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Merlin's Magic Square Enhanced

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1 Introduction.

The hand-held battery-operated computer game Merlin offers a game called Magic Square. Merlin has nine buttons and lights located at the grid points of a 3×3 square. The pattern of lights of the grid is altered by touching any one of the buttons, and the topology of the buttons/lights serves as a guide to remembering how the pattern will change.

- If a *corner* button is touched, all lights in the 2×2 corner that button belongs to are inverted.
- If an *edge* button is touched, all lights in the (outer) row or column that button belongs to are inverted.
- If the *center* button is touched, its light together with those of the four edge buttons are inverted.

After Merlin displays a random starting pattern of lights, the player's objective is to the push buttons so as to achieve the pattern where all lights but the center one are lit.

The first published solution to Merlin's Magic Square that we are able to reference can be found in Marc Konvisser's 1981 book, *Elementary Linear Algebra* [3, pages 243–252]. A more widely known, and more often cited solution, virtually identical to Konvisser's, was published by Don Pelletier in 1987 in the *American Mathematical Monthly* [6]. Inquiries about how this sequence of events occurred produced the following excerpt from a letter sent to us by Konvisser [4].

> I devised the solution somewhere over the Atlantic Ocean ... My big problem was inverting the 9×9 matrix. I remember using checkers or coins to perform the row operations in my mother-in-law's house in Jerusalem. A number of years later an article appeared in the Monthly with the same solution, but I was not referenced. I had some rather rancorous communications with the editor of the Monthly about citing my work. Eventually there was a note mentioning my book.

Sure enough, on page 994 of the December 1987 issue of the Monthly, Item 3 under a list headed "Addenda, Errata, Etcetera, for 1987" acknowledges Marc's prior solution.

This paper first considers questions about games related to Merlin's Magic Square from the point of view of group actions. At this juncture, little beyond the formal model is new, but the exposition sets the stage for considering certain "enhanced" versions of these games. The analysis of enhanced games, with the aid of semigroup actions, is carried out in complete detail for an ostensibly simpler ($k = 3$) game before turning to a Merlin ($k = 4$) game. Concluding sections discuss various ways to generalize our games.

To review the solution to Merlin's Magic Square, we begin by introducing our formal model. As usual, we use 1 to represent an ON light and 0 to represent an OFF light. When light and button are one unit as in Merlin we shall also speak of the button itself as being ON or OFF.

2 A Formal Model.

Let $\mathcal{S}$ be a set of *labels* for the lights to be controlled by the buttons. Then the current pattern or *configuration* of the lights can be represented as a binary string of length $|\mathcal{S}|$, where the bits are indexed by the elements of $\mathcal{S}$. Equivalently, we may view a configuration as a vector in the vector space $V(\mathcal{S})$, of dimension $|\mathcal{S}|$ over the field with two elements, Z_2, where the coordinates are indexed by the elements of $\mathcal{S}$. For $x \in \mathcal{S}$, we let e_x be the usual standard basis vector in $V(\mathcal{S})$. For $v \in V(\mathcal{S})$, we implement the bitwise flip, clear, and set operations, denoted f_x, p_x, and b_x respectively, which affect only bit x as follows:

$$
\begin{aligned}
f_x(v) &= e_x + v, \\
p_x(v) = p_x(\sum_{y \in \mathcal{S}} \epsilon_y e_y) &= \sum_{y \neq x} \epsilon_y v_y, \\
b_x(v) &= (f_x \circ p_x)(v).
\end{aligned}
$$

Henceforth we shall write $b_x = f_x p_x$, and in general when we write operators as products, we mean that they are products under *composition*. We list some properties of these operators, whose proofs are omitted as they are routine.

PROPOSITION 2.1. Let $x, y \in \mathcal{S}$ be distinct, and let I be the the identity operator on $V(\mathcal{S})$. Then

1. $f_x b_x = p_x$.
2. p_x is *linear* on $V(\mathcal{S})$.

3. $o_x o_y = o_y o_x$, for o freely chosen from the set $\{p, f, b\}$.

4. $p_x o_x = p_x$ and $b_x o_x = b_x$, for o freely chosen from the set $\{p, f, b\}$.

5. $f_x^2 = I$.

6. $(f_x p_x)(v) = p_x(v) + e_x$, $(f_x b_x)(v) = b_x(v) + e_x$.

Returning briefly to Merlin's Magic Square, let the lights and buttons be labeled using the set $\{1, \ldots, 9\}$ as in Figure 1, and let the result of touching button i be denoted by F_i. Observe that for a corner button, say F_1,

$$F_1 = f_1 f_2 f_4 f_5,$$

while for an edge button, say F_6,

$$F_6 = f_3 f_6 f_9,$$

and for the center button F_5,

$$F_5 = f_2 f_4 f_5 f_6 f_8.$$

Evidently, a sequence of button presses is a *word* W in the group generated by $\{F_i\}$ subject to the relations induced by the $\{f_i\}$. Moreover, $W(v)$ represents the configuration obtained by evaluating this word on the initial configuration v. Thus we can restate the objective of Merlin's Magic Square as follows: *Given u, a solution to Merlin's Magic Square is a word W such that $W(u) = \sum_{i \neq 5} e_i$.*

Continuing with our formal development, we construct from any subset $\mathcal{R} \subseteq \mathcal{S}$ the operator $F_{\mathcal{R}}$ of F-type as

$$F_{\mathcal{R}} = \prod_{r \in \mathcal{R}} f_r.$$

Operators $P_{\mathcal{R}}$ and $B_{\mathcal{R}}$ are constructed similarly. Our first two results are straightforward and their proofs are omitted.

LEMMA 2.2. For any $\mathcal{R}, \mathcal{T} \subseteq \mathcal{S}$ and $v, w \in V(\mathcal{S})$,

1. $F_{\mathcal{R}}(v) = v + \sum_{r \in \mathcal{R}} e_r$.

2. $F_{\mathcal{R}}^2 = I$.

3. $F_{\mathcal{R}} F_{\mathcal{T}} = F_{\mathcal{T}} F_{\mathcal{R}}$.

4. $F_{\mathcal{R}}(v + w) = F_{\mathcal{R}}(v) + w$.

PROPOSITION 2.3. For a family of subsets $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_m$ of $\mathcal{S}$, the group of operators generated by the set $\{F_{\mathcal{R}_i}\}$ is an elementary abelian 2-group of rank $r < m$.

DEFINITION 2.4. A *Merlin style game* is one where all buttons correspond to operators of F-type.

If a game has $n = |\mathcal{S}|$ lights, then there are 2^n possible configurations. Usually we would expect m, the number of buttons, not to exceed n, the number of lights. There is no harm or loss of generality in assuming that the number of buttons is different than the number of lights, nor is there any harm in assuming that some lights offer a *choice* of buttons of F-type. There are several questions one might try to answer when analyzing Merlin style games.

- *The Transitivity Problem.* Given u, v is there a word W such that $W(u) = v$?
- *The All-Ones Problem.* Does there exist a word W such that $W(0) = 1$?
- *The Orbit Problem.* Given u, describe $W(u)$ as W ranges over all words.
- *The Minimality Problem.* If $W(u) = v$, what is the minimal length word that solves this equation?
- *The Garden-of-Eden Problem.* Does there exist a $u \in V(\mathcal{S})$, such that $W(u) \neq u$ for all $W \neq I$? (*i.e.*, Does there exist a starting configuration that one cannot return to?)[1]
- *The Mentat Problem.* Are there shortcuts and mnemonics to enable a human to algorithmically solve $W(u) = v$ without recourse to pencil, paper, calculator, etc.[2]

Both Konvisser and Pelletier "solve" the Magic Square game by finding a canonical minimal solution to the Transitivity Problem. Pelletier also

[1]The Garden-of-Eden Problem anticipates games where F-type operators are not allowed.

[2]The word "mentat" is adapted from the Ghola Mentats of Arthur C. Clarke's science fiction series *Dune*.

makes suggestions concerning the Mentat Problem. Improvements and further discussion of the Mentat Problem are found in a paper by Stock [9].

The solution to the Transitivity problem for Merlin's Magic Square and Merlin style games is easily derived from the observation that since W is an element in an elementary abelian 2-group, W must be of the form

$$W = \prod_{j=1}^{m} F_{\mathcal{R}_j}^{\epsilon_j},$$

where $\epsilon_j \in Z_2$. Let M be the $n \times m$ matrix whose j-th *column* is the vector $F_{\mathcal{R}_j}(0)$. We have

$$F_{\mathcal{R}_j}^{\epsilon_j}(v) = v + F_{\mathcal{R}_j}^{\epsilon_j}(0) = v + M(\epsilon_j e_j),$$

so we can solve

$$W(u) = v,$$

if and only if

$$\prod F_{\mathcal{R}_j}^{\epsilon_j}(u) = v$$

$$u + \sum M(\epsilon_j e_j) = v$$

$$M(\sum \epsilon_j e_j) = v - u,$$

which leads to:

PROPOSITION 2.5. In a Merlin style game, if the number of buttons m equals the number of lights n, and M is invertible there is a unique (minimal) solution to the Transitivity Problem.

COROLLARY 2.6. There is a unique (up to ordering) solution to the Transitivity Problem for Merlin's Magic Square. Namely, starting from configuration u, to achieve configuration v, press button j if and only if $\epsilon_j = 1$ in the solution to $M(\sum \epsilon_j e_j) = v - u$.

Proof. See [3] or [6] for the details about the invertibility of M.

A game remarkably similar to Merlin's Magic Square is Quatrainment, analyzed by Gantner [2]. This game uses a 4×4 grid with the cells row ordered as in Merlin and a light-button pair associated to each cell. As in Merlin we may identify buttons and lights. As Figure 2 helps illustrate, in Quatrainment,

- Corner buttons invert the associated 3×3 *triangle.*

- Interior buttons invert their lights and those of their *four* horizontal and vertical neighbors (so they behave like Merlin's center button).
- Edge buttons invert the lights of their *three* horizontal and vertical neighbors but NOT themselves.

The game to be played is the transitivity game: *Given u, v determine a sequence of moves, if possible, that will transform u to v.* Gantner proves Quatrainment also has a unique solution, but if we resume the usual practice of allowing an edge button to invert itself, then the resulting 16×16 matrix M has rank 12, and we conclude solutions for this *Modified* Quatrainment that are obtained using our matrix equation are unique up to disabling four buttons — for example, in the Merlin notation buttons F_1, F_2, F_{15}, F_{16} [2, page 34]. From elementary linear algebra considerations it is clear that the orbits in Modified Quatrainment form equivalence classes of size 2^{12}, and because the dimension of the null space of M is 4, that every solvable instance of the Transitivity Problem will yield $2^4 = 16$ distinct solutions of length at most 16. The Minimality Problem reduces to the well known problem of trying to determine the solution x to the matrix equation $Ax = b$ with the fewest nonzero coefficients.

Once more the historical sequence of events is confusing. Appearing in 1988, Gantner's Quatrainment makes no reference to Merlin; rather it traces its origins to Think-a-Dot [1], [7] a game where an inversion operator is *dependent* in the sense that an operator's effect on certain lights in the configuration cannot be calculated without prior knowledge of the status of some of the other lights in the configuration. Technically, this amounts to saying some of its operators do not satisfy the following independence property.

DEFINITION 2.7. An operator W of $V(\mathcal{S})$ is *independent* if and only if

$$W(v) = W(\sum \epsilon_x e_x) = \sum W_x(\epsilon_x e_x),$$

where each induced operator W_x is not required to be linear.

Think-a-Dot does inspire both the Minimality and Mentat Problems, and it is noteworthy for marking the introduction of automata theory into the analysis of these inversion games. The fusion of Merlin style games and linear bounded automata occurs in Sutner's papers [10], [11] only now Pelletier is referenced but Gantner and his predecessors are not! Sutner is motivated by an All-Ones Problem where lights and buttons are identified, and all buttons affect themselves and their horizontal and vertical neighbors (Merlin's center button again). Sutner considers many All-Ones and Garden-of-Eden variations and is responsible for the most far-reaching of the results about the All-Ones Problem. But along these lines a dynamic problem formulated by Peled, together with its elegant solution by Lossers, is not to be missed [5]. We restate the Peled-Lossers version of the All-Ones Problem in terms of our model.

PROPOSITION 2.8 If $n = m$ and the matrix M of a Merlin style game is (a) symmetric and (b) has all ones on the diagonal, then the All-Ones Problem has a solution.

Note that from a design point of view, since $M = (m_{i,j})$ where

$$m_{i,j} = \begin{cases} 1 & \text{if button } j \text{ inverts light } i \\ 0 & \text{otherwise} \end{cases}$$

the conditions stated are merely (a) that if button i inverts light j then button j inverts light i, and (b) button i inverts light i.

3 An Enhancement.

The enhanced game we wish to consider was motivated by a problem that appeared in the November 1990 Monthly [8] and is here produced verbatim.

> E 3406. *Proposed by Jeffrey Shallit, Dartmouth College, Hanover, NH*
>
> Consider three circles in the plane that intersect to form seven regions. In each region there is a token that is white on one side and black on the other. At any stage the following two operations are permissible: (a) we can invert (flip over) all four tokens inside one of the three circles, or (b) we can invert those tokens showing black inside one of the three circles so that afterwards all tokens in the circle show white. From the starting configuration in which all tokens show white, can we reach the configuration in which all tokens show white except that the central region common to these three discs shows black?

To begin our analysis of Shallit's game, let the *fixed* orientation of the three circles be labeled A, B, C as in Figure 3. The seven regions referred to will be our seven lights. White is interpreted as OFF and black as ON. The set of labels is $\{A, B, C, AB, AC, BC, ABC\}$. It is necessary to identify labels and regions.

- X is the label for the isolated or exterior region $X - (Y \cup Z)$.
- XY is the label for the overlap region $(X \cap Y) - (X \cap Y \cap Z)$
- ABC is the label for the central region $A \cap B \cap C$.

We recognize, of course, that ordering within our labels does not matter *i.e.*, YX is also a valid label for XY. The distinction between circle or set X and the labeled exterior region X should be clear from the context. To each light with a label of the form X we associate *two* buttons:

$$F_X = f_X f_{XY} f_{XZ} f_{XYZ},$$

and

$$P_X = p_X p_{XY} p_{XZ} p_{XYZ}.$$

We form words from the *semigroup* generated by the set

$$\{F_X, P_X : X = A, B, C\},$$

and restate Shallit's problem as: *Is there a word W such that $W(0) = e_{A,B,C}$?*

Warning. Because our semigroup is not abelian, and because W is a composition operator, to implement $W(v)$ as a sequence of button presses the word W must be read from *right to left.* If $W = O_1 O_2 \ldots O_t$ the sequence of button presses is $O_t, \ldots, O_2, O_1$.

A solution to Shallit's problem is not difficult. We let the canonical ordering for a *light* configuration be

$$\epsilon_A e_A + \epsilon_B e_B + \epsilon_C e_C + \epsilon_{AB} e_{AB} + \epsilon_{AC} e_{AC} + \epsilon_{BC} e_{BC} + \epsilon_{ABC} e_{ABC}.$$

LEMMA 3.1. The subgroup generated by $\{F_X\}$ is an elementary abelian 2-group of rank 3 *i.e, isomorphic to* $C_2 \times C_2 \times C_2$.

Proof. The 7×3 matrix M for the operators $\{F_X\}$ under the canonical ordering has the property that its transpose $M^t = [\, I_3 \mid * \,]$, so M has rank three and the result follows.

For future reference we record the configurations that can be obtained from this subgroup acting on the zero vector. We call this the F-class of 0.

$$\begin{aligned} F_X(0) &= e_X + e_{XY} + e_{XZ} + e_{XYZ} \\ F_X F_Y(0) &= e_X + e_Y + e_{XZ} + e_{YZ} \\ F_X F_Y F_Z(0) &= e_X + e_Y + e_Z + e_{XYZ} \end{aligned}$$

LEMMA 3.2. $(W_1 P_X W_2 O_X W_3)(v) = (W_1 P_X W_2 W_3)(v)$ where O is either F, P, or B. In particular, a semigroup word never requires F_X to follow P_X.

Proof. Because our operators are coordinatewise independent it suffices to consider $v = \epsilon_i e_i$. If i is not a label associated with set X, then P_X and O_X have no influence, and if i is such a label, then both expressions evaluate to $W_1(v)$.

COROLLARY 3.3. If $W(u) = v$, then $(W_F W_P)(u) = v$ where $W_F = F_A^{\epsilon_A} F_B^{\epsilon_B} F_C^{\epsilon_C}$, and either $W_P = I$, or for some set X, W_P begins with P_X and does not contain F_X.

Proof. Immediate from the previous lemmas.

CLAIM 3.4 It is not possible to solve $W(0) = e_{ABC}$.

Proof. Suppose $W(0) = e_{ABC}$. We write $W = W_F W_P$ as described above. With reference to the F-class of 0, it is clear $W_P \neq I$, so assume W_P begins in say P_X. If F_X precedes P_X then e_X is a component of $W(0)$. If $W = F_Y P_X \ldots$ then e_{XY} is a component of $W(0)$. And if $W = F_Y F_Z P_X \ldots$ then e_{ABC} is not a component of $W(0)$. This establishes that no such W exists.

CLAIM 3.5. It is not possible to solve $W(0) = e_{AB} + e_{AC} + e_{BC}$.

Proof. Arguing as in the previous claim, we quickly see that some P_X operator must be used, and since e_{ABC} is not a component of $W(0)$, $W = F_Y F_Z P_X \ldots$ But then e_{XZ} is not a component. This contradiction completes the proof.

With an eye towards the Transitivity Problem, we calculate the F-classes for these two unobtainable configurations, recording 16 configurations that are not in the orbit of 0. For the purpose of symmetry it is often convenient to use XYZ as an unspecified ordering for ABC. We compute the F-class of e_{ABC}.

1. $I(e_{ABC}) = e_{ABC}$.

2. $F_X(e_{XYZ}) = e_X + e_{XY} + e_{XZ}$.

3. $F_X F_Y(e_{XYZ}) = e_X + e_Y + e_{XZ} + e_{YZ} + e_{XYZ}$.

4. $F_X F_Y F_Z(e_{ABC}) = e_A + e_B + e_C$.

We compute the F-class of $e_{AB} + e_{AC} + e_{BC}$.

1. $I(e_{AB} + e_{AC} + e_{BC}) = e_{AB} + e_{AC} + e_{BC}$.

2. $F_X(e_{AB} + e_{AC} + e_{BC}) = e_X + e_{YZ} + e_{XYZ}$.

3. $F_X F_Y(e_{AB} + e_{AC} + e_{BC}) = e_X + e_Y + e_{XY}$.

4. $F_A F_B F_C(e_{AB} + e_{AC} + e_{BC}) = e_A + e_B + e_C + e_{AB} + e_{AC} + e_{BC} + e_{ABC}$.

Note from this last equation that the All-Ones Problem is not solvable for Shallit's Game. For visual cues to the patterns found above whose labels involve indeterminates, see Figures 4 and 5.

We will now derive our solutions to the Orbit and Minimality Problems simultaneously. A lemma sets-up the key relations we will need.

LEMMA 3.6. For $X \neq Y$, $(P_X F_Y)(v) = (F_Y P_X)(v) + e_{XY} + e_{XYZ}$.

Proof. By independence we may assume v has no e_Z component. For $v = \sum_{i \neq Z} \epsilon_i e_i$,

$$(P_X F_Y)(v) = (1 + \epsilon_Y)e_Y + (1 + \epsilon_{YZ})e_{YZ},$$

and

$$(F_Y P_X)(v) = (1 + \epsilon_Y)e_Y + (1 + \epsilon_{YZ})e_{YZ} + e_{XY} + e_{XYZ}.$$

The result is now clear.

PROPOSITION 3.7. For $X \neq Y$,

1. $(F_X F_Y P_Y F_X)(0) = (P_X F_Y)(0)$.

2. $(F_X P_X F_Y)(0) = (F_Y P_Y F_X)(0)$.

3. $(F_Y P_X F_Y)(0) = (F_X P_Y F_X)(0)$.

Proof. Using the previous lemma twice, we find

$$\begin{aligned}(F_X F_Y P_Y F_X)(0) &= (F_X F_Y)(P_Y F_X(0)) \\ &= (F_X F_Y)(F_X P_Y(0) + e_{XY} + e_{XYZ}) \\ &= (F_X F_Y(F_X P_Y(0))) + e_{XY} + e_{XYZ} \\ &= (F_Y P_Y)(0) + e_{XY} + e_{XYZ} \\ &= (F_Y P_X)(0) + e_{XY} + e_{XYZ} \\ &= (P_X F_Y)(0),\end{aligned}$$

which proves the first assertion. Then

$$(F_X P_X F_Y)(0) = (F_X F_X F_Y P_Y F_X)(0) = (F_Y P_Y F_X)(0),$$

and

$$(F_Y P_X F_Y)(0) = (F_Y F_X F_Y P_Y F_X)(0) = (F_X P_Y F_X)(0),$$

as desired.

THEOREM 3.8. There are 14 distinct configurations of the form $W_P(0)$, and therefore under F-equivalence $8 \cdot 14 = 112$ configurations in the orbit of 0. The *minimal* non-identity words producing distinct non-zero configurations are:

1. $P_A F_B$, $P_A F_C$, $P_B F_C$.
2. $P_A F_B F_C$, $P_B F_A F_C$, $P_C F_A F_B$.
3. $P_A P_B F_C$, $P_A P_C F_B$, $P_A P_C F_B$.
4. $P_A F_B P_C F_B$, $P_B F_C P_A F_C$, $P_C F_A P_B F_A$.
5. $P_A F_B P_B F_C$.

Proof. Because $P_X(0) = 0$, we do not want W_P to end in P_X, and since

$$(P_X P_Y P_Z)(v) = 0,$$

we know W_P will not contain this subsequence. We also know neither F_X nor P_X can follow P_X in W_P so minimality can be established by showing F-inequivalence for the words listed *i.e.*, showing $W_F W_P(0) = W'_P(0)$ implies

$W_P = W'_P$. Observe that since

$$\begin{aligned}(P_X F_Y)(0) &= e_Y + e_{YZ} \\ (P_X F_Y F_Z)(0) &= e_Y + e_Z \\ (P_X P_Y F_Z)(0) &= e_Z \\ (P_X F_Y P_Z F_Y)(0) &= e_{YZ} \\ (P_X F_Y P_Y F_Z)(0) &= e_Y + e_Z + e_{YZ},\end{aligned}$$

and

$$(W_F W_P)(0) = W_F(W_P(0)) = W_F(0) + W_P(0),$$

a glance at the listing of the F-class of 0 convinces us that the words listed under different numerals in the statement of the theorem cannot be F-equivalent. Within numerals, the arguments are more tedious. For (1), the previous proposition shows the F-equivalence of $P_X F_Y$ and $P_Y F_X$ and the form $(P_X F_Y)(0)$ takes shows F-equivalence cannot arise in any other way. An F-translate of a word in (2) is not of that form, so by commutativity of the F's we have distinct inequivalences. For (3) or (4), F-translates are again forbidden and the fact that there are only three distinct configurations obtainable verifies the listings. Finally, for (5), using the identity

$$(F_X F_Y)((P_X F_Y P_Y F_Z)(0)) = e_X + e_Z + e_{X,Z} = (P_Y F_X P_X F_Z)(0),$$

and the identity

$$(F_X P_X F_Y)(0) = (F_Y P_Y F_X)(0),$$

one shows *all* words of the form $(P_X F_Y P_Y F_Z)(0)$ are F-equivalent to the one listed.

To solve the Orbit Problem, let $\mathcal{R}(u)$ be the orbit of u, and let $\mathcal{F}(u)$ be the F-class of u.

COROLLARY 3.9. The three orbits for Shallit's Game are $\mathcal{R}(0)$, $\mathcal{R}(0) \cup \mathcal{F}(e_{ABC})$, and $\mathcal{R}(0) \cup \mathcal{F}(e_{AB} + e_{AC} + e_{BC})$.

Proof. By the theorem, $V(\mathcal{S})$ is the disjoint union of $\mathcal{R}(0)$ and the F-classes of the two vectors indicated. Since the zero *operator* is a word, every orbit contains $\mathcal{R}(0)$. Therefore the orbits listed are those for 0, e_{ABC}, and $e_{AB} + e_{AC} + e_{BC}$ respectively. The orbit of an arbitrary u is determined by its membership in the disjoint union.

We conclude this section with an algorithm for the Mentat Problem which is applicable to the solution of $W(0) = v$. It is somewhat hampered by the need to first apply operators of F-type to v in order to determine whether or not v is actually in the orbit of 0. The idea behind the algorithm is that a configuration $(W_F W_P)(0)$ is determined up to F-equivalence once one decides which operator of P-type, say P_Z, W_P begins with. Let

$$v_{Z^c} = v - P_Z(v) = \epsilon_X e_X + \epsilon_{XY} e_{XY} + \epsilon_Y e_Y$$

record the status of the lights in the sought for configuration v that do *not* belong to Z and set $v_Z = v - v_{Z^c}$. The algorithm is devised by by evaluating v_{Z^c} at each of $W_F = F_X$, $F_X F_Y$, $F_X F_Z$ and $F_X F_Y F_Z$; separating the result into those components belonging to Z, say w_Z, and those not belonging Z, say w_{Z^c}; then setting up the chart shown in Figure 6 by matching w_Z to v_Z. (Note that by careful labeling $W_F = F_Y F_Z$ is not required, and the trivial cases $W_F = I, F_Z$ require no special consideration.) At this stage we have

$$W_F(v_{Z^c}) = w_Z + w_{Z^c} = v_Z + w_{Z^c},$$

or

$$v_{Z^c} = W_F(v_Z + w_{Z^c}) = v_Z + W_F(w_{Z^c}),$$

which allows us to write

$$v = v_{Z^c} + v_Z = W_F(w_{Z^c}).$$

Knowledge of the forms of the minimal W_P expressions that were obtained in the proof of the Theorem allows us in *all* cases to select W_P satisfying $W_P(0) = w_{Z^c}$, hence $(W_F W_P)(0) = v$.

For example, consider the argument for $W_P = F_X F_Z$, using

$$(F_X F_Z)(v_{Z^c}) = (1 + \epsilon_X)e_X + (1 + \epsilon_{XY})e_{XY} + \epsilon_Y e_Y + e_{YZ} + e_Z.$$

The presence of e_Z and e_{YZ} tells us: If we orient the device so the bottom circle (to be labeled Z) of the v configuration has its exterior light together with one of the lights in an overlapping region (to be labeled Y) ON, and those of the other overlapping region (to be labeled X) and the center OFF, then we will need a W_P that gives the configuration that matches the final state of v in Y but has lights in the X and XY regions inverted from their final state. Now, W_P applied to 0 leaves the lights in set Z OFF, but applying $F_X F_Z$ turns ON lights labeled Z and YZ while simultaneously inverting lights labeled X and XY — precisely, the ones that W_P had incorrectly set.

Algorithm 3.10. A Solution to the Mentat Problem for $W(0) = v$ in Shallit's Game.

Step 1 By using F-type operators extinguish the exterior lights in v but *remember* which X's were used.

Step 2a If this configuration is not obtainable, restore v, announce IMPOSSIBLE, and HALT. Otherwise,

Step 2b Restore v and rotate the the device until the bottom circle of v matches one that appears in the heavily outlined circles of the chart in Figure 6.

Step 3 Identify X and Y consistent with the chart's labeling, and enter the sequence necessary to set the lights, or their inversions when indicated by the presence of the symbol c, for regions X, XY, and Y.

Step 4 Apply the specified F-sequence found in the chart.

4 Merlin Enhanced.

To enhance Merlin, label the four 2×2 *corner* squares of the grid as A, B, C, and D as in Figure 7, and consider our "inversion" and "clear" operators F_X and P_X for each of these 2×2 squares. More precisely, label the lights in the corner squares with $i \in Z_4$, the light in the center with c, and the edge lights with the unordered pairs $(i, i+1)$ (see Figure 8). To each corner light we associate the two buttons

$$F_i = f_{i-1,i} f_i f_{i,i+1} f_c,$$

and

$$P_i = p_{i-1,i} p_i p_{i,i+1} p_c.$$

We quickly observe this Shallit $k = 4$ game with 9 lights and 8 buttons is quite different from Shallit's $k = 3$ game with 7 lights and 6 buttons as the computation

$$(F_i P_{i+2} F_i)(0) = e_c,$$

lights the center light, and shows this configuration is now obtainable from zero. In fact, perhaps surprisingly, the Transitivity Problem can be solved for this game. Since zero is in every orbit, we shall prove this by showing the orbit of zero is all of $V(\mathcal{S})$. The idea behind the proof is to mimic the approach we used to devise the Mentat Algorithm in the last section.

(12)

LEMMA 4.1. If $P_4(v) = v$ then the equation

$$(P_4 F_3^{\alpha_{-3}} F_2^{\alpha_{+2}} P_3 F_2^{\alpha_{-2}} F_1^{\alpha_{+1}} P_2 F_1^{\alpha_{-1}})(0) = v$$

has a unique mod 2 solution for $\{\alpha_i\}$.

Proof. Let W be the word suggested. $P_4(W(0)) = W(0)$ and it is clear that for $i \neq 4$, the e_i component of $W(0)$ has coefficient $\alpha_{-i} + \alpha_{+i}$, where it is understood $\alpha_{+3} = 0$. For $i \neq 4$, it is true that the $e_{i,i+1}$ component of $W(0)$ has coefficient $\alpha_{+i} + \alpha_{-(i+1)}$. Therefore, using ϵ's for the coefficients of v, back substitution allows one to solve the linear system

$$\begin{aligned} \alpha_{-3} &= \epsilon_3 \\ \alpha_{+2} + \alpha_{-3} &= \epsilon_{2,3} \\ \alpha_{-2} + \alpha_{+2} &= \epsilon_2 \\ \alpha_{+1} + \alpha_{-2} &= \epsilon_{1,2} \\ \alpha_{-1} + \alpha_{+1} &= \epsilon_1 . \end{aligned}$$

THEOREM 4.2. $\mathcal{R}(0)$, the orbit of zero, is all of $V(\mathcal{S})$ for the Enhanced Merlin Game.

Proof. To solve $W(0) = v$, locate the F-sequence, W_F, in Figure 9 for which $W_F(0)$ matches v_4, the configuration that matches the lower left 2×2 corner of the configuration v but has zeroes at other labels. Figure 9 is condensed so that if an inversion is required, W_F is replaced by $F_4 W_F$. Use the lemma to construct W satisfying

$$\begin{aligned} W(0) &= P_4(v - W_F(0)) \\ &= P_4(v) - P_4(W_F(0)) \\ &= (v - v_4) - (W_F(0) - v_4) \\ &= v - W_F(0). \end{aligned}$$

Then

$$(W_F W)(0) = W(0) + W_F(0) = v.$$

As the reader may have anticipated there is an easy generalization to an enhanced $k \geq 4$ game with $n = 2k + 1$ lights and $m = 2k$ buttons. At each vertex i of a regular k-gon, the two buttons F_i, P_i are defined exactly as they were at the beginning of this section, so they affect the light at vertex i, the center light c, and two lights at the midpoints of the chords joining vertex i to vertices $i - 1$ and $i + 1$ with labels formed in the obvious way. The proof of a suitably modified Lemma 4.1 where 4 is replaced by k is straightforward and this task is left as an exercise for the reader. In the theorem no changes are required if we replace 4 by k. Topologically, it may make more sense to imagine Shallit's $k \geq 4$ game as being played on an annulus with alternating regions as suggested in Figure 10.

5 Changing the Modulus.

It is also natural to generalize by changing the modulus, which is equivalent to having buttons operate on d-way bulbs. (It may be useful to substitute an ordered sequence of colored bulbs located at label x for the basic operator f_x to step through in favor of a d-way bulb.) For the puposes of generalization we did not appeal to the mod 2 nature of the matrix equation derived for handling operators of F-type, $M(\sum \epsilon_x e_x) = v - u$. Before we proceed, we should mention the obvious: Now operators of F-type have order d and the abelian subgroup of our semigroup is a rank $r < t$ elementary abelian d-group, where t is the number of operators of F-type in the generating set.

As we have already pointed out, thanks to our matrix equation, d-way bulbs do not effect our results for Merlin style games. Turning to Shallit's game when $k \geq 4$, the linear system in the lemma of the previous section was to be solved mod 2, but can equally well be solved mod d. Thus we need only discern whether or not the four lights at vertex i can be controlled using only the F_i buttons. This, in turn, means we will need to be able to solve the linear system induced from the equation

$$(F_k^{\alpha_k} F_1^{\alpha_1} F_2^{\alpha_2} F_{k-1}^{\alpha_{k-1}})(0) = \epsilon_k e_k + \epsilon_{1,k} e_{1,k} + \epsilon_{k-1,k} e_{k-1,k} + \epsilon_c e_c + w,$$

where $P_k(w) = w$. The system we want to solve mod d is

$$\begin{aligned} \alpha_k &= e_k \\ \alpha_1 + \alpha_k &= \epsilon_{1,k} \\ \alpha_{k-1} + \alpha_k &= \epsilon_{k-1,k} \\ \alpha_c + \alpha_{k-1} + \alpha_k + \alpha_1 &= \epsilon_c, \end{aligned}$$

which can be solved by inspection. This extends our theorem about the orbit of zero to the case where $k \geq 4$, and we state this formally.

THEOREM 5.1. $\mathcal{R}(0) = V(\mathcal{S})$ for the Shallit game with $k \geq 4$, and $d \geq 2$. That is, every configuration is in the orbit of zero.

Returning to the original $k = 3$ Shallit game we must exercise greater care in the analysis.

PROPOSITION 5.2. A configuration v for Shallit's $k = 3$, $d \geq 2$ game is in the orbit of zero if and only if for some region Z the coefficients of the components of v belonging to Z,

$$v_Z = v - P_Z(v) = \epsilon_Z e_Z + \epsilon_{XZ} e_{XZ} + \epsilon_{YZ} e_{YZ} + \epsilon_{XYZ} e_{XYZ},$$

satisfy the mod d relation

$$\epsilon_Z + \epsilon_{XYZ} = \epsilon_{XZ} + \epsilon_{YZ}.$$

Proof. If $W(0) = (W_F W_P)(0) = v$, where as usual W_P begins with P_Z, then the inverse of W_F must zero out the components under consideration. Ignoring ϵ_{XYZ} for the moment, we are forced to choose

$$W_F^{-1} = F_X^{-\epsilon_{XZ}+\epsilon_Z} F_Y^{-\epsilon_{YZ}+\epsilon_Z} F_Z^{-\epsilon_Z}$$

whence the coefficient of the e_{XYZ} component of $W_F^{-1}(v)$ will be ϵ_{XYZ} plus the sum of the exponents of W_F^{-1} taken modulo d, giving rise to the desired equation. To complete the proof we must again show that there is a W_P for which

$$W_P(0) = \epsilon_X e_X + \epsilon_Y e_Y + \epsilon_{XY} e_{XY}.$$

Is is routine to verify that

$$W_P = P_Z F_X^{\epsilon_X} F_Y^{\epsilon_{XY}-\epsilon_X} P_X F_Y^{\epsilon_Y+\epsilon_X-\epsilon_{XY}}$$

will do the job.

Remark. This proposition allows us to streamline our alogorithm for solving the Mentat Problem for Shallit's $k = 3$ game, because it points out a fact that was not clearly evident when we were devising that algorithm. We can combine Steps 1 and 2a of Algorithm 3.10 into a single step which does not require temporarily modifying v at all. In the new Step 1, we announce IMPOSSIBLE precisely when the parity of the exterior and central regions does not match the parity of the two overlapping regions in *each* of the circles A, B, and C.

We shall not try to find the *minimal* W_P words giving rise to the F-inequivalent configurations for d-way bulbs, but we shall count the number of F-inequivalences. To facilitate the counting, we let

$$\mathcal{P} = \{v : P_Z(v) = v \text{ for some } Z\}.$$

Our task is to decide under what conditions $W_F(W_P(0)) \in \mathcal{P}$. Assume W_P begins in P_Z. Since the e_{XYZ} component of any $v \in \mathcal{P}$ must be zero we know by independence $W_F = F_X^{\alpha_X} F_Y^{\alpha_Y} F_Z^{\alpha_Z}$, where $\alpha_X + \alpha_Y + \alpha_Z = 0$ in Z_d. At most one of α_X, α_Y, α_Z can be zero otherwise we will have the trivial solution $W_F = I$. Moreover, if $\alpha_Z = 0$, then $\alpha_X \neq 0$ and $\alpha_Y \neq 0$ give nonzero e_{XZ} and e_{YZ} components, and it will not be the case that $W_F(W_P(0)) \in \mathcal{P}$. On the other hand if $\alpha_X, \alpha_Y, \alpha_Z \neq 0$ then $\alpha_X + \alpha_Z$ and $\alpha_Y + \alpha_Z$ are nonzero, and they will be the coefficients of e_{XY} and e_{YZ} so again $W_F(W_P(0))$ cannot be in $\mathcal{P}$. This means, by relabeling if necessary, $W_F = F_X^{-\alpha_Z} F_Z^{\alpha_Z}$. We write $W_P(0)$ canonically as

$$\epsilon_X e_X + \epsilon_Y e_Y + \epsilon_{XY} e_{XY}.$$

$W_F(W_P(0))$ adds to this

$$(-\alpha_Z)e_X + \alpha_Z e_Z + (-\alpha_Z)e_{XY} + \alpha_Z e_{YZ}.$$

The result will be in $\mathcal{P}$ if and only if

$$\alpha_Z = \epsilon_X = \epsilon_{XY}.$$

This calculation helps us make inequivalent assignments to the triples

$$(\epsilon_X, \epsilon_{XY}, \epsilon_Y),$$

where without loss of generality region Y follows X under clockwise ordering.

PROPOSITION 5.3. For Shallit's $k = 3$, $d \geq 2$ game, the F-inequivalent classes in $\mathcal{P}$ are enumerated as follows. There are

1. d F-inequivalences of the form (i, i, i).
2. $3(d^2 - d)$ F-inequivalences of the form (i, j, i) with $i \neq j$.
3. $3(d^2 - d)$ F-inequivalences of the form (i, j, j) with $i \neq j$.
4. $3d(d-1)(d-2)$ F-inequivalences arising from triples in Z_d with all entries distinct.

Proof. We must prove all possible assignments to triples are accounted for and the listed assignments are inequivalent. Under clockwise labeling, the orderings for XY are AB, BC, and CA. All XY assignments for (1) are F-equivalent to say the AB assignment. For (2), The assignments are clearly not F-equivalent. For (3), assignments (i,j,j) to AB, BC, CA are inequivalent, and assignment (j,j,i) with $i \neq j$ at XY is accounted for because it is equivalent to assignment (i,j,j) at YZ! For (4), it is again clear that the assignments are inequivalent, and the counting is simple.

To summarize, after generating d^3 configurations comprising each F-class of the $d(3d^2-3d+d)$ F-inequivalent configurations enumerated above, we see that the orbit of zero contains $d^4(3d^2-3d+d)$ of the d^7 possible configurations for Shallit's $k=3$ game. To solve the Minimality Problem for these F-inequivalent configurations seems daunting.

6 Design Considerations.

We pose a rhetorical question: *Would anyone care to implement Shallit's game in hardware or software?* We believe that the solvable $k \geq 4$ games could turn out to be very amusing, especially since a "solve" button could skillfully hide from a perplexed player the details of the algorithm we have developed. For example, since labeling of the vertices is not important, each time a "solve" is requested a random vertex could serve as the key vertex we labeled k. There are also alternatives to the linear system we established that could be exchanged and manipulated to further obscure the algorithm.

The reader may wonder why we introduced operators of B-type, which we have hardly made mention of at all. Some have found it convenient to think of the $B_X = F_X P_X$ composition as a separate "Make Black" operation that supplements "Invert" and "Make White" operations.

We experimented briefly with a generalization of Shallit's game that preserved the intersecting sets theme. A topological layout for illustrating all possible intersections between sets A, B, C, and D is shown in Figure 11. For this game F_X and P_X would affect all regions that use X in their labels. We made no dramatic progress towards analyzing this game.

There are a wealth of games that can be designed based on compositions of the independent f_x, p_x, and b_x operators. In this paper we have not explored individual operators constructed using a *mix* of p and f operations though one would expect interesting properties and relationships could be developed. The challenge, of course, is to make compelling, simple to understand, yet fiendishly difficult to solve games.

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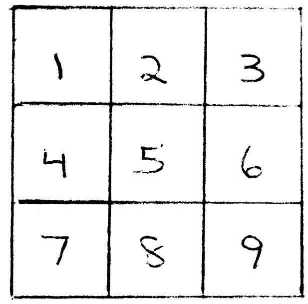


Figure 1.

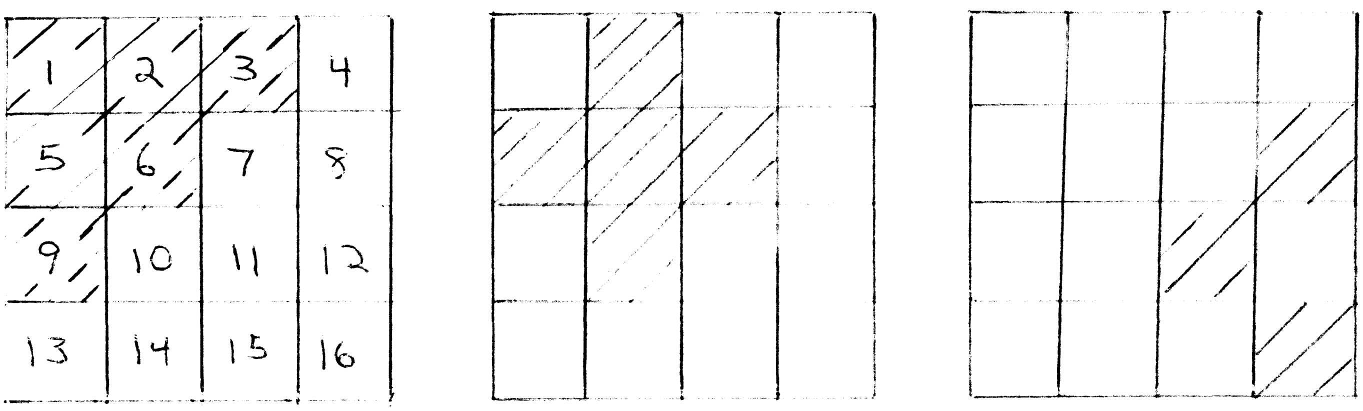


Figure 2.

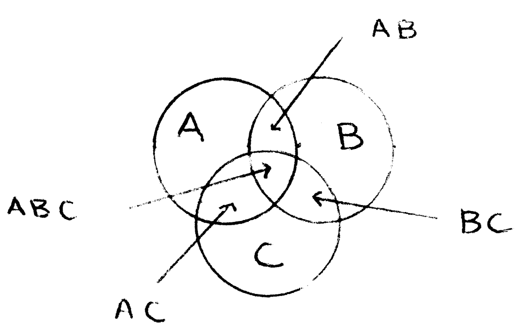


Figure 3.

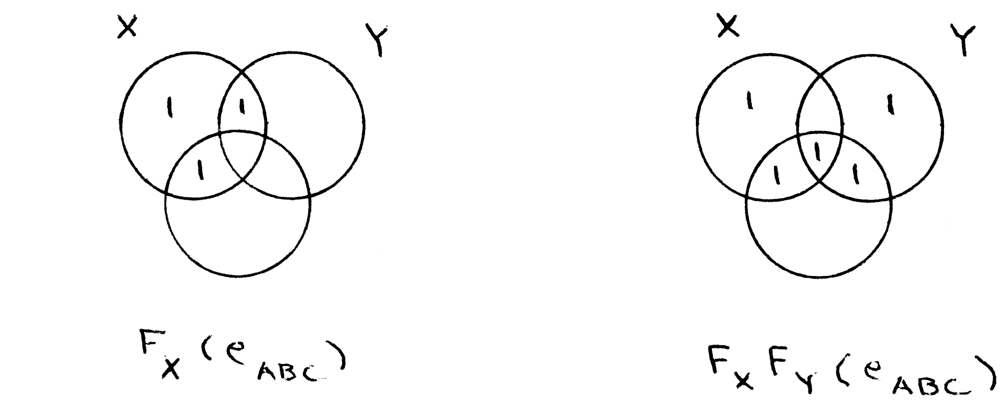


Figure 4.

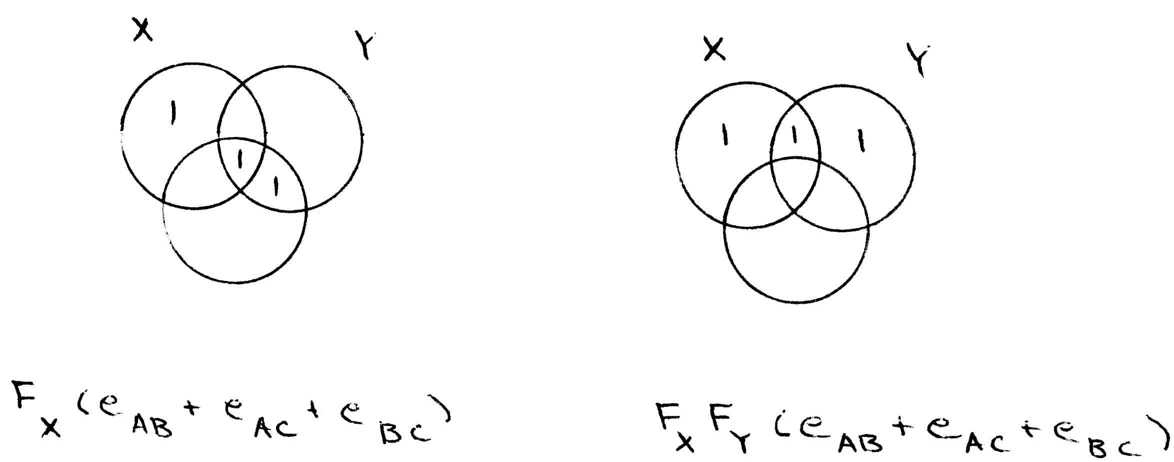


Figure 5.

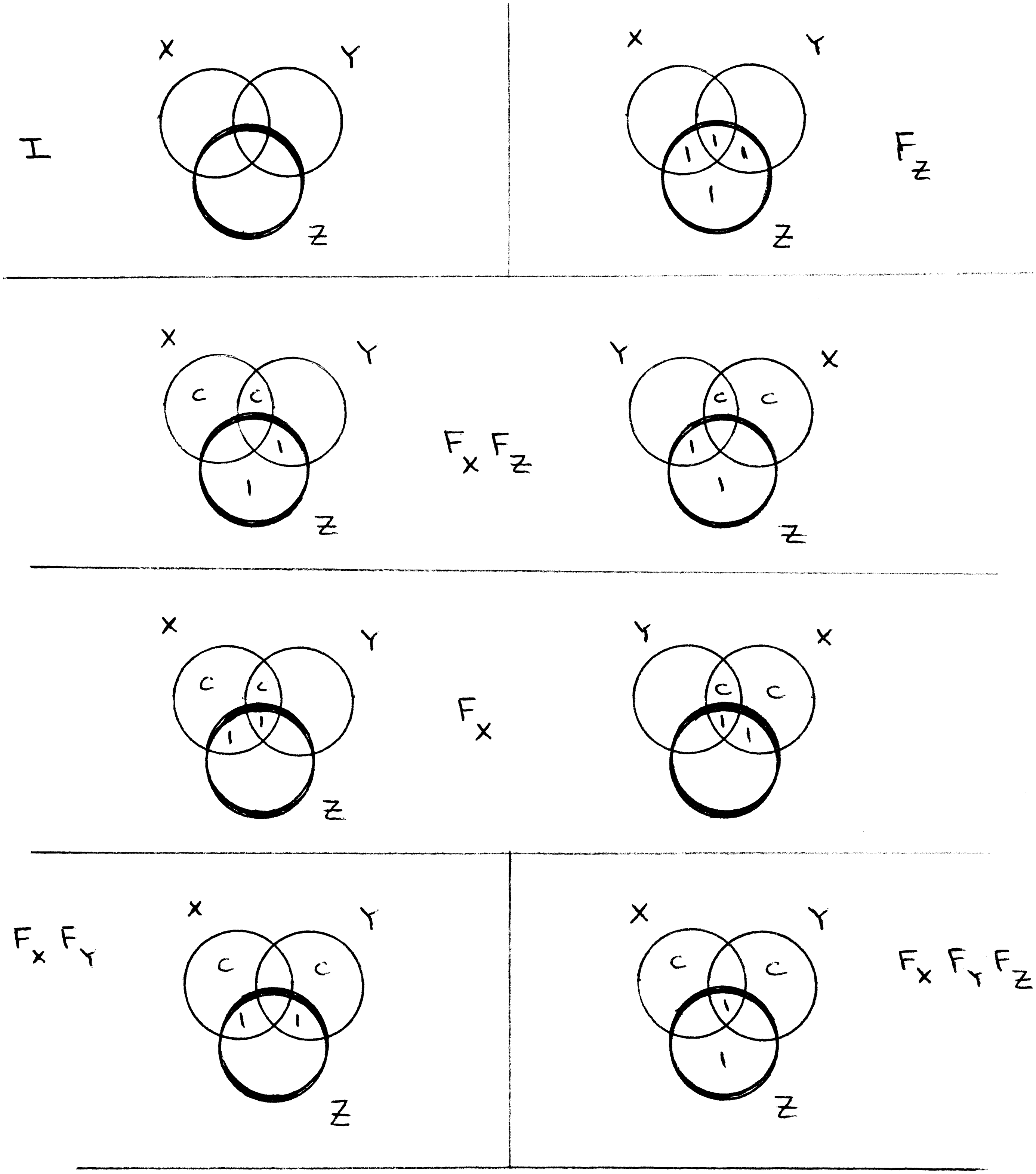


Figure 6.

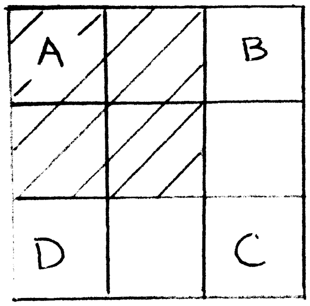


Figure 7.

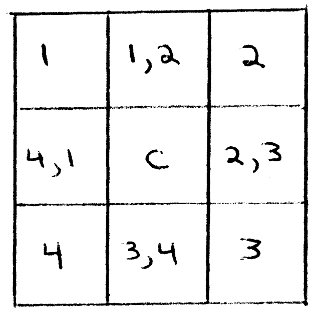
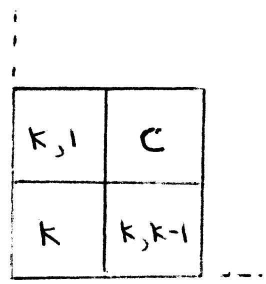
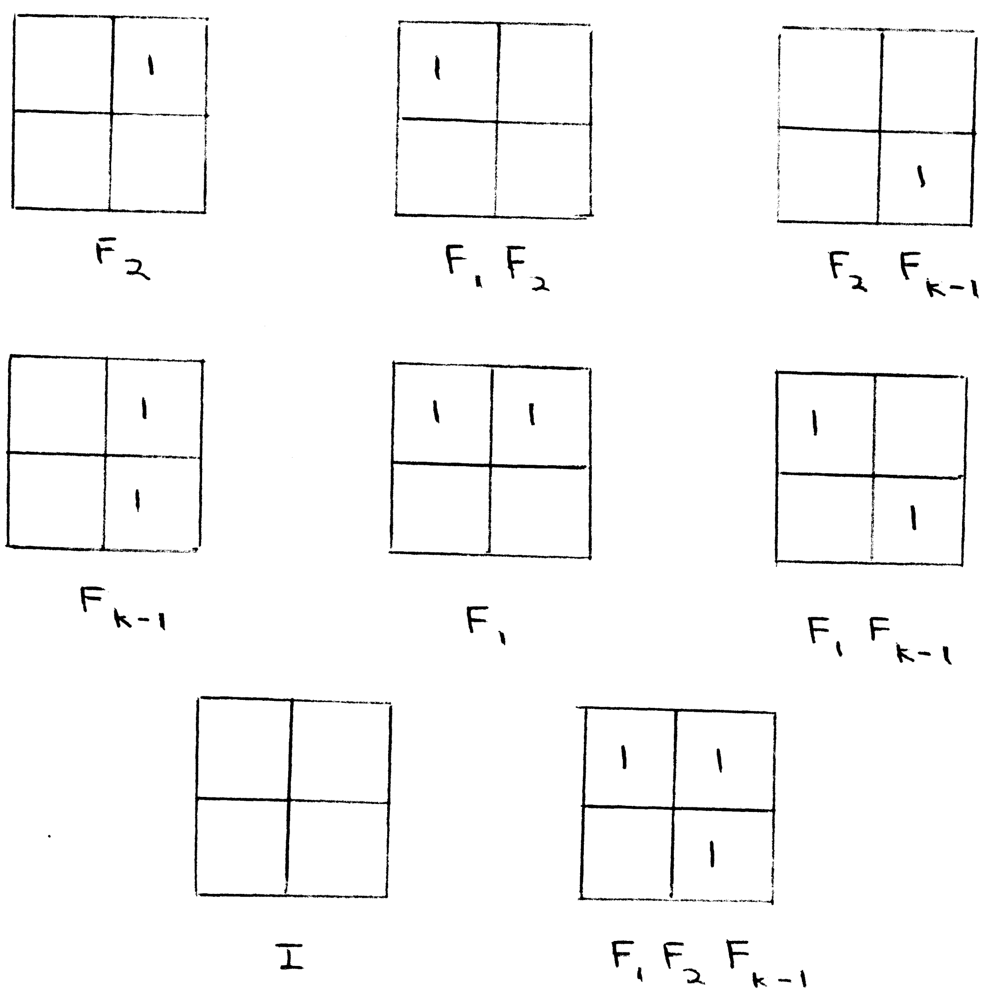


Figure 8.

Figure 9.





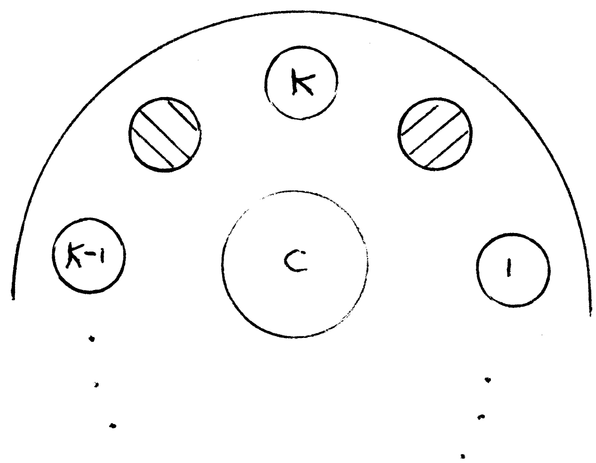


Figure 10.

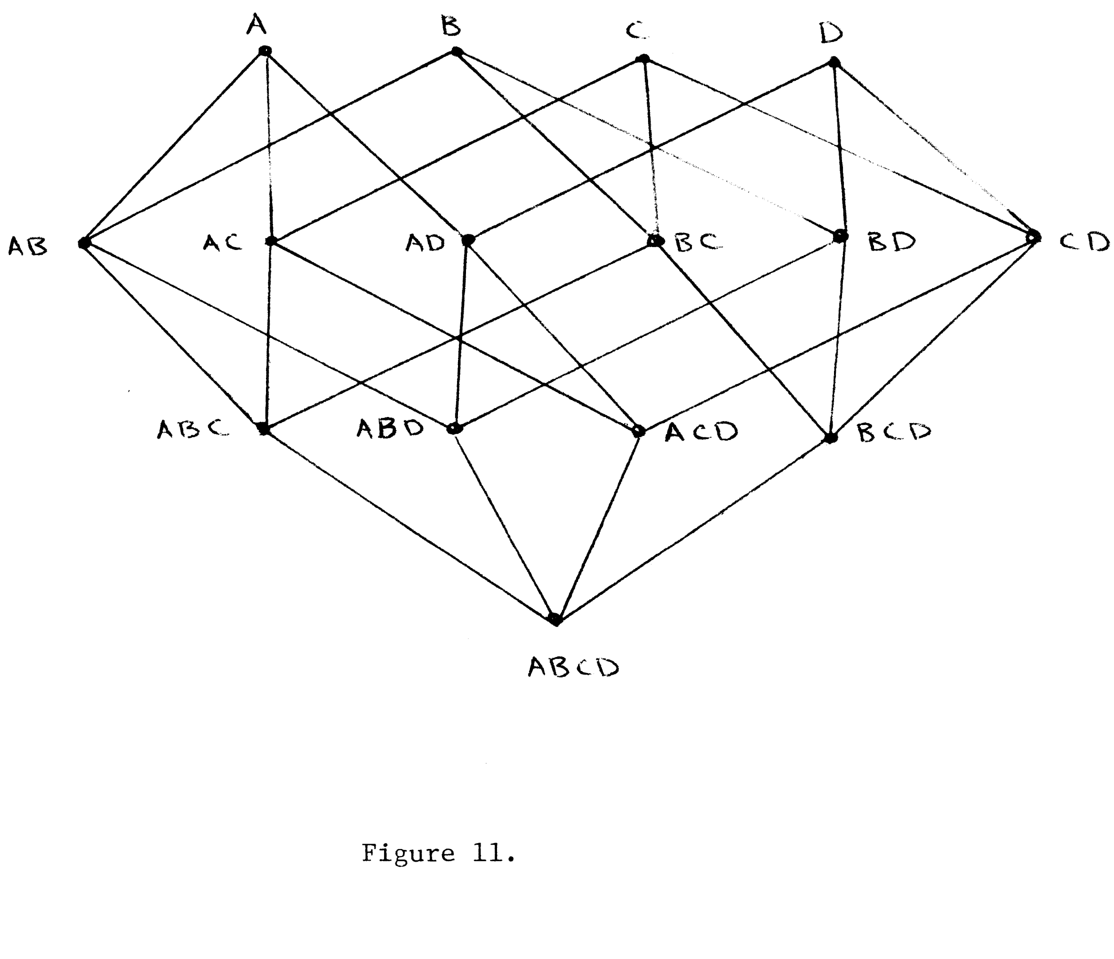


Figure 11.