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Abstract: Composition operators C_φ induced by a selfmap φ of some set S are operators acting on a space consisting of functions on S by composition to the right with φ, that is $C_\varphi f = f \circ \varphi$. In this paper, we consider the Hilbert Hardy space H^2 on the open unit disk and find exact formulas for distances $\|C_\varphi - C_\psi\|$ between composition operators. The selfmaps φ and ψ involved in those formulas are constant, inner, or analytic selfmaps of the unit disk fixing the origin.

Key words: Composition operators, norm–distance.

AMS *Subject Class.* (2000): 47B33, 47B38.

1. Introduction

Let H^p denote the Hardy space of index p on the open unit disk $\mathbb{U}$, that is the space of all functions f analytic in $\mathbb{U}$ satisfying the condition

$$\|f\|_p := \sup_{0<r<1} \left(\int_{\partial\mathbb{U}} |f(r\zeta)|^p \, dm(\zeta) \right)^{1/p} < +\infty, \tag{1}$$

where m is the normalized Lebesgue measure and p is fixed $0 < p < +\infty$.

It is well known that $\| \ \|_2$ is a Hilbert norm on H^2 with alternative description

$$\|f\|_2 = \sqrt{\sum_{n=0}^{\infty} |c_n|^2}, \tag{2}$$

where $\{c_n\}$ is the sequence of Maclaurin coefficients of f. The Hilbert Hardy space H^2 is our space of choice in this paper.

The space H^∞ is the space of all bounded analytic functions on $\mathbb{U}$ endowed with the supremum norm $\| \ \|_\infty$. It is easy to see that $H^\infty \subseteq H^p$, $0 < p < +\infty$. Another well known fact about H^p–functions is the fact that, by a classical result of P. Fatou [5, Theorem 1.3], eventually extended by F. and M. Riesz, those functions have nontangential limits a.e. on $\partial\mathbb{U}$. The nontangential limit function of any f in H^p will be denoted by the same symbol as the function itself. It is known that it is an $L^p_{\partial\mathbb{U}}$–function and

$$\|f\|_p = \left(\int_{\partial\mathbb{U}} |f(\zeta)|^p \, dm(\zeta)\right)^{1/p}, \qquad f \in H^p. \tag{3}$$

An analytic selfmap of $\mathbb{U}$ is called an inner function if it has unimodular nontangential limits a.e. on $\partial\mathbb{U}$.

For each analytic selfmap φ of $\mathbb{U}$ *the composition operator of symbol* φ is the following operator

$$C_\varphi f = f \circ \varphi, \qquad f \in H^2. \tag{4}$$

Such operators are bounded, as a consequence of Littlewood's Subordination Principle, [5, Theorem 1.7], saying that composition operators whose symbol fixes the origin are contractions. If φ is a conformal automorphism of $\mathbb{U}$, we call C_φ an automorphic composition operator.

The numerical range of a Hilbert space operator T is the set $W(T) = \{< Tf, f >: \|f\| = 1\}$. It is well known that numerical ranges are convex subsets of the complex plane whose closure contains the spectrum of the given operator, [6, Chapter 22]. The quantity $w(T) = \sup\{| < Tf, f > | : \|f\| = 1\}$ is called the numerical radius of the operator T.

A Hilbert space operator T that satisfies an equation of the form $T^2 + \lambda T + \mu I = 0$ where I is the identity operator and λ and μ are constants is called a quadratic operator.

In Section 2 of this paper we are able to complete the description of numerical ranges of quadratic, automorphic composition operators by showing that the numerical range of such an operator is open, unless the operator is $C_z = I$ or $C_{(-z)}$. The description of the closure of the aforementioned numerical range was obtained in [1] and [3].

Quadratic Hilbert space operators are known to have elliptical numerical ranges, [19, Theorem 2.1]. We apply the aforementioned result and the theorems characterizing the numerical ranges of quadratic composition operators to calculate distances between composition operators. The symbols of the composition operators involved in these distance computations are constant or inner. We use different methods to calculate the distance $\|C_0 - C_\varphi\|$ when $\varphi(0) = 0$. We show that the distance above equals some Hardy norm $\|\varphi\|_p$, of φ, for some $2 \le p \le +\infty$. We are able to show that any $2 \le p \le +\infty$ works if and only if φ is a scalar multiple of an inner function. Otherwise, we show the choice of p is unique, and that $p = 2$ if and only if φ is orthogonal to the set $\{\varphi^2, \varphi^3, \dots\}$. These distance computations are in Section 3.

2. Description of a numerical range

By the spectral mapping theorem [6, Chapter 9], the spectrum of a quadratic operator can consist of at most 2 points. Quadratic operators with spectrum consisting of two points are known to have elliptical numerical ranges. More exactly, the following is proved in [19, Theorem 2.1].

Theorem 1. *The numerical range of a quadratic operator having spectrum consisting of the two distinct points a and b is an open or a closed elliptical disk, possibly degenerate, (that is, reduced to its focal axis). The major axis of the disk has length $\|T-aI\|$ and the length of the minor axis is $\sqrt{(\|T-aI\|^2-|a-b|^2)}$. The elliptical disk is closed if and only if T attains its norm or equivalently, if and only if it attains its numerical radius.*

Above, the statement that T attains its norm, respectively its numerical radius, means that there is some norm–one vector f so that $\|T\| = \|Tf\|$, respectively $| < Tf, f > | = w(T)$.

It is easy to determine the quadratic automorphic composition operators. Indeed, by [12], the only automorphic composition operators with spectrum consisting of at most 2 points are the identity operator and the composition operators whose automorphic symbol φ fixes a point in $\mathbb{U}$ and is conformally conjugated to $-z$, that is φ should be of the form

$$\varphi(z) = \alpha_p(z) = \frac{p-z}{1-\overline{p}z}, \qquad z \in \mathbb{U},$$

where p is any fixed point in $\mathbb{U}$. Visibly such operators are quadratic because $\alpha_p \circ \alpha_p(z) = z$, $z \in \mathbb{U}$ and hence $C_{\alpha_p}^2 = I$.

The closure of $W(C_{\alpha_p})$ was characterized by the authors of [3]. They showed it is a closed elliptical disc of foci -1 and 1. That disk is reduced to its focal axis if and only if $p = 0$. The authors of [3] gave a formula for the length of the major axis of that disk. That formula is hard to use in practical problems. Therefore, very recently, the author of [1] found the following practical formula for the length of the aforementioned major axis.

Theorem 2. *For each $p \in \mathbb{U}$, the length of the major axis of the closure of $W(C_{\alpha_p})$ is $2/\sqrt{1-|p|^2}$.*

Except the case when $p = 0$, it is not known if $W(C_{\alpha_p})$ is open or closed. We prove in the sequel that it is open. According to Theorem 1, we should check if C_{α_p} attains its norm or not.

Norms of composition operators are not easy to calculate. The commonest things known about them are the following.

$$\frac{1}{\sqrt{1-|\varphi(0)|^2}} \le \|C_\varphi\| \le \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}} \tag{5}$$

Visibly the lower and upper bound coincide to 1 if φ fixes the origin. If $\varphi(0) \neq 0$, then the lower bound is attained if and only if the symbol is constant [13, Theorem 4], whereas the upper bound is attained if and only if the symbol is an inner function [16, Theorem 5.2]. We are now ready to prove that the numerical range of a quadratic, automorphic composition operator, other than $C_{(-z)}$ or C_z is an open elliptical disk. We obtain this result, as a consequence of the following.

Proposition 1. *A composition operator having inner symbol φ attains its norm if and only if $\varphi(0) = 0$.*

Proof. Let

$$P(z,u) = \Re \frac{u+z}{u-z}, \qquad u \in \partial\mathbb{U},\ z \in \mathbb{U},$$

be the usual Poisson kernel.

Let φ be an inner function. The following formula is established in [12]

$$\int_{\partial\mathbb{U}} |f\circ\varphi(u)|^2\, dm(u) = \int_{\partial\mathbb{U}} |f(u)|^2 P(\varphi(0),u)\, dm(u), \qquad f \in H^2. \tag{6}$$

An immediate consequence is the fact that composition operators whose symbols are inner functions fixing the origin are isometries.

It is well known and easy to prove that

$$\frac{1+|z|}{1-|z|} \ge P(z,u), \qquad u \in \partial\mathbb{U},\ z \in \mathbb{U}. \tag{7}$$

Given (6), if $f \in H^2$ has norm 1, the relation $\|C_\varphi f\|_2 = \|C_\varphi\|$ is equivalent to

$$\int_{\partial\mathbb{U}} |f(u)|^2 \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|} - P(\varphi(0),u)\right) dm(u) = 0.$$

By (7), it follows that

$$|f(u)|^2 \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|} - P(\varphi(0),u)\right) = 0 \qquad \text{a.e.}$$

Since $\|f\|_2 = 1$, we deduce

$$\frac{1+|\varphi(0)|}{1-|\varphi(0)|} = P(\varphi(0), u) \qquad \text{a.e.},$$

a condition that is satisfied if and only if $\varphi(0) = 0$. Since composition operators with inner symbol fixing the origin are isometric, they obviously attain their norm. ∎

Corollary 1. *If $p \neq 0$, $W(C_{\alpha_p})$ is the open elliptical disk of foci ± 1 and major axis of length $2/\sqrt{1-|p|^2}$.*

Proof. This is a direct consequence of Theorem 1, Proposition 1, and the computation of the length of the major axis contained by Theorem 2. ∎

3. Distance computations

Composition operators of constant symbol C_p, $p \in \mathbb{U}$ are obviously idempotent and hence quadratic. As one can see, we denote by p both the function on $\mathbb{U}$ constantly equal to p and the complex number p itself. Since composition operators having constant symbol are rank-one operators, they are compact on the infinite dimensional space H^2. Thus their spectrum contains 0. The evident relation $C_\varphi 1 = 1$, valid for any composition operator, shows that the spectrum of C_p consists of 0 and 1. Then the following theorem originally proved in [9], (see also [4]), becomes very easy to obtain as a consequence of Theorem 1, the formula for the norm of composition operators of constant symbol, and the fact that composition operators of constant symbol attain their norms, (which is not hard to prove).

Theorem 3. *The numerical range of C_p is the closed elliptical disk of foci 0 and 1 and major axis $\frac{1}{\sqrt{1-|p|^2}}$. The disk is reduced to its focal axis if and only if $p = 0$.*

From now on, our basic problem will be finding the norm of a difference of two distinct composition operators.

First, recall that H^2 is a reproducing kernel Hilbert space, (see [6, Chapter 4] for the basics on this kind of spaces), that is the functions $k_p(z) = 1/(1-\overline{p}z)$, (called the kernel–functions of H^2) have the property below, known as "the reproducing property"

$$f(p) = < f, k_p >, \qquad p \in \mathbb{U},\ f \in H^2.$$

By the “reproducing property”, for any $p_1, p_2 \in \mathbb{U}$, the following holds

$$\|k_{p_1} - k_{p_2}\| = \sqrt{\frac{1}{1-|p_1|^2} + \frac{1}{1-|p_2|^2} - 2\Re\frac{1}{1-\overline{p_1}p_2}}. \tag{8}$$

Next, let us introduce more terminology. For an analytic map ψ on $\mathbb{U}$, M_ψ denotes the multiplication operator of symbol ψ, that is the operator

$$M_\psi f = \psi f, \qquad f \in H^2.$$

It is well known that M_ψ is bounded on H^2 if and only if $\psi \in H^\infty$ and, in that case, $\|M_\psi\| = \|\psi\|_\infty$. For an analytic map ψ on $\mathbb{U}$ and an analytic selfmap φ of $\mathbb{U}$, the operator $T_{\psi,\varphi} = M_\psi C_\varphi$ is called the weighted composition operator of symbols ψ and φ. Clearly such operators are bounded if the first symbol is bounded.

Denote $H_0^2 = zH^2 = H^2 \ominus \mathbb{C}$. A theorem that will be extensively cited in the sequel is the following, [16, Theorem 5.1].

Theorem 4. *Let φ be an analytic selfmap of $\mathbb{U}$ that fixes the origin. Then $\|C_\varphi|H_0^2\| = 1$ if and only if φ is inner.*

We give a new proof of Theorem 4, based on the following result which appears in [16] with a proof independent of that of Theorem 4 and was recently given an alternative short proof in [10].

The function $\varphi : \mathbb{U} \to \mathbb{U}$ is inner if and only if

$$\|C_\varphi\|_e = \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}},$$

where $\|C_\varphi\|_e$ denotes the essential norm of C_φ.

Before giving the announced new proof to Theorem 4 we record in a lemma a fact that is probably known as folklore and equally easy to prove.

Lemma 1. *If T is a Hilbert–space operator on the space H and L is a closed invariant subspace of T, then $\|T|L\|_e \le \|T\|_e$.*

Proof. Indeed, if P is the orthogonal projection of H onto L and K any compact operator on H, one can write

$$\|T+K\| \ge \|P(T+K)|L\| = \|(T|L) + (PK|L)\| \ge \|T|L\|_e$$

since $PK|L$ is a compact operator on L. ∎

Proof of Theorem 4. As we already noted, if φ is an inner function fixing the origin, then C_φ is isometric, hence $\|C_\varphi|H_0^2\| = 1$. The delicate part is the converse implication. Note that, if $\varphi(0) = 0$, then $C_\varphi^* 1 = 1$, thus H_0^2 is an invariant subspace of C_φ. Therefore, if $\|C_\varphi|H_0^2\| = 1$, then we distinguish between two cases. If $\|C_\varphi|H_0^2\|_e = 1$, then by Lemma 1, $\|C_\varphi\|_e = 1$, since C_φ is a contraction. Thus, $\|C_\varphi\|_e = \sqrt{(1+|\varphi(0)|)/(1-|\varphi(0)|)}$ and hence φ must be inner. The other case is when $\|C_\varphi|H_0^2\|_e < 1 = \|C_\varphi|H_0^2\|$. In this second case, it is very easy to prove that $C_\varphi|H_0^2$ is a norm–attaining operator, [8, Proposition 2.2]. Hence, one can consider a norm–one $f \in H^2$ so that

$$1 = \|C_\varphi(zf(z))\|_2^2 = \|\varphi f \circ \varphi\|_2^2 \leq \|f \circ \varphi\|_2^2 \leq \|f\|_2^2 = 1.$$

One gets that

$$\int_{\partial \mathbb{U}} |f \circ \varphi(\zeta)|^2 (1 - |\varphi(\zeta)|^2)\, dm(\zeta) = 0.$$

Since, $|\varphi(\zeta)| \leq 1$ a.e., one deduces $|f \circ \varphi(\zeta)|^2(1 - |\varphi(\zeta)|^2) = 0$ a.e., which implies that $|\varphi(\zeta)| = 1$ a.e., that is φ is inner. ∎

In the next proposition we prove that the norm of the restriction to H_0^2 of a composition operator is always equal to the distance between itself and certain composition operators having constant symbols. We also show that the distance between a composition operator C_φ and the orthogonal projection C_0 onto the subspace $\mathbb{C}$ of constant functions equals the norm of the weighted composition operator of identical symbols, $T_{\varphi,\varphi}$.

Proposition 2. *For each composition operator on H^2 the following relations hold*

$$\|\varphi\|_2 \leq \|C_\varphi - C_0\| = \|C_\varphi|H_0^2\| = \|T_{\varphi,\varphi}\| \leq \|\varphi\|_\infty \|C_\varphi\|. \tag{9}$$

For any analytic selfmap φ of $\mathbb{U}$

$$\|C_\varphi - C_{\varphi(0)}\| \leq \|T_{\varphi,\varphi}\| \quad \textit{hence} \quad \|C_\varphi - C_{\varphi(0)}\| \leq \|C_\varphi\|. \tag{10}$$

The second inequality in (10) *is strict if $\|\varphi\|_\infty < 1$ or if φ is a non–inner function fixing the origin. If φ is inner, then*

$$\|C_\varphi - C_{\varphi(0)}\| = \|C_\varphi - C_0\| = \|C_\varphi|H_0^2\| = \|T_{\varphi,\varphi}\| = \|C_\varphi\|. \tag{11}$$

Proof. Clearly $\|C_\varphi - C_0\| = \|C_\varphi|H_0^2\|$, since $(C_\varphi - C_0)1 = 0$ and $C_0|H_0^2 = 0$. Each function g in H_0^2 factors as $g(z) = zf(z)$ where f is an H^2–function of norm equal to the norm of g, and obviously $C_\varphi g = T_{\varphi,\varphi} f$. Hence $\|C_\varphi - C_0\| = \|C_\varphi|H_0^2\| = \|T_{\varphi,\varphi}\|$. Obviously

$$\|T_{\varphi,\varphi}\| \leq \|M_\varphi\|\|C_\varphi\| = \|\varphi\|_\infty \|C_\varphi\|,$$

which proves the upper estimate in (9). To prove the lower estimate, take the norm–one function z and note that $(C_\varphi|H_0^2)z = \varphi$.

Note now that the range $R(C_\varphi - C_{\varphi(0)})$ of $C_\varphi - C_{\varphi(0)}$ is contained in H_0^2. Therefore, $C_\varphi - C_{\varphi(0)}$ leaves H_0^2 invariant. Any difference of composition operators transforms 1 into 0, thus H_0^2, actually reduces $C_\varphi - C_{\varphi(0)}$, and $C_\varphi - C_{\varphi(0)} = 0 \oplus ((C_\varphi - C_{\varphi(0)})|H_0^2)$. Keeping this in mind, observe that

$$\|(C_\varphi - C_{\varphi(0)})f\|_2^2 = \sum_{n=1}^{\infty} |< f \circ \varphi - f(\varphi(0)), z^n >|^2$$

$$= \sum_{n=1}^{\infty} |< f \circ \varphi, z^n >|^2 = \|C_\varphi f\|_2^2 - |f(\varphi(0)|^2 \leq \|C_\varphi f\|_2^2, \qquad f \in H^2.$$

Hence $\|C_\varphi - C_{\varphi(0)}\| \leq \|C_\varphi\|$. Also, substitute f by $zf(z)$ above getting

$$\|(C_\varphi - C_{\varphi(0)})(zf(z))\|_2^2 = \|C_\varphi(zf(z))\|_2^2 - |\varphi(0)f(\varphi(0))|^2 \leq \|T_{\varphi,\varphi} f\|_2^2, \qquad f \in H^2.$$

Since $C_\varphi - C_{\varphi(0)} = 0 \oplus ((C_\varphi - C_{\varphi(0)})|H_0^2)$, one gets $\|C_\varphi - C_{\varphi(0)}\| \leq \|T_{\varphi,\varphi}\|$ which proves (10) and, by (9), also shows that the second inequality in (10) is a strict inequality if $\|\varphi\|_\infty < 1$. The fact that the same inequality is strict if $\varphi(0) = 0$ and φ is not an inner function is a direct consequence of Theorem 4 and (9).

As we noted, $\|C_\varphi\|_e = \sqrt{(1 + |\varphi(0)|)/(1 - |\varphi(0)|)} = \|C_\varphi\|$ if φ is inner. On the other hand $\|C_\varphi\|_e \leq \|C_\varphi - C_{\varphi(0)}\| \leq \|C_\varphi\|$, hence (11) holds if φ is inner. ∎

Corollary 2. *For any analytic selfmap φ of $\mathbb{U}$ the equality*

$$\|C_\varphi - C_0\| = \sqrt{\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}} \tag{12}$$

holds if and only if φ is inner. Hence composition operators having inner symbol attain their essential norm.

Proof. If φ is inner, then (12) is a consequence of (11) and the formula for the norm of a composition operator of inner symbol. Conversely, if the equality in (12) holds, then by (9), one has $\sqrt{(1+|\varphi(0)|)/(1-|\varphi(0)|)} = \|C_\varphi|H_0^2\| \le \|C_\varphi\| \le \sqrt{(1+|\varphi(0)|)/(1-|\varphi(0)|)}$, and hence φ is an inner function if $\varphi(0) \neq 0$. If $\varphi(0) = 0$ and (12) holds, then φ is inner, by (9) and Theorem 4. ∎

The following proposition contains two very simple distance–formulas.

PROPOSITION 3. *Let $p_1, p_2 \in \mathbb{U}$, $\lambda, \mu \in \overline{\mathbb{U}}$, and φ be an inner function, fixing the origin. Then*

$$\|C_{p_1} - C_{p_2}\| = \sqrt{\frac{1}{1-|p_1|^2} + \frac{1}{1-|p_2|^2} - 2\Re\frac{1}{1-\overline{p_1}p_2}} \tag{13}$$

and

$$\|C_{\lambda\varphi} - C_{\mu\varphi}\| = \sup\{|\lambda^n - \mu^n| : n = 1, 2, 3, \dots\}. \tag{14}$$

Proof. Since, for each $f \in H^2$,

$$\|(C_{p_1} - C_{p_2})f\|_2 = |f(p_1) - f(p_2)| = |< k_{p_1} - k_{p_2}, f >|,$$

(13) is a consequence of (8).

To prove (14), note that $\|C_{\lambda\varphi} - C_{\mu\varphi}\| = \|C_\varphi(C_{\lambda z} - C_{\mu z})\| = \|C_{\lambda z} - C_{\mu z}\|$. The operator $(C_{\lambda z} - C_{\mu z})$ has a diagonal matrix in the standard Hilbert base $B = \{1, z, z^2, z^3, \dots, z^n, \dots\}$ of H^2. The diagonal entries are $\{0, \lambda - \mu, \lambda^2 - \mu^2, \lambda^3 - \mu^3, \dots\}$ and hence, (14) follows, by the well known formula for the norm of a diagonal operator, [6, Chapter 7]. ∎

Formula (13) is practically a reformulation of [5, Ch. 8, Exercise 6], since composition operators of constant symbol can be identified with point–evaluations. This brings up the fact that this very particular class of composition operators relates to Hardy space extremal linear problems, whose theory reached full elegance by the addition of functional theoretical methods, due to contributions of S. Ya. Havison, W. W. Rogosinski, H. S. Shapiro, and others, (see [5, Notes to Ch. 8]).

COROLLARY 3. *If φ is an inner function fixing the origin and λ, $\mu \in \partial\mathbb{U}$, $\lambda \neq \mu$, then $\|C_{\lambda\varphi} - C_{\mu\varphi}\| = 2$ if λ/μ is a root of unity of even order or if λ/μ is not a root of unity. If λ/μ is a root of unity of odd order k, then $\|C_{\lambda\varphi} - C_{\mu\varphi}\| = |1 - e^{\pi(k-1)/k}|$.*

Proof. If λ/μ is not a root of unity, then the set of powers of λ/μ is a dense subset of $\partial\mathbb{U}$. This, the identity $|\lambda^n - \mu^n| = |\lambda^n/\mu^n - 1|$, and formula (14) imply that $\|C_{\lambda\varphi} - C_{\mu\varphi}\| = 2$, when λ/μ is not a root of 1. If λ/μ is a root of unity, the formulas for $\|C_{\lambda\varphi} - C_{\mu\varphi}\|$ in this corollary are direct consequences of formula (14) and the geometric representation of the roots of unity as vertices of a regular polygon inscribed in $\partial\mathbb{U}$, having a vertex at 1. ▮

Corollary 4. *Denote* $\varphi^{[n]} = \varphi \circ \overset{n}{\cdots} \circ \varphi$. *If* φ *has a fixed point in* $\mathbb{U}$ *and is not an inner function, then the second inequality in* (10) *is strict for all* $\varphi^{[n]}$ *starting some* n.

Proof. Choose any φ, non–inner, with a fixed point $p \in \mathbb{U}$. If $p = 0$, then one has $\|C_{\varphi^{[n]}} - C_{\varphi^{[n]}(0)}\| < \|C_{\varphi^{[n]}}\|$ for all n, by (9) and Theorem 4.

If $p \neq 0$, note that, by Schwarz's lemma in classical complex analysis, $\varphi^{[n]} \to p$ uniformly on compacts, (see also [15]). This combined with formula (13) proves that $\|C_{\varphi^{[n]}(0)} - C_p\| \to 0$. On the other hand, it is shown in [11] that $\|C_{\varphi^{[n]}} - C_p\| \to 0$. Thus, if, arguing by contradiction, one assumes $\|C_{\varphi^{[n]}} - C_{\varphi^{[n]}(0)}\| \geq \|C_{\varphi^{[n]}}\|$ for infinitely many values of n, one gets the contradiction $0 \geq \|C_p\|$. ▮

The next group of distance–formulas we prove are immediate consequences of Theorems 1, 2, 3, and of the fact that an inner function fixing the origin induces an isometric composition operator.

Proposition 4. *If* φ *is an inner function fixing* 0 *and* p *a constant in* $\mathbb{U}$ *then*

$$\|C_\varphi - C_p\| = \frac{1}{\sqrt{1 - |p|^2}} \tag{15}$$

and

$$\|C_{\alpha_p \circ \varphi} \pm C_\varphi\| = \frac{2}{\sqrt{1 - |p|^2}}. \tag{16}$$

If φ *is any inner function, then*

$$\|C_{\alpha_{\varphi(0)} \circ \varphi} \pm C_\varphi\| = \frac{2}{\sqrt{1 - |\varphi(0)|^2}}. \tag{17}$$

Proof. Let φ be an inner function fixing the origin and note that

$$\|C_p - I\| = \|C_\varphi(C_p - I))\| = \|C_\varphi - C_p\|.$$

By Theorem 1, $\|C_p - I\|$ equals the major axis of the ellipse in Theorem 3, hence (15) holds.

Given that C_φ is isometric, if φ is inner and $\varphi(0) = 0$,

$$\|C_{\alpha_p \circ \varphi} \pm C_\varphi\| = \|C_\varphi(C_{\alpha_p} \pm I)\| = \|C_{\alpha_p} \pm I\|.$$

By Theorems 1 and 2, $\|C_{\alpha_p} \pm I\| = 2/\sqrt{1 - |p|^2}$, hence (16) holds. For an arbitrary inner function φ, consider the inner function fixing the origin $\alpha_{\varphi(0)} \circ \varphi$ and let $p = \varphi(0)$. Applying (16), one gets (17). ∎

Consider an analytic selfmap φ of $\mathbb{U}$ with the property $\varphi(0) = 0$. It is easy to see that the space $\mathbb{C}$ of constant functions reduces C_φ. Indeed, on one hand, the evident relation $C_\varphi 1 = 1$ shows that $\mathbb{C}$ is left invariant by any composition operator, on the other, as was noted in the proof of Theorem 4, $C_\varphi^* 1 = 1$, so C_φ^* leaves $\mathbb{C}$ invariant. We study in the following the quantity $\|C_\varphi - C_0\| = \|C_\varphi | H_0^2\| = \|T_{\varphi,\varphi}\|$.

THEOREM 5. *If φ is not constant and $\varphi(0) = 0$, then the norm of the restriction $C_\varphi | H_0^2$ satisfies the estimates*

$$\|\varphi\|_2 \le \|C_\varphi | H_0^2\| \le \|\varphi\|_\infty \tag{18}$$

and the following are equivalent:

(i) $\frac{1}{\|\varphi\|_\infty}\varphi$ *is an inner function,*

(ii) $\|\varphi\|_\infty = \|C_\varphi | H_0^2\|$,

(iii) $\|\varphi\|_2 = \|\varphi\|_\infty = \|C_\varphi | H_0^2\|$.

Proof. The estimates in (18) are direct consequences of (9) and the fact that $\|C_\varphi\|$ is a contraction when $\varphi(0) = 0$.

Let us note that the situation $\|\varphi\|_2 = \|\varphi\|_\infty = \|C_\varphi | H_0^2\|$ occurs if and only if φ is a scalar multiple of an inner function, that is if and only if there exist λ in the closure of $\mathbb{U}$ and an inner function ϕ such that $\varphi = \lambda\phi$. The fact that, if φ has the form above, then $\|\varphi\|_2 = \|\varphi\|_\infty = \|C_\varphi | H_0^2\| = |\lambda|$ is immediate by (18). The converse is a consequence of the fact that obviously $|\varphi(e^{i\theta})| \le \|\varphi\|_\infty$ a.e. and if $\|\varphi\|_2 = \|\varphi\|_\infty$, then $\frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\|\varphi\|_\infty^2 - |\varphi(e^{i\theta})|^2\right) d\theta = 0$ so $|\varphi(e^{i\theta})| = \|\varphi\|_\infty$ a.e., that is $\phi := \varphi/\|\varphi\|_\infty$ is inner, so setting $\lambda := \|\varphi\|_\infty$, one has the desired representation $\varphi = \lambda\phi$. Visibly, φ is a scalar multiple of an inner function if and only if (i) holds. Thus (i) $\Leftrightarrow$ (iii). Clearly (iii) $\Rightarrow$ (ii). To finish the proof we show that (ii) fails if (i) fails.

The fact that, if φ is not a scalar multiple of an inner function then, the upper estimate in (18) fails to be an equality is an immediate consequence of Theorem 4.

Indeed, let us consider the connection between the norms $\|C_\phi|H_0^2\|$ and $\|C_\varphi|H_0^2\|$, where $\varphi = \lambda\phi$, $\lambda = \|\varphi\|_\infty$. If one denotes by D the diagonal operator on H_0^2 having diagonal $\{\lambda, \lambda^2, \dots, \lambda^n, \dots\}$ with respect to the standard basis $\{z, z^2, \dots, z^n, \dots\}$ of H_0^2 then one has $C_\varphi|H_0^2 = (C_\phi|H_0^2)D$ and since, $\|D\| = |\lambda|$, one gets $\|C_\varphi|H_0^2\| \le \|\varphi\|_\infty\|C_\phi|H_0^2\|$, which leads to $\|C_\varphi|H_0^2\| < \|\varphi\|_\infty$ if ϕ is not inner. ∎

Note that, if $\varphi(0) = 0$ and φ is a scalar multiple of an inner function, then

$$\|C_\varphi|H_0^2\| = \|\varphi\|_p, \qquad 2 \le p \le \infty.$$

Relative to that, we prove in the following that $\|C_0 - C_\varphi\|$ is always equal to some H^p–norm of φ, the choice of p, being unique if φ is not a scalar multiple of an inner function.

Theorem 6. *For each φ there is a $p(\varphi)$, $2 \le p(\varphi) \le \infty$ such that*

$$\|C_\varphi|H_0^2\| = \|\varphi\|_{p(\varphi)} \tag{19}$$

and $p(\varphi)$ is finite and uniquely determined unless $\frac{1}{\|\varphi\|_\infty}\varphi$ is an inner function.

Proof. The existence of a number $p(\varphi)$ satisfying (19) is a direct consequence of estimate (18), the continuity of the map

$$p \to \|\varphi\|_p, \qquad 2 \le p < \infty,$$

and the fact that, $\lim_{p\to\infty} \|\varphi\|_p = \|\varphi\|_\infty$, [14, pp 70]. As we noted above, any value $p(\varphi)$, $2 \le p(\varphi) \le \infty$ satisfies (19) if $\frac{1}{\|\varphi\|_\infty}\varphi$ is an inner function.

Assume now that $\frac{1}{\|\varphi\|_\infty}\varphi$ is not an inner function, then $\|C_\varphi|H_0^2\| < \|\varphi\|_\infty$, by Theorem 4, so a number $p(\varphi)$ satisfying (19) must be finite. To show that there is only one such number, assume, arguing by contradiction, that there exist $2 \le p < r < \infty$ such that

$$\|\varphi\|_p = \|\varphi\|_r = \|C_\varphi|H_0^2\|.$$

Applying Hölder's inequality to the functions $|\varphi|^p$ and 1, one can write

$$\int_{\partial\mathbb{U}} |\varphi(\zeta)|^p \, dm(\zeta) \le \left(\int_{\partial\mathbb{U}} |\varphi(\zeta)|^r \, dm(\zeta)\right)^{\frac{p}{r}}, \qquad \text{i.e.,} \qquad \|\varphi\|_p \le \|\varphi\|_r.$$

But Hölder's inequality above is an equality, under our assumptions, so there must be a constant $c > 0$ so that $|\varphi|^p = c$ a.e., ([14], comments following Theorem 3.5), that is $\frac{1}{\|\varphi\|_\infty}\varphi$ is an inner function, a contradiction. ∎

In the case when $\frac{1}{\|\varphi\|_\infty}\varphi$ is not inner we describe in the sequel when $p(\varphi) = 2$.

THEOREM 7. *Let φ be an analytic selfmap of $\mathbb{U}$ fixing the origin. Then the following are equivalent.*

$$\|C_\varphi|H_0^2\| = \|\varphi\|_2 \tag{20}$$

$$C_\varphi^* C_\varphi(z) = C_\varphi^*(\varphi) = \|\varphi\|_2^2 z \tag{21}$$

The coordinate function z is an eigenfunction of $C_\varphi^* C_\varphi$. (22)

$$< \varphi, \varphi^n >= 0, \qquad n \geq 2. \tag{23}$$

Proof. Assume $\|C_\varphi|H_0^2\| = \|\varphi\|_2$. Applying the Cauchy-Schwartz inequality one gets

$$\|C_\varphi|H_0^2\|^2 = \|\varphi\|_2^2 = | < C_\varphi^* C_\varphi(z), z > | \leq \|C_\varphi^* C_\varphi(z)\|\|z\| \leq \|\varphi\|_2^2.$$

The Cauchy-Schwartz inequality being an equality if and only if the vectors involved in it are colinear, we get that $C_\varphi^* C_\varphi(z) = \lambda z$ for some scalar λ, which is easy to determine, since

$$\lambda =< \lambda z, z >=< C_\varphi^* C_\varphi(z), z >= \|\varphi\|_2^2.$$

We established (20) ⇒ (21). Obviously (21) ⇒ (22).

If (22) holds, that is, if $C_\varphi^* C_\varphi(z) = \lambda z$ for some scalar λ, then

$$< \varphi, \varphi^n >=< C_\varphi^*(\varphi), z^n >=< \lambda z, z^n >= 0, \qquad n = 0, 2, 3, 4, \ldots$$

Hence (22) ⇒ (23).

To finish, we show (23) ⇒ (20). Assume that

$$\varphi \perp \varphi^n, \qquad n = 0, 2, 3, 4, \ldots$$

For any polynomial p in H_0^2, we have

$$\|C_\varphi p\|_2^2 \leq \|\varphi\|_2^2 \|p\|_2^2. \tag{24}$$

Inequality (24) is evident if the degree of p is 1. Arguing by induction, assume it is true for all polynomials of degree at most $n-1$. Consider an arbitrary polynomial $p(z)=\sum_{j=1}^{n} c_j z^j$ of degree n and note that

$$p\circ\varphi(z)=c_1\varphi(z)+\varphi(z)q\circ\varphi(z), \quad \text{where} \quad q(z)=\sum_{j=2}^{n} c_j z^{j-1}.$$

Since $\varphi\perp\varphi(q\circ\varphi)$ and the degree of q is $n-1$, one gets

$$\begin{aligned}\|C_\varphi p\|_2^2 &= |c_1|^2\|\varphi\|_2^2+\|\varphi(q\circ\varphi)\|_2^2\\ &\le |c_1|^2\|\varphi\|_2^2+\|\varphi\|_\infty\|q\circ\varphi\|_2^2 \le |c_1|^2\|\varphi\|_2^2+\|q\circ\varphi\|_2^2\\ &\le \big(|c_1|_2^2+\|q\|_2^2\big)\|\varphi\|_2^2=\|\varphi\|_2^2\|p\|_2^2.\end{aligned}$$

Because the polynomials in H_0^2 are dense in H_0^2, we get $\|C_\varphi|H_0^2\|\le\|\varphi\|_2$ hence $\|C_\varphi|H_0^2\|=\|\varphi\|_2$. ∎

Condition (23) recalls *Rudin's orthogonality condition*. We say φ satisfies Rudin's orthogonality condition if the family $\{\varphi^n : n=0,1,2,\dots\}$ is orthogonal in H^2. It is easy to see that any φ which is the scalar multiple of an inner function fixing 0 satisfies the aforementioned condition. Recent results of Sundberg [18], show that there exist examples of symbols satisfying Rudin's orthogonality condition, other than the multiples of inner functions fixing 0, (thus answering a question raised by Walter Rudin in 1988). Independently, Bishop [2] obtained similar results using the pull–back measure induced by φ. Obviously, the symbols fixing the origin and satisfying (23) form a superset of those that satisfy Rudin's orthogonality condition. Elementary examples show that the aforementioned superset is strictly larger: consider, for instance, $\varphi(z)=(z^2+z^3)/2$.

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