL THEOREMS. The Π function was introduced into analysis by Gauss during his investigation of the possibilities of summation of the hypergeometric series. Aside from its use in summation of the series, this function has other interesting applications as we shall see later. The summation of $F(\alpha, \beta, \gamma, x)$ is of course impossible unless the series is convergent. Convergence depends upon the value of x and upon the relations between the elements α, β, and γ as shown in Article 2 of Chapter III.

Consider first the ordinary geometric series, whose sum will be denoted by S.

$$S = 1 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_{n-1} x^{n-1} + c_n x^n + \cdots$$

Multiply by $(1 - x)$

$$(1-x)S = 1 + (c_1 - 1)x + (c_2 - c_1)x^2 + (c_3 - c_2)x^3 + \cdots + (c_n - c_{n-1})x^n + \cdots$$

When $x = 1$, the sum of the right member is c_n. We then have

$$(1-x)S = c_n$$

In the series $F(\alpha, \beta, \gamma, x)$ there is convergence for $x = 1$ provided $\gamma - \alpha - \beta > 0$. Let us impose this condition: then $F(\alpha, \beta, \gamma, x)$ is convergent. Moreover its terms are getting smaller and smaller and approach zero as a limit since in all convergent series the nth term approaches zero as a limit. For the present purpose it is desirable to use the series $F(\alpha, \beta, \gamma - 1, x)$. Now, provided $\gamma - \cancel{1} - \alpha - \beta > 0$, we may write

$$(1-x)\,F(\alpha,\beta,\gamma-1,x) = 0 \qquad \text{for } x = 1 .$$

By (3.29) we have

$$\gamma(\alpha+\beta-\gamma)\,F(\alpha,\beta,\gamma,1) + (\gamma-\alpha)(\gamma-\beta)\,F(\alpha,\beta,\gamma+1,1) = 0 \qquad \text{or}$$

$$F(\alpha,\beta,\gamma,1) = \frac{(\gamma-\alpha)(\gamma-\beta)}{\gamma(\gamma-\alpha-\beta)}\,F(\alpha,\beta,\gamma+1,1)$$

From this

$$F(\alpha,\beta,\gamma+1,1) = \frac{(\gamma+1-\alpha)(\gamma+1-\beta)}{(\gamma+1)(\gamma+1-\alpha-\beta)}\,F(\alpha,\beta,\gamma+2,1)$$

$$F(\alpha,\beta,\gamma+2,1) = \frac{(\gamma+2-\alpha)(\gamma+2-\beta)}{(\gamma+2)(\gamma+2-\alpha-\beta)}\,F(\alpha,\beta,\gamma+3,1)$$

. . . .

$$F(\alpha,\beta,\gamma+K,1) = \frac{(\gamma+K-\alpha)(\gamma+K-\beta)}{(\gamma+K)(\gamma+K-\alpha-\beta)}\,F(\alpha,\beta,\gamma+K,1)$$

Hence

$$F(\alpha,\beta,\gamma,1) = \frac{(\gamma-\alpha)(\gamma-\beta)}{\gamma(\gamma-\alpha-\beta)}\cdot\frac{(\gamma+1-\alpha)(\gamma+1-\beta)}{(\gamma+1)(\gamma+1-\alpha-\beta)}\cdot\frac{(\gamma+2-\alpha)(\gamma+2-\beta)}{(\gamma+2)(\gamma+2-\alpha-\beta)}$$

$$\cdots\cdots\frac{(\gamma+K-\alpha)(\gamma+K-\beta)}{(\gamma+K)(\gamma+K-\alpha-\beta)}\,F(\alpha,\beta,\gamma+K,1) \tag{4.1}$$

It is convenient to introduce a new notation at this point. Let

$$\Pi(K, z) = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots\cdots K}{(z+1)(z+2)(z+3)(z+4)\cdots(z+K)} K^{z} \tag{4.2}$$

where K is a positive integer. In this notation Equation (4.1) becomes

$$F(\alpha, \beta, \gamma, 1) = \frac{\Pi(K, \gamma-1)\,\Pi(K, \gamma-\alpha-\beta-1)}{\Pi(K, \gamma-\alpha-1)\,\Pi(K, \gamma-\beta-1)} F(\alpha, \beta, \gamma+K, 1) \tag{4.3}$$

When z is a negative integer, the Π function takes on an indefinitely great value.

For positive integral values of z we have

$$\begin{aligned}
\Pi(K, 0) &= 1 \\
\Pi(K, 1) &= \frac{1}{1+\frac{1}{K}} \\
\Pi(K, 2) &= \frac{1 \cdot 2}{\left(1+\frac{1}{K}\right)\left(1+\frac{2}{K}\right)} \\
&\cdots\cdots\cdots \\
\Pi(K, z) &= \frac{1 \cdot 2 \cdot 3 \cdots\cdots z}{\left(1+\frac{1}{K}\right)\left(1+\frac{2}{K}\right)\left(1+\frac{3}{K}\right)\cdots\left(1+\frac{z}{K}\right)}
\end{aligned} \tag{4.4}$$

Further

$$\Pi(k, z+1) = \frac{1 \cdot 2 \cdot \cdots \cdots z \cdot z+1}{\left(1+\frac{1}{k}\right)\left(1+\frac{2}{k}\right) \cdots\cdots \left(1+\frac{z}{k}\right)\left(1+\frac{z+1}{k}\right)}$$

$$= \Pi(k, z) \cdot \frac{z+1}{1+\frac{z+1}{k}} \tag{4.5}$$

and since $\Pi(1, z) = \frac{1}{z+1}$,

$$\Pi(k+1, z) = \Pi(k, z) \left\{ \frac{\left(1+\frac{1}{k}\right)^{z+1}}{1+\frac{1+z}{k}} \right\} \tag{4.6}$$

By means of this we are enabled to write

$$\Pi(2, z) = \Pi(1, z) \left\{ \frac{2^{z+1}}{2+z} \right\}$$

But $\Pi(1, z) = \frac{1}{z+1}$ hence this becomes

$$\Pi(2, z) = \frac{1}{z+1} \left\{ \frac{2^{z+1}}{2+z} \right\}$$

$$\Pi(3, z) = \Pi(2, z) \left\{ \frac{\left(\frac{3}{2}\right)^{z+1}}{\frac{1}{2}(3+z)} \right\} = \frac{1}{z+1} \cdot \frac{2^{z+1}}{2+z} \cdot \frac{3^{z+1}}{2^{z}(3+z)}$$

$$\Pi(4, z) = \Pi(3, z) \left\{ \frac{\left(\frac{4}{3}\right)^{z+1}}{\frac{4+z}{3}} \right\} = \frac{1}{z+1} \cdot \frac{2^{z+1}}{2+z} \cdot \frac{3^{z+1}}{2^{z}(3+z)} \cdot \frac{4^{z+1}}{3^{z}(4+z)}$$

This leads to

$$\Pi(k, z) = \frac{1}{z+1} \cdot \frac{2^{z+1}}{1^{z}(2+z)} \cdot \frac{3^{z+1}}{2^{z}(3+z)} \cdot \frac{4^{z+1}}{3^{z}(4+z)} \cdot \cdots \frac{k^{z+1}}{(k-1)^{z}(k+z)}$$

Equation (4.4) shows that $\lim_{k \to \infty} \Pi(k, z)$ is finite, provided z is finite, and is equal to $z!$. That is, in the limit,

$$\Pi(k, z) = 1 \cdot 2 \cdot 3 \cdot 4 \cdots\cdots (z-1)z \tag{4.7}$$

It is evident that $\Pi(\infty, z)$ depends on z alone or on a function of z, there fore the symbol Π_z will hereafter denote $\Pi(\infty, z)$. That is

$$\Pi_z = \lim_{k \to \infty} \left[\frac{1 \cdot 2 \cdot 3 \cdots\cdots k}{(z+1)(z+2)(z+3) \cdots\cdots (z+k)} k^z \right] \tag{4.8}$$

By Equation (4.5) we have

$$\Pi(z+1) = (z+1)\Pi_z$$

$$\Pi(z+2) = (z+2)\Pi_z$$

or generally

$$\Pi(z+n) = (z+1)(z+2) \cdots\cdots (z+n)\Pi_z \tag{4.9}$$

By definition[1]

$$\Pi(k, z) = \frac{1 \cdot 2 \cdots\cdots k}{(z+1)(z+2) \cdots\cdots (z+k)} k^z$$

Multiply the right member by $\frac{z!}{z!}$. The result is

$$\Pi(k, z) = \frac{k!\, k^z z!}{(z+1)(z+2)(z+3) \cdots\cdots (z+k) z!} = \frac{k^z \Pi_k \Pi_z}{\Pi(k+z)} \tag{4.10}$$

since by (4.8) and (4.9),

1. Equation (4.2)

$$\Pi_z = z!$$

$$\Pi_k = k!$$

$$\Pi(z+k) = (z+1)(z+2)\cdots(z+k)\Pi_z = (z+1)(z+2)\cdots\cdots(z+k)z!$$

We now return to Equation (4.3),

$$F(\alpha,\beta,\gamma,1) = \frac{\Pi(k,\gamma-1)\Pi(k,\gamma-\alpha-\beta-1)}{\Pi(k,\gamma-\alpha-1)\Pi(k,\gamma-\beta-1)}F(\alpha,\beta,\gamma+k,1) \qquad (4.3)$$

Since the limit of $F(\alpha,\beta,\gamma+k,1)$ as k increases without limit is 1, we may write

$$F(\alpha,\beta,\gamma,1) = \frac{\Pi(\gamma-1)\Pi(\gamma-\alpha-\beta-1)}{\Pi(\gamma-\alpha-1)\Pi(\gamma-\beta-1)} \qquad (4.11)$$

which settles the question of summation of the hypergeometric series for $x=1$.

We may express Equation (4.11) in the form

$$F(\alpha,\beta,\gamma,1) = \frac{(\gamma-1)!\,(\gamma-\alpha-\beta-1)!}{(\gamma-\alpha-1)!\,(\gamma-\beta-1)!} \qquad (4.12)$$

This enables us to write

$$F(-\alpha,-\beta,\gamma-\alpha-\beta,1) = \frac{(\gamma-\alpha-\beta-1)!\,(\gamma-\alpha-\beta+\alpha+\beta-1)!}{(\gamma-\alpha-\beta+\alpha+\beta-1)!\,(\gamma-\alpha-\beta+\beta-1)!}$$

$$= \frac{(\gamma-\alpha-\beta-1)!\,(\gamma-1)!}{(\gamma-\beta-1)!\,(\gamma-\alpha-1)!} = F(\alpha,\beta,\gamma,1)$$

or more concisely,

$$F(\alpha,\beta,\gamma,1) = F(-\alpha,-\beta,\gamma-\alpha-\beta,1)\ , \quad \gamma > 0 \tag{4.13}$$

In the same way the following formulae are derived.

$$F(\alpha,\beta,\gamma,1) \cdot F(-\alpha,\beta,\gamma-\alpha,1) = 1 \tag{4.14}$$

$$F(\alpha,\beta,\gamma,1) \cdot F(-\alpha,-\beta,\gamma-\beta,1) = 1 \tag{4.15}$$

2. RELATION BETWEEN Π AND π. From Equation $(A.10)$, we obtain the formula,

$$t = \sin t \cdot F\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, \sin^2 t\right)$$

If we set $t = \frac{\pi}{2}$[1],

$$\frac{\pi}{2} = F\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, 1\right) \qquad \text{or}$$

$$\frac{\pi}{2} = 1 + \frac{1\cdot1}{2\cdot3} + \frac{1\cdot1\cdot3}{2\cdot4\cdot5} + \frac{1\cdot1\cdot3\cdot5}{2\cdot4\cdot6\cdot7} + \frac{1\cdot1\cdot1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8\cdot9} + \cdots\cdots \tag{4.16}$$

By Equation (4.11)

$$\frac{\pi}{2} = F\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, 1\right) = \frac{\Pi_{\frac{1}{2}}\ \Pi_{-\frac{1}{2}}}{\Pi_0\ \Pi_0} \tag{4.17}$$

But $\Pi(z+1) = (z+1)\Pi_z$, hence

$$\Pi_{\frac{1}{2}} = \tfrac{1}{2}\,\Pi_{-\frac{1}{2}}$$

Further $\Pi_0 = 1$ and we get the result

$$\frac{\pi}{2} = \frac{1}{2}\left[\Pi_{(-\frac{1}{2})}\right]^2 \tag{4.18}$$

or

$$\Pi_{-\frac{1}{2}} = \sqrt{\pi} \tag{4.19}$$

or

$$\Pi_{\frac{1}{2}} = \frac{\sqrt{\pi}}{2} \tag{4.20}$$

since $\Pi_{-\frac{1}{2}} = 2\,\Pi_{\frac{1}{2}}$.

3. RELATIONS BETWEEN THE Π FUNCTION AND THE TRIGONOMETRIC FUNCTIONS. The following formula is given by Equation (7.18) .

$$\sin nt = n \sin t \; F\left(\frac{n}{2} + \frac{1}{2}, \; -\frac{n}{2} + \frac{1}{2}, \; \frac{3}{2}, \; \sin^2 t \right)$$

for $t = \frac{\pi}{2}$, this becomes

$$\sin \frac{n\pi}{2} = n \, F\left(\frac{n}{2} + \frac{1}{2}, \; -\frac{n}{2} + \frac{1}{2}, \; \frac{3}{2}, \; 1 \right)$$

and by use of (411) we obtain the relation,

$$\sin \frac{n\pi}{2} = \frac{n\,\Pi_{\frac{1}{2}} \; \Pi_{-\frac{1}{2}}}{\Pi_{-\frac{n}{2}} \; \Pi_{\frac{n}{2}}} = \frac{\frac{1}{2}\, n \left[\Pi_{-\frac{1}{2}} \right]^2}{\Pi_{-\frac{n}{2}} \; \Pi_{\frac{n}{2}}} \tag{4.21}$$

But by Equation (4.18) $\left[\Pi_{-\frac{1}{2}} \right]^2 = \pi$ and (4.21) becomes

$$\sin \frac{n\pi}{2} = \frac{\frac{1}{2}\, n\pi}{\Pi_{-\frac{n}{2}} \, \Pi_{\frac{n}{2}}}$$

or

$$\Pi_{\frac{n}{2}} \cdot \Pi_{-\frac{n}{2}} = \frac{\frac{1}{2}\, n\pi}{\sin \frac{n\pi}{2}} \tag{4.22}$$

Let $n = 2z$. Then the preceding equation becomes

$$\Pi_z \cdot \Pi_{-z} = \frac{z\pi}{\sin z\pi} \tag{4.23}$$

Since $\Pi_z = z\,\Pi_{(z-1)}$ we may express this relation in yet another form

$$\Pi_{-z} \cdot \Pi_{(z-1)} = \frac{\pi}{\sin \pi z} \tag{4.24}$$

If a relation between Π and $\cos z$ is desired, we make the substitution, $z = z + \frac{1}{2}$ in (4.24). The result is

$$\Pi(-\tfrac{1}{2}-z) \cdot \Pi(-\tfrac{1}{2}+z) = \frac{\pi}{\sin(\pi z + \frac{\pi}{2})} = \frac{\pi}{\cos \pi z} \tag{4.25}$$

Combining (4.23) with our definition of Π given by (4.2) we have

$$\frac{\pi z}{\sin \pi z} = \lim_{k \to \infty} \left[\frac{(1 \cdot 2 \cdot 3 \cdots k)^2}{(1-z^2)(4-z^2)(9-z^2)\cdots(k^2-z^2)}\right] \tag{4.26}$$

or

$$\sin \pi z = \pi z \lim_{k \to \infty} \left(\frac{1-z^2}{1^2}\right)\left(\frac{4-z^2}{2^2}\right)\left(\frac{9-z^2}{3^2}\right)\cdots\left(\frac{k^2-z^2}{k^2}\right) \tag{4.27}$$

or

$$\sin \pi z = \pi z (1-z^2)(1-\tfrac{z^2}{4})(1-z^2/9)(1-z^2/16)\cdots \tag{4.28}$$

4. THE INTEGRAL, $\int_0^1 x^{\lambda-1}(1-x^\mu)^\nu dx$, IN TERMS OF Π FUNCTIONS. If λ, and μ are positive quantities,

$$\int x^{\lambda-1}(1-x^\mu)^\nu dx = \int x^{\lambda-1}\Big[1 - \frac{\nu}{1!}x^\mu + \frac{\nu(\nu-1)}{2!}x^{2\mu} \cdots\cdots$$

$$+ (-1)^{n-1}\frac{\nu(\nu-1)\cdots\cdots(\nu-n-2)}{(n-1)!}x^{(n-1)\mu} + \cdots\Big] \tag{4.29}$$

$$= \int \left[x^{\lambda-1} - \frac{\nu}{1!} x^{\mu+\lambda-1} + \frac{\nu(\nu-1)}{2!} x^{2\mu+\lambda-1} - \cdots + (-1)^{n-1} \frac{\nu(\nu-1)\cdots(\nu-n-2)}{(n-1)!} x^{(n-1)\mu+\lambda-1} + \cdots \right]$$

$$= \frac{x^\lambda}{\lambda} - \frac{\nu x^{\mu+\lambda}}{1\cdot(\mu+\lambda)} + \frac{\nu(\nu-1)\, x^{2\mu+\lambda}}{1\cdot 2\cdot(2\mu+\lambda)} - \cdots + (-1)^{n-1} \frac{\nu(\nu-1)\cdots(\nu-n-2)}{(n-1)!\,(\overline{n-1}\,m+\lambda)} x^{(n-1)\mu+\lambda} + \cdots$$

$$= \frac{x^\lambda}{\lambda} F\left(-\nu, \frac{\lambda}{\mu}, \frac{\lambda}{\mu}+1, x^\mu\right)$$

For $x = 1$,

$$\frac{x^\lambda}{\lambda} F\left(-\nu, \frac{\lambda}{\mu}, \frac{\lambda}{\mu}+1, x^\mu\right) = \frac{1}{\lambda} F\left(-\nu, \frac{\lambda}{\mu}, \frac{\lambda}{\mu}, 1\right) = \frac{\Pi\left(\frac{\lambda}{\mu}\right)\Pi\nu}{\lambda\,\Pi\left(\frac{\lambda}{\mu}+\nu\right)} \tag{4.30}$$

This theorem has many interesting applications of which the following is an example. Consider the integrals,

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = A \qquad\qquad \int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} = B$$

$$A = \int_0^1 x^0 (1-x^4)^{-\frac{1}{2}} dx \qquad\qquad B = \int_0^1 x^2 (1-x^4)^{-\frac{1}{2}} dx$$

Here $\lambda - 1 = 0$ or $\lambda = 1$, $\mu = 4$, $\nu = -\frac{1}{2}$. Substituting these values in (4.30) we obtain

$$A = \frac{\Pi_{\frac{1}{4}}\,\Pi_{-\frac{1}{2}}}{1\cdot \Pi_{-\frac{1}{4}}} \qquad \text{for } x = 1. \tag{4.31}$$

We repeat this process for B .

$$B = \int_0^1 x^2 (1-x^4)^{-\frac{1}{2}} dx$$

Here $\lambda - 1 = 2$ or $\lambda = 3$, $\mu = 4$, $\nu = -\frac{1}{2}$. The substitution of these values in (4.30) gives

$$\beta = \frac{\Pi_{\frac{3}{4}}\, \Pi_{-\frac{1}{2}}}{3\, \Pi_{\frac{1}{4}}} \tag{4.32}$$

But $\Pi_{\frac{1}{4}} = \Pi_{(-\frac{1}{4}+1)} = \frac{3}{4}\, \Pi_{(-\frac{1}{4})}$ and therefore

$$\beta = \frac{\Pi_{-\frac{1}{4}}\, \Pi_{-\frac{1}{2}}}{4\, \Pi_{\frac{1}{4}}} \tag{4.33}$$

and

$$A\beta = \frac{\Pi_{\frac{1}{4}}\, \Pi_{-\frac{1}{2}}}{\Pi_{-\frac{1}{4}}} \cdot \frac{\Pi_{-\frac{1}{4}}\, \Pi_{-\frac{1}{2}}}{4\, \Pi_{-\frac{1}{4}}} = \frac{1}{4}\left[\Pi_{-\frac{1}{2}}\right]^2 = \frac{\pi}{4} \tag{4.34}$$

The values of A and β have been computed by Stirling.[1] His results are

$A = 1.31102877714605987$

$\beta = 0.59907011736779611$

Gauss' computation of β is slightly different, being

$\beta = 0.5990701173677961037 2$

Now that the values of A and β are known we proceed to determine the value of $\Pi_{\frac{1}{4}}$ and $\Pi_{-\frac{1}{4}}$. By means of (4.23) we write

$$\Pi_{\frac{1}{4}} \cdot \Pi_{-\frac{1}{4}} = \frac{\frac{\pi}{4}}{\sin \frac{\pi}{4}} = \frac{\pi}{\sqrt{8}}$$

or

$$\Pi_{-\frac{1}{4}} = \frac{\pi}{\sqrt{8}} \cdot \frac{1}{\Pi_{\frac{1}{4}}}$$

1. These figures of Stirling and those of Gauss are found in Volume III of Gauss' Werke.

A substitution of this value of $\Pi_{-\frac{1}{4}}$ in (4.31) gives

$$A = \frac{\sqrt{8}\,(\Pi_{\frac{1}{4}})^2}{\sqrt{\pi}} \quad \text{or} \quad A^2 = \frac{8}{\pi}\left[\Pi_{\frac{1}{4}}\right]^4$$

Solving for $\Pi_{\frac{1}{4}}$ we get

$$\Pi_{\frac{1}{4}} = \sqrt[4]{\frac{\pi A^2}{8}}$$

Similarly we obtain for $\Pi_{-\frac{1}{4}}$,

$$\Pi_{-\frac{1}{4}} = \sqrt[4]{\frac{\pi^3}{8A^2}} = \sqrt[4]{2\pi B^2}$$

5. $\dfrac{(2\pi)^{\frac{1}{2}(n-1)}}{\sqrt{n}}$ IN TERMS OF Π FUNCTIONS. Let us denote by the symbol, M, the value of

$$\frac{n^{nz}\,\Pi(k,z)\cdot\Pi(k,z-\frac{1}{n})\cdot\Pi(k,z-\frac{2}{n})\cdots\Pi(k,z-\frac{n-1}{n})}{\Pi(nk,\ nz)} \tag{4.35}$$

or

$$\frac{(1\cdot 2\cdot 3\ \cdots\cdots\ k)^n\cdot n^{nk}}{(1\cdot 2\cdot 3\ \cdots\cdots\ nk)\cdot k^{\frac{1}{2}(n-1)}}$$

This expression is independent of z and its value will remain the same whatever z may be. Let us choose zero for our value of z. Since $\Pi(k,0) = \Pi(nk,0) = 1$, we have for (4.35) ,

$$\Pi(k,-\tfrac{1}{n})\cdot\Pi(k,-\tfrac{2}{n})\,\Pi(k,-\tfrac{3}{n})\cdots\cdots\Pi(k,-\tfrac{n-1}{n}) \tag{4.36}$$

As $k \to \infty$ this becomes

$$\Pi_{-\frac{1}{n}} \cdot \Pi_{-\frac{2}{n}} \cdot \Pi_{-\frac{3}{n}} \cdot \cdots\cdots\cdots \cdot \Pi_{-\frac{n-1}{n}} \tag{4.37}$$

There follows the relation

$$\Pi_{-\frac{1}{n}} \Pi_{-\frac{2}{n}} \cdots \Pi_{-\frac{n-1}{n}} = \lim_{K\to\infty} \frac{n^{nz}\,\Pi(K,z)\,\Pi(K,z-\frac{1}{n}) \cdots \Pi(K,z-\frac{n-1}{n})}{\Pi(nK,nz)}$$

or

$$\Pi_{-\frac{1}{n}} \Pi_{-\frac{2}{n}} \Pi_{-\frac{3}{n}} \cdots \Pi_{-\frac{n-1}{n}} = \frac{n^{nz}\,\Pi_z\,\Pi_{z-\frac{1}{n}} \cdots \Pi_{(z-\frac{n-1}{n})}}{\Pi_{nz}} \tag{4.38}$$

If, in Equation (4.24), we set $z = \frac{1}{n}$, we have

$$\Pi_{-\frac{1}{n}} \Pi_{-\frac{n-1}{n}} = \frac{\pi}{\sin\frac{\pi}{n}}$$

But the factors of the left member are respectively the first and last factors of (4.36). Moreover, on setting $z = \frac{2}{n}$, we get

$$\Pi_{-\frac{2}{n}} \Pi_{-\frac{n-2}{n}} = \frac{\pi}{\sin\frac{2\pi}{n}}$$

in which the factors of the left member are the second and next to the last factors of (4.36). This process continued gives

$$\Pi_{-\frac{1}{n}} \Pi_{-\frac{2}{n}} \Pi_{-\frac{3}{n}} \cdots \Pi_{-\frac{n-2}{n}} \Pi_{-\frac{n-1}{n}} \tag{4.39}$$

$$= \frac{\pi}{\sin\frac{\pi}{n}} \cdot \frac{\pi}{\sin\frac{2\pi}{n}} \cdot \frac{\pi}{\sin\frac{3\pi}{n}} \cdots\cdots \frac{\pi}{\sin\left[\left(\frac{n-1}{n}\right)\pi\right]} = \frac{(2\pi)^{n-1}}{n}$$

and there follows from (4.38)

$$\frac{n^{nz}\Pi_z \Pi_{z-\frac{1}{n}} \cdots\cdots \Pi_{z-\frac{n-1}{n}}}{\Pi_{nz}} = \frac{(2\pi)^{\frac{1}{2}(n-1)}}{\sqrt{n}} \tag{4.40}$$

6. THE INTEGRAL, $\int_0^\infty z^\beta e^{-z^\alpha} dz$. In Article 4 of this chapter, Equation (4.30), it was shown that

$$\int_0^1 x^{\lambda-1}(1-x^\mu)^\nu dx = \frac{\Pi(\frac{\lambda}{\mu})\,\Pi(\nu)}{\lambda\,\Pi(\frac{\lambda}{\mu}+\nu)} \tag{4.30}$$

where λ, μ, and ν are constants.

Let us make the substitution, $x = \frac{y}{\nu}$, and set $\mu = 1$. We then have

$$\int_0^\nu \frac{y^{\lambda-1}(1-\frac{y}{\nu})^\nu dy}{\nu^\lambda} = \frac{\Pi(\lambda)\,\Pi(\nu)}{\lambda\,\Pi(\lambda+\nu)} \tag{4.41}$$

since $\displaystyle\int_0^\nu \frac{y^{\lambda-1}(1-\frac{y}{\nu})^\nu}{\nu^\lambda} dy = \int_0^1 x^{\lambda-1}(1-x^\mu)^\nu dx = \frac{\Pi(\lambda)\,\Pi(\nu)}{\lambda\,\Pi(\frac{\lambda}{\mu}+\nu)}$

Writing (4.41) in the form,

$$\int_0^\nu y^{\lambda-1}(1-\frac{y}{\nu})^\nu dy = \frac{\nu^\lambda\,\Pi_\lambda\,\Pi_\nu}{\lambda\,\Pi(\lambda+\nu)} \tag{4.42}$$

and remembering that $\Pi(\kappa, z) = \frac{\kappa^z\,\Pi_\kappa\,\Pi_z}{\Pi(\kappa+z)}$ where z is an integer, we obtain

$$\int_0^\nu y^{\lambda-1}(1-\frac{y}{\nu})^\nu dy = \frac{\Pi(\nu,\lambda)}{\lambda} \quad \text{where } \nu \text{ is an integer.} \tag{4.43}$$

Now let ν increase without limit. $\Pi(\nu,\lambda)$ becomes Π_λ and $(1-\frac{y}{\nu})^\nu$

becomes e^{-y} where e denotes the base of natural logarithms. Therefore if λ is positive $\frac{\Pi_\lambda}{\lambda} = (\Pi_{(\lambda-1)}$ expresses the value of the integral, $\int_0^\infty y^{\lambda-1} e^{-y} dy$. Or letting $\lambda = \lambda+1$, we have

$$\Pi_\lambda = \int_0^\infty y^\lambda e^{-y} dy \qquad \text{if} \qquad \lambda + 1 > 0 \tag{4.44}$$

In order to obtain a general case place $y = z^\alpha$, $\alpha\lambda + \alpha - 1 = \beta$, where α, β, are constants. Then

$$\int_0^\infty y^\lambda e^{-y} dy \qquad \text{becomes} \qquad \int_0^\infty \alpha z^\beta e^{-z^\alpha} dz$$

and it follows on substitution of $\lambda = \frac{\beta - \alpha + 1}{\alpha}$ that

$$\int_0^\infty z^\beta e^{-z^\alpha} dz = \frac{\Pi\left(\frac{\beta+1}{\alpha} - 1\right)}{\alpha} \tag{4.45}$$

if α and $\beta + 1$ are positive quantities. If either α or β is negative (4.45) becomes

$$\int_0^\infty z^\beta e^{-z^\alpha} dz = -\frac{\Pi\left(\frac{\beta+1}{\alpha} - 1\right)}{\alpha} \tag{4.46}$$

7. EVALUATION OF $\int_0^\infty e^{-z^2} dz$. If in (4.45) we set $\alpha = 2$, $\beta = 0$,

$$\int_0^\infty e^{-z^2} dz = \frac{\Pi_{-\frac{1}{2}}}{2} = \frac{\sqrt{\pi}}{2} \tag{4.47}$$

CHAPTER V .

THE TWENTY-FOUR INTEGRALS OF THE HYPERGEOMETRIC EQUATION

1. NORMAL FORM BF THE GENERAL LINEAR DIFFERENTIAL EQUATION OF THE SECOND ORDER. Consider the general linear differential equation of the second order,

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \tag{5.1}$$

Set $y = ve^{-\frac{1}{2}\int P dx}$. Then $y' = \left(-\frac{Pv}{2} + \frac{dv}{dx}\right)e^{-\frac{1}{2}\int P dx}$ and

$$y'' = \left(\frac{1}{4}P^2v - P\frac{dv}{dx} - \frac{1}{2}v\frac{dP}{dx} + \frac{d^2v}{dx^2}\right)e^{-\frac{1}{2}\int P dx}$$

On substituting these values in (5.1) we have

$$\left(\frac{P^2v}{4} - P\frac{dv}{dx} - \frac{v}{2}\frac{dP}{dx} + \frac{d^2v}{dx^2} - \frac{P^2v}{2} + P\frac{dv}{dx} + Qv\right)e^{-\frac{1}{2}\int P dx} = 0$$

If v' denote $\frac{dv}{dx}$ and v'' denote $\frac{d^2v}{dx^2}$, the equation becomes

$$v'' + \frac{1}{4}\left(4Q - 2\frac{dP}{dx} - P^2\right)v = 0 \tag{5.2}$$

Let us represent the expression, $\frac{1}{4}\left(4Q - 2\frac{dP}{dx} - P^2\right)$ by ψ. (5.2) then assumes the form

$$v'' + \psi v = 0 \tag{5.3}$$

This is called the normal form of (5.1). There is a great advantage in using this form as will later be appasent.

2. THE INVARIANT OF THE COEFFICIENTS OF THE LINEAR DIFFERENTIAL EQUATION OF THE SECOND ORDER. Let us make the substitution $y = z\phi(x)$ in (5.1). Denoting by primes differentiation with respect to x we have

$$y' = z'\phi + \phi' z$$
$$y'' = z''\phi + 2z'\phi' + z\phi''$$

When these are substituted in (5.1) there results

$$\phi z'' + (2\phi' + P\phi)z' + (\phi'' + P\phi' + Q\phi)z = 0 \qquad \text{or}$$

$$z'' + \left(\frac{2\phi'}{\phi} + P\right)z' + \left(\frac{\phi''}{\phi'} + \frac{P\phi'}{\phi} + Q\right)z = 0$$

$$z'' + P_1 z' + Q_1 z = 0 \qquad \text{or}$$

where (5.4)

where $\quad P_1 = \frac{2\phi'}{\phi} + P\,, \quad Q_1 = \frac{\phi''}{\phi} + \frac{P\phi'}{\phi} + Q$

Let the normal form of (5.4) be

$$\omega'' + \psi_1 \omega = 0 \qquad \text{where} \quad \omega = z e^{\frac{1}{2}\int P_1 dx}$$

Then

$$\psi_1 = \frac{1}{4}\left(4Q_1 - 2\frac{dP_1}{dx} - P_1^2\right)$$
$$= \frac{\phi''}{\phi} + P\frac{\phi'}{\phi} + Q - \frac{1}{2}\frac{d}{dx}\left(2\frac{\phi'}{\phi} + P\right) - \frac{1}{4}\left(2\frac{\phi'}{\phi} + P\right)^2$$
$$= \frac{1}{4}\left(4Q - 2\frac{dP}{dx} - P^2\right)$$

which shows that ψ is an invariant of the coefficients of the equation, (5.1), under the transformation $y = z\phi(x)$. It follows taht that any two linear equations such as (5.1) and (5.4) can be transformed into one another if the normal form of each is the same. This fact is of importance in obtaining the twenty-four integrals of the hypergeometric equation.

3. THE SCHWARZIAN DERIVATIVE. Let y_1 and y_2 be two particular integrals of (5.1) and let v_1 and v_2 be two particular integrals of (5.3), the normal form of (5.1). Then

$$y_1 = v_1 e^{-\frac{1}{2}\int P\,dx}, \qquad y_2 = v_2 e^{-\frac{1}{2}\int P\,dx}$$

$$\frac{y_1}{y_2} = \frac{v_1}{v_2} = s$$

Let us now determine the differential equation which is satisfied by s. Each of the quantities v or y may consist of two terms, each containing an arbitrary constant factor. Since we can divide by one of thses constants without changing the value of the quotient or altering its generality, the quotient contains at most three arbitrary constants. Therefore the equation satisfied by s is of the third order.

Since v_1 and v_2 are solutions of $v'' + \psi v = 0$,

$$v'' + \psi v_1 = 0 \tag{5.5}$$

$$v_2'' + \psi v_2 = 0 \tag{5.6}$$

Now $s = \dfrac{v_1}{v_2}$ or in logarithmic language

$$\log s = \log v_1 - \log v_2 \tag{5.7}$$

Differentiating this we have

$$\frac{s'}{s} = \frac{v_1'}{v_1} - \frac{v_2'}{v_2} \qquad \text{where } s' = \frac{ds}{dx},\ v_1' = \frac{dv_1}{dx}, \text{ etc.} \tag{5.8}$$

Differentiating again,

$$\frac{s''}{s} - \left(\frac{s'}{s}\right)^2 = \frac{v_1''}{v_1} - \left(\frac{v_1'}{v_1}\right)^2 - \left(\frac{v_2''}{v_2}\right) + \left(\frac{v_2'}{v_2}\right)^2$$

But $\frac{V_1''}{V_1} = \frac{V_2''}{V_2} = -\psi$ by (5.5) and (5.6).

$$\therefore \frac{S''}{S} - \left(\frac{S'}{S}\right)^2 = -\psi - \left(\frac{V_1'}{V_1}\right)^2 + \psi + \left(\frac{V_2'}{V_2}\right)^2$$

$$= -\left(\frac{V_1'}{V_1} - \frac{V_2'}{V_2}\right)\left(\frac{V_1'}{V_1} + \frac{V_2'}{V_2}\right)$$

Divide through by $\frac{S'}{S}$ which is equal to $\left(\frac{V_1'}{V_1} - \frac{V_2'}{V_2}\right)$ by (5.8). We obtain

$$\frac{S''}{S'} - \frac{S'}{S} = -\left(\frac{V_1'}{V_1} + \frac{V_2'}{V_2}\right)$$

or

$$\frac{S''}{S'} = \frac{S'}{S} - \left(\frac{V_1'}{V_1} + \frac{V_2'}{V_2}\right) = \frac{V_1'}{V_1} - \frac{V_2'}{V_2} - \left(\frac{V_1'}{V_1} + \frac{V_2'}{V_2}\right) = -2\,\frac{V_2'}{V_2} \tag{5.9}$$

On differentiating a third time:

$$\frac{S'S''' - S''^2}{S'^2} = \frac{-2V_2V_2'' + 2V_2'^2}{V_2^2}$$

$$\frac{S'''}{S'} - \left(\frac{S''}{S'}\right)^2 = -2\,\frac{V_2''}{V_2} + 2\left(\frac{V_2'}{V_2}\right)^2$$

But $\frac{V_2''}{V_2} = -\psi$ and $\frac{V_2'}{V_2} = -\frac{1}{2}\frac{S''}{S'}$ by (5.9). Therefore we may write

$$\frac{S'''}{S'} - \left(\frac{S''}{S'}\right)^2 = 2\psi + \frac{1}{2}\left(\frac{S''}{S'}\right)^2$$

or

$$\frac{S'''}{S'} - \frac{3}{2}\left(\frac{S''}{S'}\right)^2 = 2\psi \tag{5.10}$$

which is the differential equation satisfied by S. The left member of this equation is called the Schwarzian Derivative and will hereafter be denoted by $\{s,x\}$.

4. RELATION BETWEEN THE DEPENDENT VARIABLES IN EQUATIONS WHOSE INDEPENDENT VARIABLES ARE DIFFERENT ON THE HYPOTHESIS THAT THE EQUATIONS ULTIMATELY DETERMINE THE SAME FUNCTION. Consider the equations

$$\frac{d^2y}{dx^2} + 2P\,dy/dx + Qy = 0 \tag{5.11}$$

$$\frac{d^2v}{dz^2} + 2R\frac{dv}{dz} + Sv = 0 \tag{5.12}$$

in which P and Q are functions of x : R and S, functions of z.

Let y_1 be an integral of (5.11). Substitute $y = y_1 e^{-\int P dx}$ in the equation and let $\left(Q - \frac{dP}{dx} - P^2\right)$ be denoted by ψ. We then get the normal form of (5.11):

$$\frac{d^2y_1}{dx^2} + \psi y_1 = 0 \tag{5.13}$$

Similarly in (5.12) let $v = v_1 e^{-\int P dz}$, $\psi_1 = \left(S - \frac{dR}{dz} - R^2\right)$: then we have the normal form of (5.12),

$$\frac{d^2v_1}{dz^2} + \psi_1 v_1 = 0 \tag{5.14}$$

Changing the independent variable in (5.13) from x to z we get

$$\frac{d^2y_1}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy_1}{dz}\left(\frac{d^2z}{dx^2}\right) + \psi y_1 = 0 \qquad \text{or}$$

$$\frac{d^2y_1}{dz^2} + \frac{dy_1}{dz}\cdot\frac{z''}{z'^2} + \frac{\psi}{z'^2}y_1 = 0 \tag{5.15}$$

where $z' = \frac{dz}{dx}$, $z'' = \frac{d^2z}{dx^2}$.

Now reduce (5.15) to its normal form by means of the substitution

$y_1 = y_2 e^{-\frac{1}{2}\int \frac{z''}{z'^2}dz} = y_2 z'^{-\frac{1}{2}}$. (5.15) then becomes

$$\frac{d^2y_2}{dz^2} + g\,y_2 = 0 \tag{5.16}$$

where
$$\begin{aligned} g &= \frac{\psi}{z'^2} - \frac{1}{4}\left(\frac{z''}{z'^2}\right)^2 - \frac{1}{2}\frac{d}{dz}\left(\frac{z''}{z'^2}\right) \\ &= \frac{\psi}{z'^2} - \frac{1}{4}\frac{z''^2}{z'^4} - \frac{1}{2}\left(\frac{z'''}{z'^2} - 2\frac{z''^2}{z'^3}\right)\frac{1}{z'} \\ &= \frac{\psi}{z'^2} - \frac{1}{2z'^2}\left\{\frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2\right\} \\ &= \frac{\psi}{z'^2} - \frac{1}{2}\cdot\frac{\{z,x\}}{z'^2} \end{aligned}$$

But the normal forms of (5.11) and (5.12), namely, (5.13) and(5.14) will be the same expressed in terms of the same independent variable if (5.11) and (5.12) are mutually transformable into one another. Comparing (5.14) with (5,16), which is the normal form of (5.13) after x is changed to z in (5.13),

$$\frac{d^2v_1}{dz^2} + \psi_1 v_1 = 0 \tag{5.14}$$

$$\frac{d^2y_2}{dz^2} + g\,y_2 = 0 \tag{5.16}$$

we have

$$y_2 = v_1$$
$$g = \psi_1$$

But
$$g = \frac{\psi}{z'^2} - \frac{1}{2}\frac{\{z,x\}}{z'^2}$$

$$\therefore\ \psi - \frac{1}{2}\{z,x\} = \psi_1 z'^2 \tag{5.17}$$

or, recalling $\psi = Q - \frac{dP}{dx} - P^2$, $\psi_1 = S - \frac{dR}{dz} - R^2$,

$$\frac{1}{2}\{z, x\} + \left(\frac{dz}{dx}\right)^2\left(S - \frac{dR}{dz} - R^2\right) - \left(Q - \frac{dP}{dx} - P^2\right) = 0 \tag{5.18}$$

From the former equations of substitution, namely

$$y\,e^{\int P dx} = y_1\,, \quad y_1 z'^{\frac{1}{2}} = y_2\,, \quad v_1 = v\,e^{\int R dz}$$

we have

$$y_2 = y\left(\frac{dz}{dx}\right)^{\frac{1}{2}} e^{\int P dx}$$

and the relation , $y_2 = v_1$

becomes

$$y\left(\frac{dz}{dx}\right) e^{\int P dx} = v\,e^{\int R dz} \tag{5.19}$$

Thus the following theorem has been established:

THEOREM: The necessary condition that the two equations,

$$\frac{d^2y}{dx^2} + 2P\frac{dy}{dx} + Qy = 0$$

and

$$\frac{d^2v}{dz^2} + 2R\frac{dv}{dz} + Sv = 0$$

where P and Q are functions of x , R and S are functions of z be mutually transformable into one another is that between the independent variables exists the relation,

$$y\left(\frac{dz}{dx}\right)^{\frac{1}{2}} e^{\int P dx} = v\,e^{\int R dz}$$

and between the dependent variables, the relation,

$$\frac{1}{2}\{z, x\} + \left(\frac{dz}{dx}\right)^2\left(S - \frac{dR}{dz} - R^2\right) - \left(Q - \frac{dP}{dx} - P^2\right) = 0$$

where $\{z, x\}$ denotes the Schwarzian Derivative.

5. REDUCTION OF THE HYPERGEOMETRIC EQUATION TO NORMAL FORM.

On comparing the hypergeometric equation,

$$\frac{d^2y}{dx^2} + \frac{\gamma-(\alpha+\beta+1)x}{x(1-x)}\frac{dy}{dx} - \frac{\alpha\beta}{x(1-x)}y = 0 \tag{5.20}$$

with
$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$$

we see that

$$P = \frac{\gamma-(\alpha+\beta+1)x}{x(x-1)} = \frac{\gamma}{x} + \frac{\gamma-\alpha-\beta-1}{1-x}$$

$$Q = \frac{-\alpha\beta}{x(1-x)}$$

$$\psi = \frac{1}{4}\left\{4Q - 2\frac{dP}{dx} - P^2\right)$$

Let us substitute the values of Q, P, and $\frac{dP}{dx}$ in the expression for ψ and denote $1-\gamma$ by λ, $\alpha-\beta$ by μ, and $\gamma-\alpha-\beta$ by ν. Then

$$\psi = \frac{1}{4}\left(\frac{1-\lambda^2}{x^2}\right) + \frac{1}{4}\left(\frac{1-\nu^2}{(x-1)^2}\right) + \frac{1}{2}\left(\frac{\lambda^2-\mu^2+\nu^2-1}{x(x-1)}\right) \tag{5.21}$$

Since ψ is a function of x, we will denote it by $\psi(x)$. Now let us make the substitution $y = ve^{-\frac{1}{2}\int P dx}$ in (5.20) where

$$e^{-\frac{1}{2}\int P dx} = e^{-\frac{1}{2}\int\left(\frac{\gamma}{x} + \frac{\gamma-\alpha-\beta-1}{1-x}\right)dx} = e^{-\frac{1}{2}\gamma\log x + \frac{1}{2}(\gamma-\alpha-\beta-1)\log(1-x)}$$

$$= x^{-\frac{\gamma}{2}}(1-x)^{-\frac{1}{2}(\alpha+\beta+1-\gamma)}$$

When this is done, (5.20) becomes

$$\frac{d^2v}{dx^2} + v\,\psi(x) = 0 \tag{5.22}$$

6. APPLICATION OF THE CONDITIONS OF MUTUAL TRANSFORMABILITY TO THE HYPERGEOMETRIC EQUATION. Consider the equations,

$$\frac{d^2v}{dx^2} + v\,\psi(x) = 0 \qquad (5.23)$$

$$\frac{d^2z}{dt^2} + z\,\psi_1(t) = 0 \qquad (5.24)$$

If these are mutually transformable into one another

$$v\left(\frac{dt}{dx}\right)^{\frac{1}{2}} = z \quad \text{or} \quad v = zu \quad \text{where} \quad u = \left(\frac{dt}{dx}\right)^{-\frac{1}{2}} \qquad (5.25)$$

$$\frac{1}{2}\{t,x\} + \left(\frac{dt}{dx}\right)^2\psi_1(t) - \psi(x) = 0 \qquad (5.26)$$

$\psi_1(t)$ is a function of t, hence (5.26) will give t in terms of x. When this value of t is obtained (5.25) will give the relation of v to z.

We now impose the condition that $\psi_1(t)$ is such as to make (5.24) the normal form of the equation satisfied by the hypergeometric series, $F(\alpha', \beta', \gamma', t)$. Then if (5.26) can be solved for t in terms of x, $u \left(= \left(\frac{dt}{dx}\right)^{-\frac{1}{2}}\right)$ will be derivable at once from this value of t and a solution of (5.23) is

$$v = zu = u\,t^{\frac{1}{2}\gamma'}(1-t)^{\frac{1}{2}(\alpha'+\beta'-\gamma'+1)}F(\alpha', \beta', \gamma', t)$$

The general solution of (5.24) will give $t = f(x, \alpha', \beta', \gamma', \alpha, \beta, \gamma)$. Select those forms of f which make t dependent on x alone and independent of the two sets of constant elements.

To make such a selection set

$$\{t, x\} = 0$$

$$\psi_1(t)\left(\frac{dt}{dx}\right)^2 = \psi(x)$$

or since by definition $\{t,x\} = \frac{t'''}{t'} - \frac{3}{2}\left(\frac{t''}{t'}\right)^2$

$$\frac{t'''}{t'} - \frac{3}{2}\left(\frac{t''}{t'}\right)^2 = 0$$

Multiply through by $(t')^{-\frac{1}{2}}$, getting

$$t'''t'^{-3/2} - \frac{3}{2}t''^2 t'^{-5/2} = 0$$

This differential is exact for

$$\frac{d}{dx}\left(t''t'^{-3/2}\right) = t'''t'^{-3/2} - \frac{3}{2}t''^2 t'^{-5/2}$$

Its solution is therefore

$$t''t'^{-3/2} = c$$

If p denote $\frac{dt}{dx}$,

$$\frac{dp}{dx} = c_1 p^{3/2} \qquad \text{or} \qquad \frac{dp}{p^{3/2}} = c_1\,dx$$

Integrating this we have

$$-2p^{-\frac{1}{2}} = c_1 x + c_2$$

$$p = \frac{4}{(c_1 x + c_2)^2} \quad \text{or}$$

$$p = 4(c_1 x + c_2)^{-2}$$

Integrating this, there results

$$t = c_3 - \frac{4}{c_1(c_1 x + c_2)}$$

This may be expressed, $t = \dfrac{ax+b}{cx+d}$

which is the general value of t which makes $\{t, x\} = 0$. But the conditions also require that

$$\psi_1(t)\left(\frac{dt}{dx}\right)^2 = \psi(x)$$

But $\left(\dfrac{dt}{dx}\right)^2 = \dfrac{(ad-bc)^2}{(cx+d)^4}$

and we get $\dfrac{(ad-bc)^2}{(cx+d)^4}\psi_1\left(\dfrac{ax+b}{cx+d}\right) = \psi(x)$

Arbitrary constants will not satisfy this equation and therefore a, b, c, and d must be determined.

By definition $\psi(x) = \frac{1}{4}\left(4Q - 2\frac{dP}{dx} - P^2\right)$ and is the invariant of the normal form, $\frac{d^2v}{dx^2} + v\psi(x) = 0$, of the equation $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$

In the case of the hypergeometric equation

$$\psi(x) = \frac{1}{4}\left[\frac{(1-\mu^2)x^2 + (\lambda^2+\mu^2-\nu^2-1)x + (1-\lambda^2)}{x^2(1-x)^2}\right]$$

Let

$$(1-\mu^2) = A$$

$$(\lambda^2-\mu^2-\nu^2-1) = B$$

$$(1-\lambda^2) = C$$

Then $$\psi(x) = \frac{1}{4}\left[\frac{Ax^2+Bx+C}{x^2(1-x)^2}\right]$$

Since (5.24) is the normal form of the equation satisfied by $F(\alpha_1', \beta_1', \gamma_1', t)$ we may write for $\psi_1(t)$, its invariant,

$$\psi_1(t) = \frac{1}{4}\left[\frac{A't^2+B't+C'}{t^2(1-t)^2}\right]$$

where $A' = (1-\mu'^2)$, $B' = \lambda'^2-\mu'^2-\nu'^2-1$, $C' = (1-\lambda'^2)$

and $\lambda' = 1-\gamma'$, $\mu' = \alpha'-\beta'$, $\nu' = \gamma'-\alpha'-\beta'$

But by hypothesis equations (5.23) and (5.24) are mutually transformable into one another and we may write with the aid of (5.27)

$$\frac{Ax^2+Bx+C}{x^2(1-x^2)} = (ad-bc)^2\left[\frac{A'(ax+b)^2+B'(ax+b)(cx+d)+C'(cx+d)^2}{(ax+b)^2(cx+d)^2[\overline{c-a}\,x+d-b]^2}\right] \tag{5.28}$$

Here α, β, and γ are arbitrary and therefore A, B, and C which are functions of α, β, and γ are arbitrary. Therefore the numerator and denominator of the left member can have no common factor other than a constant, i. e., the two sides are equal except possibly for a constant factor. The same is true for the right member. We may write, therefore, for the numerator

$$m(Ax^2+Bx+C) = (ad-bc)^2\left\{[A'(ax+b)^2+B'(ax+b)(cx+d)] + C'(cx+d)^2\right\} \tag{5.29}$$

and for the denominator

$$m\,x^2(1-x^2) = (ax+b)^2(cx+d)^2[\overline{c-a}\,x+d-b]^2 \tag{5.30}$$

where m is the constant previously mentioned.

Comparing coefficients of like powers of x in (5.30), we find that the following values satisfy the identity.

$$\begin{array}{lll}
a-d=0 & d-b=0 & c-b=0 \\
b=0 & a+b=0 & a=0 \\
c=0 & c=0 & d=0 \\
m=a^6 & m=a^6 & m=b^6
\end{array}$$

$$\begin{array}{lll}
d-b=0 & c-a=0 & c-a=0 \\
c+d=0 & c+d=0 & a+b=0 \\
a=0 & b=0 & d=0 \\
m=b^6 & m=a^6 & m=b^6
\end{array}$$

Let us substitute these values in $t=\frac{ax+b}{cx+d}$

Using (i) we obtain

$$t=\frac{ax}{d}=\frac{ax}{a}=x$$

Similarly by (ii), (iii), (iv), (v), and (vi), respectively, we have:

$$t=1-x$$
$$t=\frac{1}{x}$$
$$t=\frac{1}{1-x}$$
$$t=\frac{x}{x-1}$$
$$t=\frac{x-1}{x}$$

On substitution of these values of a, b, c, and d given by (1) in (5.29) the following relation is obtained

$$a^6(Ax^2+Bx+C) \equiv (a^2)^2\left[A'a^2x^2+B'a^2x+C'a^2\right]$$

$$a^6(Ax^2+Bx+C) \equiv a^6(A'x^2+B'x+C')$$

It follows that $A = A'$, $B = B'$, $C = C'$

or $1-\mu^2 = 1-\mu'^2$; $\lambda^2-\mu^2-\nu^2-1 = \lambda'^2-\mu'^2-\nu'^2-1'$, $1-\lambda^2 = 1-\lambda'^2$

from which we get

$$\mu = \mu', \quad \lambda = \lambda', \quad \nu = \nu'$$

Expressing these in terms of α, β, γ, and α', β', γ' :

$$1-\gamma' = 1-\gamma$$
$$\alpha'-\beta' = \alpha-\beta$$
$$\gamma'-\alpha'-\beta' = \gamma-\alpha-\beta$$

Since an interchange of α and β in $F(\alpha,\beta,\gamma,x)$ does not change the series in any way, we may write down the following set of equalities:

(A)	$\gamma' = \gamma$	$\alpha' = \alpha$	$\beta' = \beta$
(B)	$\gamma' = \gamma$	$\alpha' = \gamma-\alpha$	$\beta' = \gamma-\beta$
(C)	$\gamma' = 2-\gamma$	$\alpha' = \alpha-\gamma+1$	$\beta' = \beta-\gamma+1$
(D)	$\gamma' = 2-\gamma$	$\alpha' = 1-\alpha$	$\beta' = 1-\beta$

When $t=x$, $u\left(=\frac{dt}{dx}\right)=1$ and a solution of $\frac{d^2v}{dx^2}+v\psi(x)=0$ is

$$v_1 = t^{\frac{1}{2}\gamma'}(1-t)^{\frac{1}{2}(\alpha'+\beta'+1-\gamma')} F(\alpha',\beta',\gamma',x)$$
$$= t^{\frac{\gamma}{2}}(1-x)^{\frac{1}{2}(\alpha+\beta+1-\gamma)} F(\alpha,\beta\gamma,x)$$

Using (B), (C), and (D), respectively, we obtain

$$V_2 = x^{\gamma/2}(1-x)^{\frac{1}{2}(\gamma-\alpha-\beta+1)}F(\gamma-\alpha,\gamma-\beta,\gamma,x)$$
$$V_3 = x^{1-\gamma/2}(1-x)^{\frac{1}{2}(\alpha+\beta+1-\gamma)}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma,x)$$
$$V_4 = x^{1-\frac{1}{2}\gamma}(1-x)^{\frac{1}{2}(\gamma-\alpha-\beta+1)}F(1-\alpha,1-\beta,2-\gamma,x)$$

These integrals V_1, V_2, V_3, and V_4 are solutions of the equation, $d^2v/dx^2+v\psi(x)=0$, the normal form of the hypergeometric equation. This normal form is obtained from the hypergeometric equation by means of the substitution,

$$V = y e^{\frac{1}{2}\int P dx} = y x^{\frac{1}{2}\gamma}(1-x)^{\frac{1}{2}(\alpha+\beta+1-\gamma)}$$

Hence, in order to obtain the integrals of the original equation, the hypergeometric equation, we multiply V_1, V_2, V_3, and V_4 by $\left[x^{\frac{1}{2}\gamma}(1-x)^{\frac{1}{2}(\alpha+\beta+1-\gamma)}\right]^{-1}$. We thus get the following integrals of the hypergeometric equation:

$$y_1 = F(\alpha,\beta,\gamma,x)$$

$$y_2 = (1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma, x)$$

$$y_3 = x^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, x)$$

$$y_4 = x^{1-\gamma}(1-x)^{\gamma-\alpha-\beta} F(1-\alpha, 1-\beta, 2-\gamma, x)$$

The same procedure carried out with $t = 1-x$ will give four additional integrals. Substituting in (5.29) the values of a, b, c, and d given by (ii) we have

$$a^6 (A x^2 + Bx + C) = (-a^2)^2 [A'(ax-a)^2 + B'(ax-a)(-a) + C(-a)^2] \text{ or}$$

$$A x^2 + Bx + C = A'x^2 - 2A'x + A' + B' - B'x + C'$$

There follows from this:

$$A = A'$$
$$B = -2A' - B'$$
$$C = A' + B' + C'$$

or the equivalent equations

$$1-\mu^2 = 1-\mu'^2$$
$$\lambda^2 + \mu^2 - \nu^2 - 1 = -2 + 2\mu'^2 - \lambda'^2 - \mu'^2 + \nu'^2 + 1$$
$$1 + \lambda^2 = 1 - \lambda'^2 + \lambda'^2 + \mu'^2 - \nu'^2 - 1 + 1 - \mu'^2$$

or

$$\mu^2 = \mu'^2$$
$$\lambda^2 - \nu^2 = \nu'^2 - \lambda'^2$$
$$\lambda^2 = \nu'^2$$

These expressed in terms of α, β, γ, α', β', and γ' become

$$(\alpha-\beta)^2 = (\alpha'-\beta')^2$$

$$(1-\gamma)^2 - (\gamma-\alpha-\beta)^2 = (\gamma'-\alpha'-\beta')^2 - (1-\gamma')^2$$

$$(1-\gamma)^2 = (\gamma'-\alpha'-\beta')^2$$

The following values of α', β' and γ' will satisfy these three equations.

(E) $\alpha' = \alpha$ $\quad \beta' = \beta$ $\quad \gamma' = \alpha+\beta-\gamma+1$

(F) $\alpha' = \alpha-\gamma+1$ $\quad \beta' = \beta-\gamma+1$ $\quad \gamma' = \alpha+\beta-\gamma+1$

(G) $\alpha' = \gamma-\alpha$ $\quad \beta' = \gamma-\beta$ $\quad \gamma' = \gamma-\alpha-\beta+1$

(H) $\alpha' = 1-\alpha$ $\quad \beta' = 1-\beta$ $\quad \gamma' = \gamma-\alpha-\beta+1$

These values lead respectively to the following particular integrals:

$$y_5 = F(\alpha, \beta, \alpha+\beta-\gamma+1, 1-x)$$
$$y_6 = x^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, \alpha+\beta-\gamma+1, 1-x)$$
$$y_7 = (1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1, 1-x)$$
$$y_8 = x^{1-\gamma}(1-x)^{\gamma-\alpha-\beta} F(1-\alpha, 1-\beta, \gamma-\alpha-\beta+1, 1-x)$$

To obtain still another set of integrals we repeat the procedure with $t = \frac{1}{x}$. On substituting the values of a, b, c, and d given by (iii) into (5.29), we have

$$b^6(Ax^2+Bx+C) = (-b^2)^2[A'b^2+B'b^2+C'b^2x^2]$$
$$Ax^2+Bx+C = A'+B'+C'x^2$$

It follows

$$A = C' \quad \text{or } 1-\mu^2 = 1-\lambda'^2 \quad \text{or } \mu^2 = \lambda'^2$$
$$B = B' \quad \text{or } \lambda^2+\mu^2-\nu^2-1 = \lambda'^2-\mu'^2-\nu'^2-1 \text{ or } \lambda^2+\mu^2+\nu^2 = \lambda'^2+\mu'^2-\nu'^2$$
$$C = A' \quad \text{or } 1-\lambda^2 = 1-\mu'^2 \quad \text{or } \lambda^2 = \mu'^2$$

or, expressing these in terms of α, β, γ, α', β', and γ':

$$(\alpha-\beta)^2=(1-\gamma')^2$$
$$(1-\gamma)^2+(\alpha-\beta)^2-(\gamma-\alpha-\beta)^2=(1-\gamma')^2+(\alpha'-\beta')^2-(\gamma-\alpha-\beta)^2$$
$$(1-\gamma)^2=(\alpha'-\beta')^2$$

These equations are satisfied by the following values of α', β', and γ':

(I)	$\alpha'=\alpha$	$\beta'=\alpha-\gamma+1$	$\gamma'=\alpha-\beta+1$
(J)	$\alpha'=\beta$	$\beta'=\beta-\gamma+1$	$\gamma'=\beta-\alpha+1$
(K)	$\alpha'=1-\alpha$	$\beta'=\gamma-\alpha$	$\gamma'=\beta-\alpha+1$
(L)	$\alpha'=1-\beta$	$\beta'=\gamma-\beta$	$\gamma'=\alpha-\beta+1$

These values lead to the integrals,

$$y_9=x^{-\alpha}F(\alpha,\alpha-\gamma+1,\alpha-\beta+1,\tfrac{1}{x})$$
$$y_{10}=x^{-\beta}F(\beta,\beta-\gamma+1,\beta-\alpha+1,\tfrac{1}{x})$$
$$y_{11}=x^{\alpha-\gamma}(1-x)^{\gamma-\alpha-\beta}F(1-\alpha,\gamma-\alpha,\beta-\alpha+1,\tfrac{1}{x})$$
$$y_{12}=x^{\beta-\gamma}(1-x)^{\gamma-\alpha-\beta}F(1-\beta,\gamma-\beta,\alpha-\beta+1,\tfrac{1}{x})$$

The foregoing procedure followed through with the values of a, b, c, d, and m given by (iv), (v), and (vi), respectively, will furnish the twelve following particular integrals which complete the set.

$$y_{13}=(1-x)^{-\alpha}F(\alpha,\gamma-\beta,\alpha-\beta+1,\tfrac{1}{1-x})$$
$$y_{14}=(1-x)^{-\beta}F(\beta,\gamma-\alpha,\beta-\alpha+1,\tfrac{1}{1-x})$$
$$y_{15}=x^{1-\gamma}(1-x)^{\gamma-\alpha-1}F(\alpha-\gamma+1,1-\beta,\alpha-\beta+1,\tfrac{1}{1-x})$$
$$y_{16}=x^{1-\gamma}(1-x)^{\gamma-\beta-1}F(\beta-\gamma+1,1-\alpha,\beta-\alpha+1,\tfrac{1}{1-x})$$

$$y_{17} = (1-x)^{-\alpha} F\left(\alpha, \gamma-\beta, \gamma, \frac{x}{x-1}\right)$$

$$y_{18} = (1-x)^{-\beta} F\left(\beta, \gamma-\alpha, \gamma, \frac{x}{x-1}\right)$$

$$y_{19} = x^{1-\gamma}(1-x)^{\gamma-\alpha-1} F\left(\alpha-\gamma+1, 1-\beta, 2-\gamma, \frac{x}{x-1}\right)$$

$$y_{20} = x^{1-\gamma}(1-x)^{\gamma-\beta-1} F\left(\beta-\gamma+1, 1-\alpha, 2-\gamma, \frac{x}{x-1}\right)$$

$$y_{21} = x^{-\alpha} F\left(\alpha, \alpha-\gamma+1, \alpha+\beta-\gamma+1, \frac{x-1}{x}\right)$$

$$y_{22} = x^{-\beta} F\left(\beta, \beta-\gamma+1, \alpha+\beta-\gamma+1, \frac{x-1}{x}\right)$$

$$y_{23} = x^{\alpha-\gamma}(1-x)^{\gamma-\alpha-\beta} F\left(1-\alpha, \gamma-\alpha, \gamma-\alpha-\beta, \frac{x-1}{x}\right)$$

$$y_{24} = x^{\beta-\gamma}(1-x)^{\gamma-\alpha-\beta} F\left(1-\beta, \gamma-\beta, \gamma-\alpha-\beta+1, \frac{x-1}{x}\right)$$

CHAPTER VI

RELATIONS BETWEEN THE TWENTY-FOUR INTEGRALS

The integrals $y_1, y_2, y_3, \ldots\ldots, y_{23}, y_{24}$ of the preceding chapter are not independent for by the ordinary property of the linear differential equation of the second order, of which these integrals are solutions, there exists between any three of them a relation of the form

$$y_\lambda = A y_s + B y_\rho$$

Certain cases arise in which A or B is zero and therefore the corresponding integrals will differ from one another by a constant factor only. One way of recognizing such cases is given by the following lemma.

LEMMA: If the differential equation,

$$y'' + \frac{\gamma - (\alpha+\beta+1)x}{x(x-1)} y' - \frac{\alpha\beta}{x(1-x)} = 0$$

has two solutions developed in the same ascending powers of x and both be converging then the solutions differ from one another by a constant factor only.

PROOF: Let the solution developed in ascending powers of

be

$$y = \sum c_n x^n \qquad \text{Then}$$

$$y' = \sum n c_n x^{n-1} \qquad \text{and}$$

$$y'' = \sum n(n-1) c_n x^{n-2}$$

On substitution of these values in the differential equation we have

$$
\begin{array}{l|l|l|l|l|l}
x^2y'' = 2c_2 & +6c_3 & 12c_4 & x^4 + \cdots + n(n-1)c_n & x^n + \cdots \\
-xy'' = \cdots -2c_2 & -3c_3 & -4c_4 & + \cdots - nc_n & + \cdots \\
(\alpha+\beta+1)xy' = \cdots 2c_2(\alpha+\beta+1) & +3c(\alpha+\beta+1) & +4c_4(\alpha+\beta+1) & + \cdots -(\alpha+\beta+1)nc_n & + \cdots \\
-\gamma y' = \cdots -3\gamma c & -4\gamma c_4 & -5\gamma c_5 & + \cdots -\gamma(n+1)c_{n+1} & + \cdots \\
\alpha\beta y = \cdots \alpha\beta c & +\alpha\beta c_3 & +\alpha\beta c_4 & + \cdots + \alpha\beta c_n & + \cdots
\end{array}
$$

$$c_3 = \frac{2(\alpha+\beta+1)+\alpha\beta}{3\gamma}c_2$$

$$c_4 = \frac{3+3(\alpha+\beta+1)+\alpha\beta}{4\gamma}c_3$$

$$\cdots\cdots\cdots$$

$$c_n = \frac{n^2-2n+1+(n-1)(\alpha+\beta+1)+\alpha\beta}{\gamma n}c_{n-1}$$

It follows that

$$c_0 = c_0$$

$$c_1 = \frac{\alpha\beta}{\gamma}c_0$$

$$c_2 = \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot \gamma(\gamma+1)}c_0$$

$$\cdots\cdots\cdots$$

$$c_n = \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)\beta(\beta+1)(\beta+2)\cdots(\beta+n-1)}{n!\,\gamma(\gamma+1)(\gamma+2)\cdots\cdots(\gamma+n-1)}c_0$$

and our solution becomes

$$y = A\ F(\alpha, \beta, \gamma, x)$$

Thus by setting for y the form, $c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$, we have obtained from the differential equation the one integral, $y = A\ F(\alpha, \beta, \gamma, x)$ and this only. Therefore all integrals which can take the form, $c_0 + c_1 x + c_2 x^2 + \cdots\cdots + c_n x_n$, i. e., those which can be developed in the same ascending powers of , will differ from the integral, $F(\alpha, \beta, \gamma, x)$, and therefore from one another, by constant factors only. This proves the lemma.

It follows that

$$F(\alpha, \beta, \gamma, x) = A(1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma, x)$$
$$F(\alpha, \beta, \gamma, x) = A'(1-x)^{-\alpha} F\left(\alpha, \gamma-\beta, \gamma, \frac{x}{x-1}\right)$$

To determine A and A', set $x = 0$. We then find that

$$A = 1, \quad A' = 1$$

and there results the following equations:

$$F(\alpha, \beta, \gamma, x) = (1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma, x) \quad \text{or} \quad y_1 = y_2$$
$$F(\alpha, \beta, \gamma, x) = (1-x)^{-\alpha} F\left(\alpha, \gamma-\beta, \gamma, \frac{x}{x-1}\right) \quad \text{or} \quad y_1 = y_{17}$$

By permutation of α with β we get

$$F(\alpha, \beta, \gamma, x) = F(\beta, \alpha, \gamma, x) = (1-x)^{-\beta} F\left(\beta, \gamma-\alpha, \gamma, \frac{x}{x-1}\right) \text{ or } y_1 = y_{18}$$

Also, since $F(\alpha, \beta, \gamma, x) = F(\beta, \alpha, \gamma, x)$, we have

$$(1-x)^{-\alpha} F\left(\alpha, \gamma-\beta, \gamma, \frac{x}{x-1}\right) = (1-x)^{-\beta} F\left(\beta, \gamma-\alpha, \gamma, \frac{x}{x-1}\right)$$

Substitute $\frac{x}{x-1}$ for x in the above and $\gamma-\beta$ for β. The result is and our solution becomes

$$y = A\ F(\alpha, \beta, \gamma, x)$$

Thus by setting for y the form, $c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$, we have obtained from the differential equation the one integral, $y = A\ F(\alpha, \beta, \gamma, x)$ and this only. Therefore all integrals which can take the form, $c_0 + c_1 x + c_2 x^2 + \cdots\cdots + c_n x_n$, i. e., those which can be developed in the same ascending powers of , will differ from the integral, $F(\alpha, \beta, \gamma, x)$, and therefore from one another, by constant factors only. This proves the lemma.

It follows that

$$F(\alpha, \beta, \gamma, x) = A\,(1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma, x)$$
$$F(\alpha, \beta, \gamma, x) = A'(1-x)^{-\alpha} F\left(\alpha, \gamma-\beta, \gamma, \frac{x}{x-1}\right)$$

To determine A and A', set $x = 0$. We then find that

$$A = 1, \quad A' = 1$$

and there results the following equations:

$$F(\alpha, \beta, \gamma, x) = (1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma, x) \quad \text{or} \quad y_1 = y_2$$
$$F(\alpha, \beta, \gamma, x) = (1-x)^{-\alpha} F\left(\alpha, \gamma-\beta, \gamma, \frac{x}{x-1}\right) \quad \text{or} \quad y_1 = y_{17}$$

By permutation of α with β we get

$$F(\alpha, \beta, \gamma, x) = F(\beta, \alpha, \gamma, x) = (1-x)^{-\beta} F\left(\beta, \gamma-\alpha, \gamma, \frac{x}{x-1}\right) \text{ or } y_1 = y_{18}$$

Also, since $F(\alpha, \beta, \gamma, x) = F(\beta, \alpha, \gamma, x)$, we have

$$(1-x)^{-\alpha} F\left(\alpha, \gamma-\beta, \gamma, \frac{x}{x-1}\right) = (1-x)^{-\beta} F\left(\beta, \gamma-\alpha, \gamma, \frac{x}{x-1}\right)$$

Substitute $\frac{x}{x-1}$ for x in the above and $\gamma-\beta$ for β. The result is

$$F(\alpha,\beta,\gamma,x) = (1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta,\gamma,x)$$

Thus it has been shown that

$$y_1 = y_2 = y_{17} = y_{18} \qquad (6.1)$$

A glance at y_3 and y_1 will show that y_3 is derived from by the following transformations of elements. In y_3 $(= x^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma,x))$ the first and second elements, $\alpha-\gamma+1$ and $\beta-\gamma+1$, are obtained by subtracting the third element, γ, of $F(\alpha,\beta,\gamma,x)$ from the first and second elements of $F(\alpha,\beta,\gamma,x)$ and adding 1 to the result. The third element of y_3, $2-\gamma$, is obtained by subtracting the third element of y_1, γ, from 2. The result so obtained, namely, $F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma,x)$ is then multiplied by $x^{1-\gamma}$ to get y_3. By exactly similar transformations y_4 is obtained from y_2, y_{19} from y_{17}, and y_{20} from y_{18}. Therefore we may write with the aid of (6.1)

$$y_3 = y_4 = y_{19} = y_{20} \qquad (6.2)$$

In this way and by use of the lemma the followinggsets of relations are obtained.

$$y_5 = y_6 = y_{27} = y_{22}$$

$$y_7 = y_8 = y_{13} = y_{15}$$

$$y_{10} = y_{11} = y_{14} = y_{16}$$

CHAPTER VII

DEMONSTRATIONS OF THE GENERALITY OF THE HYPERGEOMETRIC SERIES

1. EXPRESSION FOR $(1+x)^p+(1-x)^p$

$$(1+x)^p+(1-x)^p = 1+px+\frac{p(p-1)}{2!}x^2+\frac{p(p-1)(p-2)}{3!}x^3+\cdots+\frac{p(p-1)(p-2)\cdots(p-\overline{n-1})}{(n-1)!}x^{n-1}+\cdots$$

$$+1-px+\frac{p(p-1)}{2!}x^2-\frac{p(p-1)(p-2)}{3!}x^3+\cdots+(-1)^{n-1}\frac{p(p-1)(p-2)\cdots(p-\overline{n-1})}{(n-1)!}x^{n-1}+\cdots$$

$$= 2\sum \frac{p(p-1)(p-2)\cdots p-\overline{2n-3}}{(2n-2)!}x^{2n-2}+\cdots$$

Identifying this with $F(\alpha,\beta,\gamma,z)$, term by term,

$$\frac{\alpha\beta}{1\cdot\gamma}z = \frac{p(p-1)}{1\cdot 2}x^2 = \frac{-\frac{p}{2}\left(-\frac{p}{2}+\frac{1}{2}\right)}{1\cdot\frac{1}{2}}x^2$$

$$\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}z^2 = \frac{p(p-1)(p-2)(p-3)}{4!}x^4 = \frac{-\frac{p}{2}\left(-\frac{p}{2}+1\right)\left(-\frac{p}{2}+\frac{1}{2}\right)\left(-\frac{p}{2}+\frac{1}{2}+1\right)}{1\cdot 2\cdot\frac{1}{2}\left(\frac{1}{2}+1\right)}(x^2)^2$$

The foregoing comparisons suggest for values of α, β, γ, and z $-\frac{p}{2}$, $-\frac{p}{2}+\frac{1}{2}$, $\frac{1}{2}$, and x^2. The nth term of $F\left(-\frac{p}{2},-\frac{p}{2}+\frac{1}{2},\frac{1}{2},x^2\right)$ is

$$\frac{-\frac{p}{2}\left(-\frac{p}{2}+1\right)\cdots\cdots\left(-\frac{p}{2}+n-2\right)\left(-\frac{p}{2}+\frac{1}{2}\right)\left(-\frac{p}{2}+\frac{1}{2}+1\right)\cdots\cdots\left(-\frac{p}{2}+\frac{1}{2}+n-2\right)}{(n-1)!\,\frac{1}{2}\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)\cdots\cdots\left(\frac{1}{2}+n-2\right)}(x^2)^{n-1}$$

Multiplying this by $\left(\frac{2^{n-1}}{2^{n-1}}\right)^2$, we have after changing the sign of each constant term in the numerator,

$$\frac{p(p-1)(p-2)\cdots(p-\overline{2n-4})(p-\overline{2n-3})}{(2n-2)!}(x^2)^{n-1}$$

which is the nth term of the series for $(1+x)^p+(1-x)^p$. There are $2(n-1)$ constant terms in the numerator: therefore the number of changes in sign effected by changing the sign of each term is always an even number and consequently the sign of the nth term itself is unaltered.

Thus we see that the nth term of $F(-\frac{p}{2}, -\frac{p}{2}+\frac{1}{2}, \frac{1}{2}, x^2)$ is identically equal to the nth term of the series expansion of $(1+x)^p+(1-x)^p$. This is true for every value of n and hence we may write

$$(1+x)^p+(1-x)^p = F\left(-\frac{p}{2}, -\frac{p}{2}+\frac{1}{2}, \frac{1}{2}, x^2\right) \tag{7.1}$$

When using such relations as (7.1) one should bear in mind the conditions of convergence.

2. EXPRESSION FOR $(1+x)^p$. Expanding $(1+x)^p$ by the binomial theorem we get

$$(1+x)^p = \sum \frac{p(p-1)(p-2)\cdots p-\overline{n-1}}{(n-1)!}x^{n-1}$$

Comparing this term by term with the hypergeometric series of form $F(\alpha, \beta, \gamma, z)$, we have

$$\frac{\alpha\beta}{1\cdot\gamma}z = px = \frac{-p(-x)\cdot\beta}{1\cdot\beta} = \frac{-p\cdot\beta}{1\cdot\beta}(-x)$$

$$\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)} = \frac{p(p-1)}{1\cdot2}x^2 = \frac{-p(-p+1)\beta(\beta+1)}{1\cdot2\cdot\beta(\beta+1)}(-x)^2$$

$$\frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-2)\beta(\beta+1)(\beta+2)\cdots(\beta+n-2)}{(n-1)!\,\gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-2)}\,z^{n-1}$$

$$=\frac{-p(-p+1)(-p+2)\cdots(-p+n-2)\beta(\beta+1)(\beta+2)\cdots(\beta+n-2)}{(n-1)!\,\beta(\beta+1)(\beta+2)\cdots(\beta+n-2)}(-x)^{n-1}$$

The product $(-p)(-p+1)(-p+2)\cdots(-p+n-2)$ contains $(n-1)$ terms. When n is even, there are an odd number of changes in sign and the product will be negative. But when n is even $(-x)^{n-1}$ will also be negative. Hence the value of the nth term is positive for all even values of n.

When n is odd there are an even number of changes in sign and $(-p)(-p+1)\cdots\cdots(-p+n-2)$ is positive. Moreover $(-x)^{n-1}$ is positive. Therefore the nth term is positive for all odd values of n.

Therefore

$$(1+x)^p = F(-p,\beta,\beta,-x) \tag{7.2}$$

3. EXPRESSION FOR $(t+u)^r$. The nth term of $(t+u)^r$ when expanded by the binomial theorem is

$$t^r\left(\frac{r(r-1)(r-2)\cdots\cdots(r-n-2)}{(n-1)!}\right)\left(\frac{u}{t}\right)^{n-1}$$

$$=\frac{-r(-r+1)(-r+2)\cdots(-r+n-2)}{(n-1)!}\left(\frac{-u}{t}\right)^{n-1}\cdot t^r$$

$$= t^r \cdot \frac{-r(-r+1)(-r+2)\ldots(-r+n-2)\beta(\beta+1)(\beta+2)\cdots\cdots(\beta+n-2)}{(n-1)!\,\beta(\beta+1)(\beta+2)\cdots\cdots(\beta+n-2)}\left(-\frac{v}{t}\right)^{n-1}$$

$=$ the nth term of $t^r F(-r, \beta, \beta, -\frac{v}{t})$

Hence $(t+v)^r = t^r F(-r, \beta, \beta, -\frac{v}{t})$ (7.3)

where β is an arbitrary quantity.

4. EXPRESSION FOR $(t+v)^r + (t-v)^r$. The nth term of $(t+v)^r + (t-v)^r$ when expanded by the binomial theorem is

$$2t^r\left(\frac{r(r-1)(r-2)\cdots\cdots(r-2n-3)}{(2n-2)!}\right)\left(\frac{v^2}{t^2}\right)^{n-1}$$

But the nth term of $F\left(-\frac{r}{2}, -\frac{r}{2}+\frac{1}{2}, \frac{1}{2}, v^2/t^2\right)$ is:

$$2t^r\frac{\left(-\frac{r}{2}\right)\left(\frac{-r+2}{2}\right)\left(\frac{-r+4}{2}\right)\cdots\cdots\left(\frac{-r+2n-4}{2}\right)\left(\frac{-r+1}{2}\right)\left(\frac{-r+3}{2}\right)\cdots\left(\frac{-r+2n-3}{2}\right)}{(n-1)!\;\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\frac{2n-3}{2}}\left(\frac{v^2}{t^2}\right)^{n-1}$$

Multiplying this by $\frac{(2^{n-1})^2}{(2^{n-1})^2}$ we have

$$2t^r\frac{(-r)(-r+2)(-r+4)\cdots(-r+2n-4)(-r+1)(-r+3)\cdots\cdots(-r+2n-3)}{2\cdot4\cdot6\cdots(2n-2)\cdot1\cdot3\cdot5\cdots 2n-3}\left(\frac{v^2}{t^2}\right)^{n-1}$$

Since there are $2(n-1)$ terms in the numerator, a change in the sign of each factor does not change the sign of the term and the nth term may be written thus:

$$2t^r \cdot \frac{r(r-1)(r-2)\ldots(r-\overline{2n+4})(r-\overline{2n+3})}{(2n-2)!}\left(\frac{v^2}{t^2}\right)^{n-1}$$

This is identically equal to the nth term of the series for $(t+v)^r+(t-u)^r$. Therefore

$$(t+v)^r+(t-v)^r = 2t^2\,F\left(-\tfrac{r}{2},\ -\tfrac{r}{2}+\tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{v^2}{t^2}\right) \tag{7.4}$$

5. EXPRESSION FOR $(t+v)^r+t^r$. The nth term of the expansion of $(t+v)^r+t^r$ by the binomial theorem is

$$(2t^r)\frac{r(r-1)(r-2)\ldots(r-n-2)}{2(n-1)!}\left(\frac{v}{t}\right)^{n-1}$$

Now the nth term of $F(-r, w, 2w, -\frac{v}{t})$ is

$$\frac{r(r-1)(r-2)(r-3)\cdots\cdots(r-n+2)(w+1)(w+2)(w+3)\cdots(w+n+2)}{(n-1)!\cdot 2\cdot(2w+1)(2w+2)(2w+3)\cdots(2w+n-2)}\left(\frac{v}{t}\right)^{n-1}$$

The limit of this term as $w \to 0$ is

$$\frac{r(r-1)(r-2)\cdots\cdots(r-n+2)}{2(n-1)!}\left(\frac{v}{t}\right)^{n-1}$$

which is the nth term of the series for $(t+v)^r+t^r$ multiplied by the factor $\frac{1}{2t^r}$

Thus we have proved the relation

$$(t+v)^r + t^r = \lim_{\omega \to 0} 2t^r F(-r, \omega, 2\omega, -v/t) \qquad (7.5)$$

6. EXPRESSION FOR $(t+v)^r - (t-v)^r$. $(t+v)^r - (t-v)^r$ may be expressed as a binimial series whose nth term is

$$2rt^{r-1}v \frac{(r-1)(r-2)\cdots(r-2n+2)}{(2n-1)!}\left(v^2/t^2\right)^{n-1}$$

But the nth term of $F\left(-\frac{r}{2}+\frac{1}{2}, -\frac{r}{2}+1, \frac{3}{2}, v^2/t^2\right)$ is

$$\frac{\left(-\frac{r}{2}+\frac{1}{2}\right)\left(-\frac{r}{2}+\frac{3}{2}\right)\left(-\frac{r}{2}+5/2\right)\cdots\left(-\frac{r}{2}+\frac{1}{2}+n-2\right)\left(-\frac{r}{2}+1\right)\left(-\frac{r}{2}+2\right)\cdots\left(-\frac{r}{2}+1+n-2\right)}{(n-1)!\ \frac{3}{2}\cdot 5/2\cdot\frac{7}{2}\cdots\left(\frac{3}{2}+n-2\right)}\left(\frac{v^2}{t^2}\right)^{n-1}$$

$$= \frac{(r-1)(r-2)(r-3)\cdots(r-2n+2)}{(2n-1)!}\left(\frac{v^2}{t^2}\right)^{n-1}$$

Hence, $(t+v)^r - (t-v)^r = 2rt^{r-1}v\, F\left(-\frac{r}{2}+\frac{1}{2}, -\frac{r}{2}+1, \frac{3}{2}, v^2/t^2\right)$ (7.6)

7. EXPRESSION FOR $(t+v)^r - t^r$, On expansion of this by the binomial theorem we have

$$(t+v)^r - t^r = \sum (-1)^{n-1}\frac{(-r+1)(-r+2)\cdots(-r+n-1)}{(n-1)!}\left(\frac{v}{t}\right)^{n-1}$$

But the nth term of $F\left(-r+1, \beta, \beta, -\frac{v}{t}\right)$ is

$$\frac{(-r+1)(-r+2)(-r+3)\cdots\cdots(-r+n-1)}{(n-1)!}\left(\frac{u}{t}\right)^{n-1}(-1)^{n-1}$$

Therefore $(t+u)^r - t^r = r t^{r-1} u \, F(-r+1, \beta, \beta, -\frac{u}{t})$ (7.7)

where β is an arbitrary quantity.

8. EXPRESSION FOR e^x. Expanding e^x by Maclaurin's Series we obtain

$$e^x = \sum \frac{x^{n-1}}{(n-1)!}$$

Now the nth term of $F(1, \beta, 1, x/\beta)$ is

$$\frac{(1+n-2)!\ \beta(\beta+1)(\beta+2)\cdots(\beta+n-2)}{(n-1)!\cdot(1+n-2)!}\left(\frac{x}{\beta}\right)^{n-1}$$

and

$$\lim_{\beta\to\infty}\frac{(1+n-2)!\ \beta(\beta+1)\cdots(\beta+n-2)}{(n-1)!\ (1+n-2)!}\left(\frac{x}{\beta}\right)^{n-1}$$

$$= \lim_{\beta\to\infty}\frac{(1+\frac{1}{\beta})(1+2/\beta)\cdots\cdots(1+\frac{n-2}{\beta})}{(n-1)!}x^{n-1}$$

$$= \frac{x^{n-1}}{(n-1)!}$$

which is the nth term in the expansion for e^x above. Since this is true for all values of n we write:

$$e^x = \lim_{\beta\to\infty} F(1, \beta, 1, \frac{x}{\beta}) \qquad (7.8)$$

9. EXPRESSION FOR $\log(1+x)$. $\log(1+x)$ may be expressed as the following series:

$$\log(1+x) = x\sum(-1)^{n-1}\frac{x^{n-1}}{n}$$

On writing out the series, $xF(1,1,2,-x)$, we find its nth term to be

$$x\cdot\frac{1(1+1)(1+2)\cdots(1+n-2)(1)(1+1)\cdots\cdots(1+n-2)}{(n-1)!\cdot 2(2+1)(2+3)\cdots(2+n-2)}(-x)^{n-1}$$

$$=\frac{x(-x)^{n-1}}{n} = (-1)^{n-1}\frac{x^{n-1}}{n}\cdot x$$

which is the nth term in the series for $\log(1+x)$. Therefore

$$\log(1+x) = xF(1,1,2,-x) \tag{7.9}$$

10. EXPRESSION FOR $\log\frac{1+x}{1-x}$.

$$\frac{1}{x+1}+\frac{1}{1-x} = 2\sum x^{2n-2}$$

$$\int\left(\frac{1}{1+x}+\frac{1}{1-x}\right)dx = \log\left(\frac{1+x}{1-x}\right) = 2x\sum\frac{x^{2n-2}}{2n-1}$$

Now the nth term of $F(\frac{1}{2},1,\frac{3}{2},x^2)$ is

$$\frac{\frac{1}{2}(\frac{1}{2}+1)(\frac{1}{2}+2)\cdots(\frac{1}{2}+n-2)(1)(2)(3)\cdots(n-1)}{1\cdot 2\cdot 3\cdots(n-1)\,\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdots(\frac{3}{2}+n-2)}(x^2)^{n-1}$$

$$\frac{\frac{1}{2}\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)\cdots\left(n-\frac{3}{2}\right)}{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdots\left(n-\frac{1}{2}\right)}\left(x^2\right)^{n-1}$$

. Multiply numerator and denominator by 2^{n+1} . The result is

$$\frac{1\cdot 3\cdot 5\cdots(2n-3)}{3\cdot 5\cdot 7\cdots(2n-1)}\,x^{2n-2}$$

or $$\frac{1\cdot 3\cdot 5\cdots(2n-5)(2n-3)}{3\cdot 5\cdot 7\cdots(2n-3)(2n-1)}\,x^{2n-2} = \frac{x^{2n-2}}{2n-1}$$

which is the nth term of the series for $\frac{1}{2}\log\frac{1+x}{1-x}$ Hence

$$\log\frac{1+x}{1-x} = 2F\left(\tfrac{1}{2}, 1, \tfrac{3}{2}, x^2\right) \tag{7.10}$$

11. EXPRESSION FOR a^x. Our series for is

$$a^x = \sum \frac{(x\log a)^{n-1}}{(n-1)!}$$

The nth term of $F\left(1, \beta, 1, \frac{x\log a}{\beta}\right)$ is

$$\frac{(n-1)!\,\beta(\beta+1)(\beta+2)\cdots(\beta+n-2)}{(n-1)!\,(n-1)!}\left(\frac{x\log a}{\beta}\right)^{n-1}$$
$$= \frac{1\left(1+\frac{1}{\beta}\right)\left(1+\frac{2}{\beta}\right)\left(1+\frac{3}{\beta}\right)\cdots\left(1+\frac{n-2}{\beta}\right)}{(n-1)!}\left(x\log a\right)^{n-1}$$

The limit of this term as $\beta\to\infty$ is $\frac{(x\log a)^{n-1}}{(n-1)!}$

Therefore
$$a^x = \lim_{\beta\to\infty} F\left(1, \beta, 1, \frac{x\log a}{\beta}\right) \tag{7.11}$$

where β is an arbitrary quantity.

12. EXPRESSION FOR $e^t + e^{-t}$ AND $\cosh t$

$$e^t + e^{-t} = 2 \sum \frac{t^{2n-2}}{(2n-2)!}$$

The nth term of $F(\alpha, \beta, \frac{1}{2}, \frac{t^2}{4\alpha\beta})$ is

$$\frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)\ldots(\alpha+n-2)\beta(\beta+1)(\beta+2)(\beta+3)\ldots(\beta+n-2)}{(n-1)!\ \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdots(\frac{1}{2}+n-2)}\left(\frac{t^2}{4\alpha\beta}\right)^{n-1}$$

$$\frac{1(1+\frac{1}{\alpha})(1+\frac{2}{\alpha})(1+\frac{3}{\alpha})\ldots(1+\frac{n-2}{\alpha})(1)(1+\frac{1}{\beta})(1+\frac{2}{\beta})(1+\frac{3}{\beta})\ldots(1+\frac{n-2}{\beta})}{2^{n-1}(n-1)!\cdot 1\cdot 3\cdot 5\cdots(2n-3)}(t^2)^{n-1}$$

The limit of this nth term as α and β increase without limit is

$$\frac{(t^2)^{n-1}}{2\cdot 4\cdot 6\cdot 8\cdots(2n-2)\cdot 1\cdot 3\cdot 5\cdot 7\cdots(2n-3)} = \frac{t^{2n-2}}{(2n-2)!}$$

which is the nth term of the series for $\frac{e^t + e^{-t}}{2}$. It follows that

$$e^t + e^{-t} = 2 \lim_{\substack{\alpha\to\infty\\ \beta\to\infty}} F\left(\alpha, \beta, \frac{1}{2}, \frac{t^2}{4\alpha\beta}\right) \qquad (7.12)$$

or if we choose

$$\cosh t = \lim_{\substack{\alpha\to\infty\\ \beta\to\infty}} F\left(\alpha, \beta, \frac{1}{2}, \frac{t^2}{4\alpha\beta}\right) \qquad (7.13)$$

13. EXPRESSION FOR $e^t - e^{-t}$ AND $\sinh t$

$$e^t - e^{-t} = 2t \sum \frac{t^{2n-2}}{(2n-1)!}$$

$$F\left(\alpha, \beta, \frac{3}{2}, \frac{t^2}{4\alpha\beta}\right) = \sum \frac{\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-2)\beta(\beta+1)(\beta+2)\ldots(\beta+n-2)}{(n-1)!\ \frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdots(\frac{3}{2}+n-2)}\left(\frac{t^2}{4\alpha\beta}\right)^{n-1}$$

$$= \sum \frac{1(1+\frac{1}{\alpha})(1+\frac{2}{\alpha})\cdots(1+\frac{n+2}{\alpha})1\cdot(1+\frac{1}{\beta})\cdots\cdot(1+\frac{n-2}{\beta})}{(2n-1)!} t^{2n-2}$$

Passing to the limit we have

$$\lim_{\substack{\alpha\to\infty\\ \beta\to\infty}} F\left(\alpha,\beta,\tfrac{3}{2},\tfrac{t^2}{4\alpha\beta}\right) = \sum \frac{t^{2n-2}}{(2n-1)!}$$

Therefore
$$e^{t}-e^{-t} = 2t\lim_{\alpha,\beta\to\infty} F\left(\alpha,\beta,\tfrac{3}{2},\tfrac{t^2}{4\alpha\beta}\right) \tag{7.14}$$

This may be expressed in the form:

$$\sinh t = \lim_{\alpha,\beta} t\,F\left(\alpha,\beta,\tfrac{2}{2},\tfrac{t^2}{4\alpha\beta}\right) \tag{7.15}$$

14. EXPRESSION FOR $\sin t$. The series for $\sin t$ is

$$\sin t = t\left(1-\frac{t^2}{3!}+\frac{t^4}{5!}-\frac{t^6}{7!}+\cdots+(-1)^{n-1}\frac{t^{2(n-1)}}{(2n-1)!}+\cdots\right.$$

The nth term is $(-1)^{n-1}\dfrac{t^{2(n-1)}}{(2n-1)!}$ or $\dfrac{(-t^2)^{n-1}}{(2n-1)!}$

But the nth term of $F\left(\varepsilon,\varepsilon',\tfrac{3}{2},\tfrac{-t^2}{4\varepsilon\varepsilon'}\right)$ is

$$\frac{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)\cdots(\varepsilon+n-2)\,\varepsilon'(\varepsilon'+1)(\varepsilon'+2)(\varepsilon'+3)\cdots(\varepsilon'+n-2)}{1\cdot2\cdot3\cdots(n-1)\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdots\left(\frac{2n-1}{2}\right)}\left(\frac{-t^2}{4\varepsilon\varepsilon'}\right)^{n-1}$$

and its limit as ε and ε' become indefinitely great is

$$\lim_{\substack{\varepsilon\to\infty\\ \varepsilon'\to\infty}} \frac{1(1+\frac{1}{\varepsilon})(1+\frac{2}{\varepsilon})\cdots(1+\frac{n-2}{\varepsilon})(1)(1+\frac{1}{\varepsilon'})(1+\frac{2}{\varepsilon'})\cdots(1+\frac{n-2}{\varepsilon'})}{(2n-1)!}(-t^2)^{n-1}$$

$$= \frac{(-t^2)^{n-1}}{(2n-1)!} = \frac{1}{t}\cdot\text{(nth term of series for } \sin t)$$

Hence

$$\sin t = \lim_{\substack{\varepsilon \to \infty \\ \varepsilon' \to \infty}} t\, F\left(\varepsilon, \varepsilon', \tfrac{3}{2}, -\frac{t^2}{4\varepsilon\varepsilon'}\right) \tag{7.16}$$

where ε and ε' are arbitrary.

15. EXPRESSION FOR $\cos t$. The series for $\cos t$ is

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots + \frac{(-t^2)^{n-1}}{(2n-2)!} + \cdots$$

But the limit of the nth term of $F\left(\alpha, \beta, \tfrac{1}{2}, \frac{-t^2}{\alpha, \beta}\right)$ as α and β increase without limit is

$$\lim_{\substack{\alpha \to \infty \\ \beta \to \infty}} \frac{1\left(1+\frac{1}{\alpha}\right)\left(1+\frac{2}{\alpha}\right)\cdots\left(1+\frac{n-2}{\alpha}\right)(1)\left(1+\frac{1}{\beta}\right)\left(1+\frac{2}{\beta}\right)\cdots\left(1+\frac{n-2}{\beta}\right)}{2\cdot 4\cdot 6 \cdots\cdots (2n-2)\cdot 1\cdot 3\cdot 5\cdot 7\cdots\cdots(2n-3)}\left(-t^2\right)^{n-1}$$

$$= \frac{(-t^2)^{n-1}}{(2n-2)!}$$

It follows that

$$\cos t = \lim_{\substack{\alpha \to \infty \\ \beta \to \infty}} F\left(\alpha, \beta, \tfrac{1}{2}, \frac{-t^2}{\alpha\beta}\right) \tag{7.17}$$

where α and β are arbitrary.

16. EXPRESSION FOR $\sin n\phi$. For $\phi \leqq \frac{\pi}{2}$, $\sin K\phi$ may be expressed as the series[1]:

$$\sin K\phi = (K \sin\phi)\left(1 - \frac{K(K^2-1^2)}{3!}\sin^2\phi + \frac{K(K^2-1^2)(K^2-3^2)}{5!}\sin^4\phi + \cdots\right.$$

$$\left.\cdots + (-1)^{n-1}\frac{K(K^2-1^2)(K^2-3^2)(K^2-5^2)\cdots(K^2-\overline{2n-3}^2)}{(2K-1)^2}\sin^{2n-2}\phi + \cdots\right)$$

Now the nth term of $F\left(\frac{n+1}{2}, -\frac{n+1}{2}, \frac{3}{2}, \sin^2\phi\right)$ is

1. The series is from Bromwich's "Theory of Infinite Series", p. 208

$$\frac{\left(\frac{n+1}{2}\right)\left(\frac{n+3}{2}\right)\left(\frac{n+5}{2}\right)\cdots\left(\frac{n+2n-3}{2}\right)\left(\frac{-n+1}{2}\right)\left(\frac{-n+3}{2}\right)\left(\frac{-n+5}{2}\right)\cdots\cdots\left(\frac{-n+2n-3}{2}\right)}{(n-1)!\,\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdots\cdots\left(\frac{2n-1}{2}\right)}\left(\sin^2\phi\right)^{n-1}$$

$$=(-1)^{n-1}\frac{(n+1)(n+3)(n+5)\cdots(n+2n-3)(n-1)(n-3)(n-5)\cdots\cdots(n-\overline{2n-3})}{2\cdot 4\cdot 6\cdots\cdots(2n-2)\cdot 3\cdot 5\cdot 7\cdots\cdots(2n-1)}\left(\sin\phi\right)^{2n-2}$$

$$=(-1)^{n-1}\frac{(n^2-1^2)(n^2-3^2)(n^2-5^2)\cdots\cdots(n-\overline{2n-3}^{2})}{(2n-1)!}\left(\sin\phi\right)^{2n-2}$$

which is $\frac{1}{n\sin\phi}$ (nth term of the series for $\sin n\phi$). Therefore

$$\sin(n\phi)=n\sin\phi\,F\left(\frac{n+1}{2},\ \frac{-n+1}{2},\ \frac{3}{2},\ \sin^2\phi\right) \tag{7.18}$$

We can obtain a second expression for $\sin n\phi$ by use of the series,[1]

$$\sin n\phi=\cos\phi\sin\phi\left[1-\frac{n^2-2^2}{3!}\sin^2\phi+\frac{(n^2-2^2)(n^2-4^2)}{5!}\sin^4\phi+\cdots\right.$$

$$\left.\cdots+(-1)^{n-1}\frac{(n^2-2^2)(n^2-4^2)\cdots\cdots(n^2-\overline{2n-2}^{2})}{(2n-1)!}\sin^{2n-2}\phi+\cdots\right]$$

But the nth term of $F\left(\frac{n}{2}+1,-\frac{n}{2}+1,\ \frac{3}{2},\sin^2\phi\right)$ is

$$\frac{\left(\frac{n+2}{2}\right)\left(\frac{n+4}{2}\right)\left(\frac{n+6}{2}\right)\cdots\left(\frac{n+2n-2}{2}\right)\left(\frac{-n+2}{2}\right)\left(\frac{-n+4}{2}\right)\left(\frac{-n+6}{2}\right)\cdots\left(\frac{-n+2n-2}{2}\right)}{(n-1)!\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdots\cdots\frac{2n-1}{2}}\left(\sin^2\phi\right)^{n-1}$$

$$=\frac{(-1)^{n-1}(n+2)(n+4)(n+6)\cdots(n+2n-2)(n-2)(n-4)(n-6)\cdots(n-2n+2)}{2\cdot 4\cdot 6\cdots\cdots(2n-2)\,3\cdot 5\cdot 7\cdots\cdots(2n-1)}\left(\sin^2\phi\right)^{n-1}$$

$$=\frac{(-1)^{n-1}(n^2-2^2)(n^2-4^2)(n^2-6^2)\cdots\cdots(n^2-\overline{2n-2}^{2})}{(2n-1)!}\left(\sin^2\phi\right)^{n-1}$$

Thus we obtain the formula,

$$\sin(n\phi)=n\cos\phi\sin\phi\,F\left(\frac{n}{2}+1,-\frac{n}{2}+1,\ \frac{3}{2},\ \sin^2\phi\right) \tag{7.19}$$

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[1] Bromwich: Infinite Series, p. 207

We obtain a third expression for $\sin r\phi$ by use of the series:[1]

$$\sin(r\phi) = r\tan\phi(\cos\phi)^r\left[1-\frac{(r-1)(r-2)}{3!}\tan^2\phi+\frac{(r-1)(r-2)(r-3)(r-4)}{5!}\tan^4\phi+\cdots\right.$$

$$\left.\cdots+(-1)^{n-1}\frac{(r-1)(r-2)\cdots(r-2n+2)}{(2n-1)!}(\tan^2\phi)^{n-1}+\cdots\right]$$

The nth term of $F(\frac{-r}{2}+1,-\frac{r}{2}+\frac{1}{2},\frac{3}{2},-\tan^2\phi)$ is

$$\frac{\left(\frac{-r+2}{2}\right)\left(\frac{-r+4}{2}\right)\cdots\left(\frac{-r+2n-2}{2}\right)\left(\frac{-r+1}{2}\right)\left(\frac{-r+3}{2}\right)\cdots\left(\frac{-r+2n-3}{2}\right)}{(n-1)!\ \frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdots\left(\frac{2n-1}{2}\right)}\left(-\tan^2\phi\right)^{n-1}$$

$$=(-1)^{n-1}\frac{(r-2)(r-4)\cdots(r-2n+2)(r-1)(r-3)\cdots(r-2n+3)}{(2n-1)!}(\tan^2\phi)^{n-1}$$

Hence

$$\sin(r\phi) = r\tan\phi(\cos\phi)^r F\left(-\frac{r}{2}+1,-\frac{r}{2}+\frac{1}{2},\frac{3}{2},-\tan^2\phi\right)$$

or

$$\sin r\phi = r\sin\phi(\cos\phi)^{r-1}F\left(-\frac{r}{2}+1,-\frac{r}{2}+\frac{1}{2},\frac{3}{2},-\tan^2\phi\right) \qquad (7.20)$$

If in this we set $r=-r$ we have

$$\sin(-r\phi) = -r\sin\phi(\cos\phi)^{-r-1}F\left(\frac{r}{2}+1,\frac{r}{2}+\frac{1}{2},\frac{3}{2},-\tan^2\phi\right)$$

or

$$\sin(r\phi) = r\sin\phi(\cos\phi)^{-r-1}F\left(\frac{r}{2}+1,\frac{r}{2}+\frac{1}{2},\frac{3}{2},-\tan^2\phi\right) \qquad (7.21)$$

17. EXPRESSION FOR $\cos k\phi$. A series for $\cos k\phi$ is[2]

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1. Bromwich: Infinite Series, p. 209
2. Ibidem, p. 207

$$\cos K\phi = 1 - \frac{K^2}{2!}\sin^2\phi + \frac{K^2(K^2-2^2)}{4!}\sin^4\phi - \cdots\cdots\cdots$$

$$\cdots\cdots + (-1)^{n-1}\frac{K^2(K^2-2^2)(K^2-4^2)\cdots\cdots(K^2-\overline{2n-4}^{\,2})}{(2n-2)!}\cos^{2n-2}\phi + \cdots\cdots$$

Now the nth term of $F\left(\frac{K}{2}, -\frac{K}{2}, \frac{1}{2}, \sin^2\phi\right)$ is

$$\frac{\frac{K}{2}\left(\frac{K}{2}+1\right)\left(\frac{K}{2}+2\right)\cdots\cdots\left(\frac{K}{2}+n-2\right)\left(-\frac{K}{2}\right)\left(-\frac{K}{2}+1\right)\left(-\frac{K}{2}+2\right)\cdots\cdots\left(-\frac{K}{2}+n-2\right)}{1\cdot 2\cdot 3\cdots\cdots(n-1)\,\frac{1}{2}\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)\cdots\cdots\left(\frac{1}{2}+n-2\right)}\left(\sin^2\phi\right)^{n-1}$$

Multiply this expression by $\left(\frac{2^{n-1}}{2^{n-1}}\right)^2$. Change the sign of the $(n-1)$ terms containing $-\frac{K}{2}$ and prefix the whole term by $(-1)^{n-1}$. These changes do not alter the value of the term. We then have

$$(-1)^{n-1}\frac{K^2(K^2-2^2)(K^2-4^2)\cdots\cdots(K^2-\overline{2n-4}^{\,2})}{(2n-2)!}\left(\sin^2\phi\right)^{n-1}$$

which is the same as the nth term of the series for $\cos K\phi$. Hence

$$\cos K\phi = F\left(\frac{K}{2}, -\frac{K}{2}, \frac{1}{2}, \sin^2\phi\right) \tag{7.22}$$

We obtain a second expression for $\cos K\phi$ by use of the series,[1]

$$\cos K\phi = \cos\phi\left[1 - \frac{K^2-1^2}{2!}\sin^2\phi + \frac{(K^2-1^2)(K^2-3^2)}{4!}\sin^4\phi - \cdots\cdots\right.$$

$$\left.\cdots\cdots + (-1)^{n-1}\frac{(K^2-1^2)(K^2-3^2)\cdots\cdots(K^2-\overline{2n-3}^{\,2})}{(2n-2)!}\left(\sin^2\phi\right)^{n-1} + \cdots\cdots\right]$$

Since the nth term of $F\left(\frac{K}{2}+\frac{1}{2}, -\frac{K}{2}+\frac{1}{2}, \frac{1}{2}, \sin^2\phi\right)$ is

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[1] Bromwich: Infinite Series, p. 205

$$\frac{\left(\frac{K}{2}+\frac{1}{2}\right)\left(\frac{K}{2}+\frac{3}{2}\right)\cdots\left(\frac{K}{2}+\frac{2n-3}{2}\right)\left(-\frac{K}{2}+\frac{1}{2}\right)\left(-\frac{K}{2}+\frac{3}{2}\right)\cdots\left(-\frac{K}{2}+\frac{2n-3}{2}\right)}{(n-1)!\ \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\cdots\left(\frac{2n-3}{2}\right)}\left(\sin^2\phi\right)^{n-1}$$

$$=(-1)^{n-1}\frac{(K+1)(K+3)(K+5)\cdots(K+2n-3)(K-1)(K-3)(K-5)\cdots(K-\overline{2n-3})}{(2n-2)!}\left(\sin^2\phi\right)^{n-1}$$

$$=(-1)^{n-1}\frac{(K^2-1^2)(K^2-3^2)(K^2-5^2)\cdots\cdots(K-\overline{2n-3}^{\,2})}{(2n-2)!}\left(\sin^2\phi\right)^{n-1}$$

there at once follows the formula,

$$\cos K\phi = \cos\phi\, F\left[\left(\frac{K}{2}+\frac{1}{2}\right),\left(-\frac{K}{2}+\frac{1}{2}\right),\left(\frac{1}{2}\right),\left(\sin^2\phi\right)\right] \qquad (7.23)$$

Consider now the series,[1]

$$\cos K\phi = (\cos\phi)^K \sum (-1)^{n-1}\frac{K(K-1)(K-2)\cdots\cdots(K-2n+3)}{(2n-2)!}\left(\tan^2\phi\right)^{n-1}$$

Further, since the nth term of $F(-\frac{K}{2}, -\frac{K}{2}+\frac{1}{2},\ \frac{1}{2},\ -\tan^2\phi)$ is

$$\frac{\left(-\frac{K}{2}\right)\left(\frac{-K+2}{2}\right)\left(\frac{-K+4}{2}\right)\cdots\left(\frac{-K+2n-4}{2}\right)\left(\frac{-K+1}{2}\right)\left(\frac{-K+3}{2}\right)\left(\frac{-K+5}{2}\right)\cdots\cdots\left(\frac{-K+2n-3}{2}\right)}{(n-1)!\ \frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdots\cdots\left(\frac{2n-1}{2}\right)}\left(-\tan^2\phi\right)^{n-1}$$

$$=(-1)^{n-1}\frac{K(K-1)(K-2)(K-3)\cdots\cdots\cdots(K-2n+3)\ \left(\tan^2\phi\right)^{n-1}}{(2n-2)!}$$

we have the result.

$$\cos K\phi = (\cos\phi)^K F\left(-\frac{K}{2}, \frac{-K+1}{2},\ \frac{1}{2},\ -\tan^2\phi\right) \qquad (7.24)$$

When we set $K=-K$ this becomes

$$\cos(-K\phi) = \cos K\phi = (\cos\phi)^{-K} F\left(\frac{K}{2}, \frac{K+1}{2},\ \frac{1}{2},\ -\tan^2\phi\right) \qquad (7.25)$$

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1. Bromwich: Infinite Series, p. 205.

18. EXPRESSION FOR $(k^2-x^2)^{-\frac{1}{2}}$. Expanding $(k^2-x^2)^{-\frac{1}{2}}$ by the binomial theorem we get

$$(k^2-x^2)^{-\frac{1}{2}} = (k^2)^{-\frac{1}{2}} + \frac{\frac{1}{2}}{1!}(k^2)^{-\frac{3}{2}}x^2 + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}(k^2)^{-\frac{5}{2}}x^4 + \cdots + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\frac{2n-3}{2}}{(n-1)!}(k^2)^{-\frac{2n-1}{2}}x^{n-1} + \cdots$$

$$= k^{-1}\left\{1 + \frac{\frac{1}{2}\beta}{1\cdot\beta}\frac{x^2}{k^2} + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\beta(\beta+1)}{1\cdot 2\cdot\beta(\beta+1)}\frac{x^4}{k^4} + \cdots + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdots\left(\frac{2n-3}{2}\right)\beta(\beta+1)\cdots(\beta+n-2)}{1\cdot 2\cdots(n-1)\beta(\beta+1)\cdots(\beta+n-2)}\left(\frac{x^2}{k^2}\right)^{n-1} + \cdots\right\}$$

$$= k^{-1}F\left(\tfrac{1}{2}, \beta, \beta, \frac{x^2}{k^2}\right) \tag{7.26}$$

where β is an arbitrary quantity.

19. EXPRESSION FOR $\arcsin\frac{x}{k}$

$$\arcsin\frac{x}{k} = \int (k^2-x^2)^{-\frac{1}{2}}dx = \int k^{-1}F\left(\tfrac{1}{2}, \beta, \beta, \frac{x^2}{k^2}\right)dx$$

where β is arbitrary, by (7.26).

But

$$\int k^{-1}F\left(\tfrac{1}{2}, \beta, \beta, \frac{x^2}{k^2}\right) = \frac{1}{k}\left(x + \frac{\frac{1}{2}x^3}{1\cdot 3\cdot k^2} + \frac{\frac{1}{2}\cdot\frac{3}{2}x^5}{1\cdot 2\cdot 5\, k^4} + \cdots\right.$$

$$\left.\cdots + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\left(\frac{1}{2}-n-2\right)}{1\cdot 2\cdot 3\cdots(n-1)(2n-1)}\frac{x^{2n-1}}{k^{2n-2}} + \cdots\right) + C$$

$$= \frac{x}{k}\left(1 + \frac{\frac{1}{2}\cdot\frac{1}{2}}{1\cdot\frac{3}{2}}\frac{x^2}{k^2} + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}}{1\cdot 2\cdot\frac{3}{2}\cdot\frac{5}{2}}\left(\frac{x^2}{k^2}\right)^2 + \cdots\right.$$

$$\left.\cdots + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\left(\frac{1}{2}+n-2\right)\ \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\left(\frac{1}{2}+n-2\right)}{1\cdot 2\cdot 3\cdots(n-1)\ \frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdots\left(\frac{2n-1}{2}\right)}\left(\frac{x^2}{k^2}\right)^{n-1} + \cdots\right) + C$$

$$= \frac{x}{k}F\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, \frac{x^2}{k^2}\right) + C$$

Therefore

$$\arcsin \frac{x}{k} = \frac{x}{k} F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{x^2}{k^2}\right) + C \tag{7.27}$$

On setting $x = 0$, we find $C = 0$. Hence

$$\arcsin \frac{x}{k} = \frac{x}{k} F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{x^2}{k^2}\right) \tag{7.27}$$

20. EXPRESSION FOR $\arccos \frac{x}{k}$. From Article 19 of this chapter we have

$$\int (k^2 - x^2)^{-\frac{1}{2}} dx = \frac{x}{k} F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{x^2}{k^2}\right) + C$$

But
$$\arccos \frac{x}{k} = -\int (k^2 - x^2)^{-\frac{1}{2}} dx = -\frac{x}{k} F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{x^2}{k^2}\right) - C$$

To determine C we set $x = 0$ and find $C = -\frac{\pi}{2}$

Therefore

$$\arccos \frac{x}{k} = -\frac{x}{k} F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{x^2}{k^2}\right) + \frac{\pi}{2} \tag{7.28}$$

21. EXPRESSION FOR $\arctan \frac{x}{k}$. By actual division

$$\left(1 + \frac{x^2}{k^2}\right)^{-1} = 1 - \frac{x^2}{k^2} + \frac{x^4}{k^4} - \frac{x^6}{k^6} + \cdots + (-1)^{n-1} \frac{x^{2n-2}}{k^{2n-2}} + \cdots$$

But

$$\arctan \frac{x}{k} = \frac{1}{k} \int \left(1 + \frac{x^2}{k^2}\right)^{-1} dx = \frac{x}{k} \left\{ 1 + \frac{1}{3}\left(-\frac{x^2}{k^2}\right) + \frac{1}{5}\left(-\frac{x^2}{k^2}\right)^2 + \cdots \right.$$
$$\left. \cdots + \frac{1}{2n-1}\left(-\frac{x^2}{k^2}\right)^{n-1} + \cdots \right\} + C$$

$$= \frac{x}{k} \left\{ 1 + \frac{\frac{1}{2} \cdot 1}{1 \cdot \frac{3}{2}} \left(-\frac{x^2}{k^2}\right) + \frac{\frac{1}{2} \cdot \frac{3}{2} \cdot 1 \cdot 2}{1 \cdot 2 \cdot \frac{3}{2} \cdot \frac{5}{2}} \left(-\frac{x^2}{k^2}\right)^2 + \cdots \right.$$

$$\left. \cdots + \frac{\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdots \left(\frac{1}{2} + n + 2\right) 1 \cdot 2 \cdot 3 \cdots (n-1)}{1 \cdot 2 \cdot 3 \cdots (n-1) \frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2} \cdots \left(\frac{2n-1}{2}\right)} \left(\frac{-x^2}{k^2}\right)^{n-1} + \cdots \right\} + C$$

On setting $x=0$, we find $C=0$ and our result is the formula,

$$\arctan\left(\frac{x}{K}\right)=\frac{x}{K}F\left(\frac{1}{2},1,\frac{3}{2},-\frac{x^2}{K^2}\right) \qquad (7.29)^1$$

1. See Appendix, Note III, page 92 for additional functions expressed in terms of the hypergeometric series.

APPENDIX

NOTE ONE

PROOF OF THE THEOREM ON CONVERGENCE OF PAGE 12

1. WEIERSTRASS INEQUALITIES. It is necessary to establish here certain inequalities on which the theorem rests. Let $a_1, a_2, \cdots \cdots, a_n$ be positive numbers less than 1.

Then

$$(1+a_1)(1+a_2) = 1+(a_1+a_2)+a_1a_2 > 1+a_1+a_2$$

By use of this we obtain

$$[(1+a_1)(1+a_2)](1+a_3) > (1+a_1+a_2)(1+a_3) > 1+a_1+a_2+a_3$$

This process continued gives

$$(1+a_1)(1+a_2)(1+a_3)\cdot\cdots\cdot(1+a_n) > 1+(a_1+a_2+a_3+\cdots+a_n)$$

or in more convenient notation

$$\prod_{i=1}^{n}(1+a_i) > 1+\sum_{i=1}^{n} a_i \tag{A.1}$$

Similarly

$$(1-a_1)(1-a_2) = 1-(a_1+a_2)+a_1a_2 > 1-(a_1+a_2)$$

$$[(1-a_1)(1-a_2)][1-a_3] > [1-(a_1+a_2)][1-a_3] > 1-(a_1+a_2+a_3)$$

and generally $(1-a_1)(1-a_2)(1-a_3)\cdots(1-a_n) > 1-(a_1+a_2+a_3+\cdots+a_n)$

i. e.,

$$\prod_{i=1}^{n}(1-a_i) > 1-\sum_{i=1}^{n} a_i \tag{A.2}$$

Further $1 + a_1 = \dfrac{1 - a_1^2}{1 - a_1} < \dfrac{1}{1 - a_1}$

Hence

$$(1+a_1)(1+a_2)\cdots(1+a_n) < \frac{1}{(1-a_1)(1-a_2)\cdots(1-a_n)} \tag{A.3}$$

But by (A.2)

$$\frac{1}{(1-a_1)(1-a_2)\cdots(1-a_n)} < \frac{1}{1-(a_1+a_2+\cdots+a_n)}$$

or

$$\frac{1}{\prod_{i=1}^{n}(1-a_i)} < \frac{1}{1-\sum_{i=1}^{n} a_i}$$

Thus if $\sum_{i=1}^{n} a_i < 1$ we have

$$\prod_{i=1}^{n}(1+a_i) < \frac{1}{1-\sum_{i=1}^{n} a_i} \tag{A.4}$$

and $\prod_{i=1}^{n}(1+a_i) > 1 + \sum_{i=1}^{n} a_i$ by (A.1)

Therefore we may write

$$1 + \sum_{i=1}^{n} a_i < \prod_{i=1}^{n}(1+a_i) < \frac{1}{\prod_{i=1}^{n}(1-a_i)}$$

or

$$1 + \sum_{i=1}^{n} a_i < \frac{1}{\prod_{i=1}^{n}(1-a_i)}$$

or

$$\prod_{i=1}^{n}(1-a_i) < \frac{1}{1+\sum_{i=1}^{n} a_i} \tag{A.5}$$

A combination of (A.4) and (A.1) gives

$$\left[1 - \sum_{i=1}^{n} a_i\right]^{-1} > \prod_{i=1}^{n}(1+a_i) > 1 + \sum_{i=1}^{n} a_i \tag{A.6}$$

and by means of (A.5) and (A.2),

$$\left[1 + \sum_{i=1}^{n} a_i\right] > \prod_{i=1}^{n}(1-a_i) > 1 - \sum_{i=1}^{n} a_i \tag{A.7}$$

We now proceed to establish the following lemma.

2. LEMMA I. If $a_1, a_2, a_3, \cdots\cdots, a_n$ are numbers between 0 and 1, the convergence of the series Σa_n is necessary and sufficient for the convergence of the products P_n and Q_n to the positive limits P and Q, respectively, as $n \to \infty$ where

$$P_n = (1+a_1)(1+a_2) \cdots\cdots (1+a_n)$$

$$Q_n = (1-a_1)(1-a_2) \cdots\cdots (1-a_n)$$

PROOF: Clearly P_n increases and Q_n decreases as n increases. Moreover, if Σa_n is convergent, a value m can be found such that

$$\sigma = a_{m+1} + a_{m+2} + \cdots\cdots + a_n + \cdots\cdots \text{ to infinity}$$

is < 1. Then by (A.6) and (A.7)

$$\frac{1}{1-\sigma} > \frac{1}{1-(a_{m+1}+a_{m+2}+\cdots+a_n)} > P_n > \frac{P_n}{P_m}$$

and
$$1-\sigma < 1-(a_{m+1}+a_{m+2}+\cdots a_n) < Q_n < \frac{Q_n}{Q_m}$$

We then have if $n > m$

$$P_n < \frac{P_m}{1-\sigma}, \quad Q_n > Q_m(1-\sigma)$$

Thus P_n and Q_n approach definite limits P and Q such that

$$P \leqq \frac{P_m}{1-\sigma}$$

$$Q \geqq Q_m(1-\sigma)$$

But if $\sum a_n$ diverges, an index m can be chosen such that

$a_1 + a_2 + \cdots + a_n > N$, where N is a quantity arbitrarily large.

In that case

$$\operatorname{Lim} P_n = \infty \quad , \quad \operatorname{Lim} Q_n = 0$$

that is, the products diverge..

3. LEMMA II. If $\dfrac{a_n}{a_{n+1}} = 1 + \dfrac{b_n}{n}$, where $b_n = b > 0$,

the $\operatorname{Lim} a_n = 0$.

PROOF: An m can be found such that

$$b_n > \tfrac{1}{2} b > 0 \qquad \text{if} \quad n \geqq m$$

Then

$$\frac{a_m}{a_{m+1}} > 1 + \frac{\frac{b}{2}}{m}, \quad \frac{a_{m+1}}{a_{m+2}} > 1 + \frac{\frac{b}{2}}{m+1}, \quad \frac{a_n}{a_{n+1}} > 1 + \frac{\frac{b}{2}}{n}$$

and $$\frac{a_m}{a_{m+1}} \cdot \frac{a_{m+1}}{a_{m+2}} \cdots\cdots \frac{a_n}{a_{n+1}} > \left(1 + \frac{b}{2m}\right)\left(1 + \frac{b}{3m}\right)\cdots\left(1 + \frac{b}{2n}\right) > 1 + \frac{b}{2}\left(\frac{1}{m} + \frac{1}{m+1} + \cdots \frac{1}{n}\right)$$

$$\therefore \operatorname{Lim} \frac{a_m}{a_{m+1}} = \infty$$

or $$\operatorname{Lim} \frac{a_m}{a_n} = \infty$$

Hence $$\operatorname{Lim} a_n = 0$$

4. THEOREM: If $\dfrac{U_n}{U_{n+1}}$ can be expressed in the form

$$\frac{U_n}{U_{n+1}} = 1 + \frac{\mu}{n} + O\left(\frac{1}{n^p}\right), \quad p > 1$$

the series $\sum (-1)^{n-1}$ is convergent if $\mu > 0$.

PROOF: If $\mu > 0$, after a certain stage

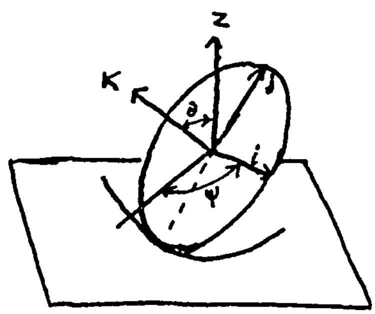
$$\frac{\mu}{n} > O\left(\frac{1}{n^p}\right) \qquad \text{so that} \quad U_n > U_{n+1}$$

But $U_n \rightarrow 0$ by Lemma II. Therefore $\sum (-1)^{n-1} U_n$ converges.

NOTE II

PROBLEMS IN PHYSICS INVOLVING HYPERGEOMETRIC SERIES

1. PROBLEM OF A ROLLING AND SPINNING COIN[1].



Let there be a movable set of axes T_1, movable with respect to a fixed point and to center of coin, but with its origin always at the center of coin, the k-axis perpendicular to the plane of the coin, and the i-axis always horizontal. Let there be a second set of axes, T_2, fixed with respect to the coin, with its origin at the center of the coin, and one of its axes perpendicular to the plane of the coin and therefore coinciding with the k-axis. If θ, ϕ, and ψ are Euler's angles for the axes that are fixed in the body: ω_k, the component of the instantaneous rotation of T_2 in the positive direction along the k-axis, the equation of motion is

$$\frac{d^2\omega_k}{d\theta^2} + \cot\theta\,\frac{d\omega_k}{d\theta} - \frac{a^2C}{A_1C_1+a^2A_1} = 0 \quad \text{where } A_1,\ C_1, \text{ and } a \text{ are constants.}$$

If the independent variable is changed by the substitution,

$$s = \cos^2\theta, \qquad p = \frac{a^2C_1}{4A_1(a^2+C_1)}$$

we have

1. Given by Appell and Kortweg in "Rendiconti del Circolo Matematico di Palermo", 1898.

$$s(1-s)\frac{d^2\omega_K}{ds^2} + \left(\frac{1}{2} - \frac{3}{2}s\right)\frac{d\omega_K}{ds} - p\,\omega_K = 0$$

or

$$s(1-s)\frac{d^2\omega_K}{ds^2} + \left\{\frac{1}{2} - \left[1 + \overline{\tfrac{1}{4} + \tfrac{1}{4}\sqrt{1-16p}} + \overline{\tfrac{1}{4} - \tfrac{1}{4}\sqrt{1-16p}}\,\right] s\right\}\frac{d\omega_K}{ds} - p\,\omega_K = 0$$

This is the hypergeometric equation of form,

$$x(1-x)\frac{d^2y}{dx^2} + \left[\gamma - \overline{1+\alpha+\beta}\,x\right]\frac{dy}{dx} - \alpha\beta y = 0$$

where

$$\alpha = \tfrac{1}{4} + \tfrac{1}{4}\sqrt{1-16p}\,, \quad \beta = \tfrac{1}{4} - \tfrac{1}{4}\sqrt{1-16p}\,, \quad \gamma = \tfrac{1}{2}$$

Hence by Article 1, Chapter II, the solution in ascending powers of the variable is

$$\omega_K = M \cdot F\left(\tfrac{1}{4} + \tfrac{1}{4}\sqrt{1-16p},\ \tfrac{1}{4} - \tfrac{1}{4}\sqrt{1-16p},\ \tfrac{1}{2},\ \cos^2\theta\right)$$
$$+ N \cdot F\left(\tfrac{3}{4} + \tfrac{1}{4}\sqrt{1-16p},\ \tfrac{3}{4} - \tfrac{1}{4}\sqrt{1-16p},\ \tfrac{3}{2},\ \cos^2\theta\right)$$

where M and N are arbitrary constants.

2. CALCULATION OF ROTATIONAL ENERGY LEVELS OF A MOLECULE HAVING A SINGLE AXIS OF SYMMETRY.[1] If internal vibration is neglected, the wave equation, obtained by means of considerations of kinetic energy, for a molecule is

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \cdot \frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin^2\theta} \cdot \frac{\partial^2\psi}{\partial\chi^2} + \left(\frac{A}{C} + \cot^2\theta\right)\frac{\partial^2\psi}{\partial\phi^2}$$
$$- 2\,\frac{\cos\theta}{\sin\theta} \cdot \frac{\partial^2\psi}{\partial\phi\,\partial\chi} + \frac{8\pi^2 A}{h^2}(E - V)\psi$$

where A is the two principal moments of inertia, and, the third moment of inertia and θ, ϕ, and ψ are Eulerian angles. The undisturbed top has no potential energy if we set $V = 0$. Let us make the

1. Reiche, Z: Physik, 39, p. 444 (1926)

substitution,

$$\psi = \Theta(\theta)\exp(i\lambda\phi + im\chi)^{1}$$

where λ and m must be integers in order for ψ to be an acceptable function. (A.8) then becomes

$$\frac{d^2\Theta}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{d\Theta}{d\theta} - \frac{(m-\lambda\cos\theta)^2}{\sin^2\theta}\Theta + \left[j(j+1)-\lambda^2\right]\Theta = 0 \qquad \text{(A.9)}$$

where $j(j+1) - \lambda^2 = \dfrac{8\pi^2 AE}{h^2} - \dfrac{A}{C}\lambda^2$

Introducing a new independent variable,

$$t = \tfrac{1}{2}(1-\cos\theta$$

and a new dependent variable,

$$\chi = t^{-\frac{d}{2}}(1-t)^{-\frac{s}{2}}\,\Theta$$

where $s = |\lambda + m|$ and $d = |\lambda - m|$, Equation (A.9) becomes

$$t(1-t)\frac{d^2\chi}{dt^2} + \left\{1+d - \left[\left(\frac{d+s}{2}+j+1\right) + \left(\frac{d+s}{2}-j\right) + 1\right]t\right\}\frac{d\chi}{dt}$$
$$- \left(\frac{d+s}{2}+j+1\right)\left(\frac{d+s}{2}-j\right)\chi = 0$$

This is a hypergeometric equation. The results of Article 1, Chapter II, enable us to write down at once the solution,

$$\chi = M\cdot F\left(\frac{d+s}{2}+j+1,\ \frac{d+s}{2}-j,\ 1+d,\ \chi\right)$$
$$+ N\cdot F\left(\frac{d+s}{2}+j-d+1,\ \frac{d+s}{2}-j-d,\ 1-d,\ \chi\right)$$

where M and N are arbitrary constants.

1. $\exp(m) = e^m$

NOTE III

THREE FORMULAE DUE TO GAUSS

1. The three following formulae are given by Gauss in his memoir, "Circa Seriem Infinitam, $1+\frac{\alpha\beta}{1\cdot\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}x^2+\cdots$" and are not included in Chapter VII.

$$t = \sin t\, F\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, \sin^2 t\right) \qquad (A.10)$$

$$t = \sin t \cos t\, F\left(1, 1, \tfrac{3}{2}, \sin^2 t\right) \qquad (A.11)$$

$$t = \tan t\, F\left(\tfrac{1}{2}, 1, \tfrac{3}{2}, -\tan^2 t\right) \qquad (A12)$$

With the exception of these three formulae Chapter V11 contains all the expressions of functions in hypergeometric series listed by Gauss and a great many others. Gauss gives no hint as to how he arrived at any of the formulae of this type listed in his memoir.

NOTE IV

THE DIFFERENTIAL EQUATION OF THE HYPERGEOMETRIC SERIES OBTAINED FROM EQUATION (3.38)

Equation (3.38) is

$$\gamma(\gamma+1)F - (\gamma+1)(\gamma - \overline{\alpha+\beta+1}\,x)F''' - (\alpha+1)(\beta+1)x(1-x)F^{IV} = 0$$

But

$$F''' = F(\alpha+1, \beta+1, \gamma+1, x) = \frac{\gamma}{\alpha\beta} \cdot \frac{dF}{dx} \qquad \text{by } (3.3)$$

and

$$F^{VI} = F(\alpha+2, \beta+2, \gamma+2, x) = \frac{\gamma(\gamma+1)}{\alpha(\alpha+1)(\beta)(\beta+1)} \frac{d^2F}{dx^2} \quad \text{by } (3.4)$$

Hence Equation (3.38) becomes on substitution of these values and clearing of fractions,

$$\frac{d^2F}{dx^2} + \frac{\gamma + \overline{\alpha+\beta+1}\,x}{x(1-x)} \frac{dF}{dx} + \frac{\alpha\beta}{x(1-x)} F = 0$$

or

$$\frac{d^2y}{dx^2} + \frac{\gamma - \overline{\alpha+\beta+1}\,x}{x(1-x)} \frac{dy}{dx} + \frac{\alpha\beta}{x(1-x)} y = 0$$

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