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The Set of Hemispheres Containing a Closed Curve on the Sphere

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Suppose you get in your car and take a drive on the sphere of radius R, so that when you return to your starting point the odometer indicates you've traveled less than $2\pi R$. Does your path, γ, have to lie in some hemisphere?

This question was presented to us by Dr. Robert Foote of Wabash College. Previous authors chose two points, A and B, on γ such that these points divided γ into two arcs of equal length. Then they took the midpoint of the great circle arc joining A and B to be the North Pole and showed that the curve must be contained in the Northern Hemisphere. This type of proof not only answers the existence question, but also yields a specific hemisphere that contains your path.

We, however, thought the problem lent itself nicely to integral geometry, which required us to consider the space whose points are hemispheres. This led to a different existence proof and to a solution of the more general question: can you describe and measure the set of all hemispheres that contain γ?

An outline of the remainder of this paper follows. In Section 2 we introduce terminology and definitions. The existence of at least one hemisphere containing γ is proved using the ideas of integral geometry in Section 3. Classifying sets of such hemispheres for a single arc, a geodesic triangle, and a geodesic quadrilateral is accomplished in Section 4. Section 5 contains a discussion of convexity on the sphere and how it relates to our question. Our main theorem is stated and proved in Section 6.

2. Terminology

In the remainder of the paper, we restrict attention to the sphere of radius one. As a subset of $\Re^3$, we write the unit sphere S^2 as the set of points x satisfying $x \cdot x = 1$.

2.1. Definitions. Let n be a unit vector in $\Re^3$. The set of points $x \in S^2$ satisfying $n \cdot x > 0$ is the *open hemisphere* determined by n. We denote this hemisphere by H_n. The *closure* $Cl(H_n)$ of H_n is the set of points x with $n \cdot x \geq 0$. The set of points satisfying $n \cdot x = 0$, which forms the *boundary* $\partial(Cl(H_n))$ of the closure of H_n, is a great circle. Arcs of great circles are *geodesics* in S^2. A subset T of S^2 is *contained in the hemisphere H_n* if $n \cdot x > 0 \ \forall\ x \in T$. Finally, we denote by H(T) the *set of all hemispheres containing T*; that is, H(T) is the set of all unit vectors n with $n \cdot x > 0 \ \forall\ x \in T$.

In this definition n is the normal vector to a plane in $\Re^3$ that intersects S^2 and passes through the origin. This plane divides the sphere into the two hemispheres H_n and H_{-n}. Since n and –n refer to the same plane, they determine the same great circle. Conversely, a pair of non-antipodal points determines a unique great circle and thus a pair of opposite hemispheres. The one-to-one correspondence between unit vectors n and hemispheres H_n shows that the space of all open hemispheres is just another copy of S^2.

2.2 Definitions. Two great circles intersect in a pair of *antipodal points* x and –x which lie on a line through the origin in $\Re^3$. The two great circles divide the sphere into four regions. Each region is a *half-lune* whose *poles* are the intersection points of the great circles. The union of two adjacent half-lunes is a hemisphere. A *lune* is the union of non-adjacent half-lunes. Finally, we introduce the *spherical cross product,* defined in terms of the usual cross product by $x \otimes y = x \times y / | x \times y |$. Note that the scalar triple products $x \otimes y \cdot z$ and $x \times y \cdot z$ have the same sign.

3. Existence

Now that the ground work has been laid, we are prepared to anwer our introductory questions. In this section we prove that there does indeed exist a hemisphere containing a closed curve of length less than 2π. Our starting point is the Cauchy-Crofton formula. For plane curves, this formula states that the measure of the set of straight lines that intersect a given curve, counted with multiplicities, is twice the length of the curve. A nice proof of this formula appears in [dC] (p. 44 - 45). The same formula holds for curves on the sphere (just replace "straight line" by "great circle").

3.1. Theorem. *If γ is a piecewise differentiable closed curve in S^2 of length $\ell < 2\pi$, then there exists a hemisphere which contains γ.*

Proof. We must show that there is a great circle g that fails to intersect γ. Since pairs of antipodal points n and –n determine the same great circle, the space whose points are great circles in S^2 is obtained from S^2 by identifying antipodal points. This space is known as the projective plane P^2. The area of a subset of P^2 is half the area in S^2 that is covered by the pairs of antipodal points that are identified to form points of the subset.

For $n \in S^2$, let N(n) be the number of times the geodesic orthogonal to n crosses γ. Then, by the Cauchy-Crofton formula, $1/2\iint_{S^2} N(n)\, dA = 2\ell$. Note that N(n) may be infinite for some points n. The assumptions on γ insure that the integral does converge (in the Lebesgue sense, see [S1], p.31). Let $X \subseteq S^2$ correspond to the set of geodesics that intersect γ at least twice. Then $\text{Area}(X) \leq 1/2\iint_{S^2} N(n)\, dA = 2\ell < 4\pi$. Since 4π is the total area of S^2, there must be some geodesic g' that intersects γ in at most one point. If g' fails to intersect γ, we're done. Otherwise, let x be the point of intersection and let n' be the unit normal to the plane of g'. Then $\gamma \subset Cl(H_{n'})$. We may parameterize γ by arclength so that, as the continuous image of the closed and bounded interval $[0,\ell]$, γ is compact. Hence the set $K = \gamma \cap Cl(H_{n'}) \cap Cl(H_{-x})$ is also compact and the function that gives the distance from a point of γ to g' achieves a nonzero minimum δ in K. Let $n = \cos(\delta/2)n' + \sin(\delta/2)x$ and let g be the great circle orthogonal to n. Then, because of the way n is rotated from n', any intersection point of g with γ must lie in K. But the distance from any point y in g to g' is at most $\delta/2$, so $g \cap \gamma$ is empty and $\gamma \subset H_n$.

4. Basic Examples of Sets H(γ)

With the existence of at least one hemisphere established, we look at a few simple examples for which the set H(γ) of hemispheres that contain γ can be described explicitly.

4.1. Theorem. *Let γ be a great circle arc of length $\ell < \pi$. Let v_1 and v_2 be the endpoints of γ so that the $\mathrm{Arccos}(v_1 \cdot v_2) = \ell$.. Then the set $H(\gamma)$ is equal to the open half-lune and centered on the midpoint of γ having poles $\pm(v_1 \otimes v_2)$ and included angle $\pi - \ell$. The area of this half-lune is* $2(\pi - \ell)$.

Proof. Without loss of generality, write $\gamma = \{(\cos(t), \sin(t), 0) \mid 0 \le t \le \ell < \pi\}$ and write $n = (\cos(u)\cos(v), \sin(u)\cos(v), \sin(v))$ with $0 \le u \le 2\pi$ and $-\pi/2 \le v \le \pi/2$. For $x \in \gamma$, we compute $n \cdot x = \cos(v)\cos(u - t)$. For this to be positive, we must have $-\pi/2 < v < \pi/2$ and $-\pi/2 < u - t < \pi/2$, i.e. we need $t - \pi/2 < u < t + \pi/2$ for all t with $0 \le t \le \ell$. Hence u must lie in the intersection of the intervals $(t - \pi/2, t + \pi/2)$ with $0 \le t \le \ell$. Since $\ell < \pi$ and each open interval has length π, the left endpoint for the intersection occurs when t is at its maximum, i.e. $t = \ell$, and the right endpoint occurs at $t = 0$. Thus, u must satisfy $\ell - \pi/2 < u < \pi/2$ or $\ell/2 - (\pi - \ell)/2 < u < \ell/2 + (\pi - \ell)/2$. Note that $u=\ell/2$ and $v=0$ corresponds to the midpoint of γ and is the center of the lune, whose included angle is $(\pi - \ell)/2 + (\pi - \ell)/2 = \pi - \ell$. Letting R denote the rectangle $\{(u,v)\colon \ell - \pi/2 < u < \pi/2 \text{ and } -\pi/2 < v < \pi/2\}$, the measure of H($\gamma$) is $\int\int_R \cos(v)\, dv\, du = 2(\pi - \ell)$.

The next case we consider has γ as the boundary of a geodesic triangle.

4.2. Theorem. *Let Δ be a non-degenerate triangle in S^2 with vertices v_1, v_2, and v_3. Then*

1) Perimeter(Δ) < 2π;

2) H($\partial\Delta$) *is the interior of the triangle having vertices $v_1 \otimes v_2$, $v_2 \otimes v_3$, and $v_3 \otimes v_1$;*

3) Area(H($\partial\Delta$)) = 2π - Perimeter(Δ) .

Proof. The assumption of non-degeneracy means that v_1, v_2, and v_3 do not lie on a great circle. These three points are certainly coplanar in $\Re^3$ however. Let $d > 0$ be the distance from the origin to this plane. Then v_1, v_2, and v_3 lie on a circle of radius $\sqrt{(1-d^2)}$ and Perimeter(Δ) < $2\pi\sqrt{(1-d^2)} < 2\pi$.

Without loss of generality, write $v_1 = (1, 0, 0)$, $v_2 = (\cos(u), \sin(u), 0)$, and $v_3 = (\cos(v)\cos(w), \sin(v)\cos(w), \sin(w))$, with $0 < u < 2\pi$, $0 < v < 2\pi$, and $0 < w < \pi/2$. From the previous theorem, we know the set of hemispheres containing an arc is a half-lune with included angle $(\pi-\ell)$, where ℓ is the length of the arc, and poles $\pm(v_1 \times v_2)$, where v_1

and v_2 are the endpoints of the arc. When we consider the three arcs of $\partial\Delta$, we see that $H(\partial\Delta)$ is the intersection of three half lunes. Since each pair of arcs of $\partial\Delta$ shares a vertex, the half lunes are created from three great circles instead of six, with each half lune that corresponds to an arc of $\partial\Delta$ sharing a great circle with another half-lune.

The pairwise intersections of the three great circles yield six intersection points, namely $\pm(v_1 \otimes v_2)$, $\pm(v_2 \otimes v_3)$, and $\pm(v_3 \otimes v_1)$, because they are the poles of the lunes created. We can use $n \cdot x > 0$ to find which of the six vertices define the region $H(\partial\Delta)$. We compute $(v_1 \times v_2) \cdot v_3 = \sin(w)$. Our range for w gives $\sin(w) > 0$, and since $(v_1 \times v_2) \cdot v_3 = (v_2 \times v_3) \cdot v_1 = (v_3 \times v_1) \cdot v_2$, the points $v_1 \otimes v_2$, $v_2 \otimes v_3$, and $v_3 \otimes v_1$ are the vertices of $H(\partial\Delta)$. Hence $Cl(H(\partial\Delta))$ is a triangle and has area equal to the sum of its interior angles minus π. Writing the length of the arc $v_i v_j$ as ℓ_{ij}, the corresponding interior angle is $(\pi - \ell_{ij})$. From this we compute

$$\text{Area}\,(H(\partial\Delta)) = (\pi - \ell_{12}) + (\pi - \ell_{23}) + (\pi - \ell_{31}) - \pi = 2\pi - \text{Perimeter}(\Delta).$$

Covering the arc in Theorem 4.1 twice and considering the new arc as the boundary of a degenerate triangle Δ having perimeter 2ℓ, we see that the measure of the set of hemispheres containing the arc is once again given by 2π - Perimeter(Δ). This interpretation unifies 4.1 and 4.2. We may also unify these results by noting that in each case the set of hemispheres containing the original figure has vertices given by cross products of the vertices in the original figure. We consider the new figure $Cl(H(\Delta))$ to be the "dual" of the original figure Δ, since the vertices of $Cl(H(\Delta))$ correspond to edges of Δ and the edges of $Cl(H(\Delta))$ correspond to vertices of Δ. Thus we are led to conjecture that the set of hemispheres containing a spherical polygon has vertices that are spherical cross products of consecutive vertices in the polygon. In the following theorem we treat the case of a spherical quadrilateral, showing that if some care is taken in the placement of vertices then this conjecture holds true. It will be apparent in the proof of the theorem that the notion of convexity is vitally important (see Figure 1).

4.3. Theorem. *Let Q be a spherical quadrilateral having consecutive vertices $v_1 \ldots v_4$. Suppose that*

a) the arc $v_1v_2v_3$ has included angle less than π and is oriented so that $v_1 \cdot v_2 \otimes v_3$ is positive;

b) v_4 lies in the open half lune with vertices $\pm v_2$ and edges v_1v_2 and v_2v_3;

and c) v_4 lies in the hemisphere determined by v_3v_1 that is opposite v_2.

Then

1) Perimeter(Q) $< 2\pi$;

2) $H(\partial Q)$ is the interior of the spherical quadrilateral having vertices $v_i \otimes v_{i+1}$ (with $v_5 = v_1$ by convention);

and 3) Area($H(\partial Q)$) $= 2\pi$ - Perimeter(Q).

Proof. We first prove Perimeter(Q) < 2π. Consider the triangle $\Delta v_1v_3v_4 \subset \Delta v_1v_3(-v_2)$. Clearly, Area(H($\Delta v_1v_3(-v_2)$)) < Area(H($\Delta v_1v_3v_4$)). By the previous theorem, this implies Perimeter($\Delta v_1v_3v_4$) < Perimeter($\Delta v_1v_3(-v_2)$). Thus, length (v_3v_4) + length (v_4v_1) < length ($v_3(-v_2)$) + length ($(-v_2)v_1$) = 2π - length (v_1v_2) - length (v_2v_3). Therefore, length (v_1v_2) + length (v_2v_3) + length (v_3v_4) + length (v_4v_1) = Perimeter(Q) < 2π.

For convenience, we refer to the spherical quadrilateral with vertices $v_i \otimes v_{i+1}$ as the quadrilateral dual to Q, denoting this dual quadrilateral by Q*. The set of points in the interior of Q* may be written as $\sum_{i=1}^{4} \mu_i y_i \,/\, \left\|\sum_{i=1}^{4} \mu_i y_i\right\|$ where $\sum \mu_i = 1$, $\mu_i > 0$, and $y_i = v_i \otimes v_{i+1}$. Similarly the points in the Cl(Q) may be represented in the form $\sum_{i=1}^{4} \lambda_i v_i \,/\, \left\|\sum_{i=1}^{4} \lambda_i v_i\right\|$ with $\sum \lambda_i = 1$, and $\lambda_i \geq 0$.

To prove that H(∂Q) contains the interior of Q*, we must show the dot product of these two terms is positive. Since the lengths won't contribute to the sign, they are ignored and we're left to show

$$\left(\sum_{i=1}^{4} \lambda_i v_i\right)\cdot\left(\sum_{i=1}^{4} \mu_i y_i\right) > 0.$$

The cyclic nature of scalar triple products allows the above expression to be simplified to

$$2[(\lambda_1\mu_2+\lambda_3\mu_1)\, v_1\cdot(v_2 \otimes v_3) + (\lambda_2\mu_3+\lambda_4\mu_2)\, v_2\cdot(v_3 \otimes v_4) + (\lambda_1\mu_3+\lambda_3\mu_4)\, v_3\cdot(v_4 \otimes v_1) + (\lambda_2\mu_4+\lambda_4\mu_1)\, v_4\cdot(v_1 \otimes v_2)].$$

By assumption a) in the statement of the theorem, the four scalar triple products are all positive. Moreover, the coefficients of these scalar triple products are nonnegative. So, the only concern is that the above expression may equal zero. However, since $\mu_i > 0$ for all i and at least one $\lambda_i > 0$, we are assured of at least one (actually two) positive terms.

Now we want to show H(∂Q) is contained in the interior of Q*. We'll do this by showing they have the same boundary. A point on the boundary of Q* lies on an arc joining a pair of consecutive dual vertices $v_i \otimes v_{i+1}$ and $v_{i+1} \otimes v_{i+2}$. Such a point clearly has dot product zero with $v_{i+1} \in \partial$Q, showing that it also lies on the boundary of H(∂Q). Thus, the two regions do share the same boundary.

Finally, we compute the area of Q*. Recall that

Area of a geodesic n-gon on the sphere = Sum of the interior angles - (n - 2) π.

As in the proofs of Theorems 4.1 and 4.2, the interior angle of Q* at $v_i \otimes v_{i+1}$ is π - length (v_iv_{i+1}). Therefore, the area of the spherical quadrilateral is given by

Anglesum - $2\pi = 4\pi$ - Perimeter (Q) - $2\pi = 2\pi$ - Perimeter (Q).

5. Convexity on S^2

Examining Figure 1, we see that a quadrilateral Q which satisfies the hypotheses of 4.3 has the following property: if x and y are points of Q, then the shorter of the great circle arcs joining x to y is contained in Q. Thus Q is "convex". In order to avoid any ambiguity in the phrase "the shorter of the great circle arcs joining x to y," and because subsets not contained in a hemisphere play no role in the remainder of this paper, we adopt the following definitions concerning convexity. Our definition of the convex hull of T is motivated by a theorem of Caratheodory ([C], p.264) which states that the convex hull of a subset of R^2 is the union of (possibly degenerate) triangles whose vertices lie in the subset.

5.1. Definition. Let T be a subset of S^2 that is contained in some hemisphere H_n. T is *convex* if for every pair of points x and y in T, the shorter of the great circle arcs joining x and y is contained in T. The *convex hull* of T is the union of all geodesic triangles whose vertices lie in T.

The results in the remainder of this section concern intersections of great circles with a given convex subset of S^2. Since the subset is assumed to lie in some hemisphere, H_n, we may perform a central projection $x \to x/(n.x)$ of the hemisphere onto the tangent plane to S^2 at n. Under this projection, the image of any great circle arc lying in H_n is a straight line segment. Thus central projection preserves the types of intersections we wish to study. We choose to work in the image plane so that we may apply our knowledge of plane topology (as found, for example in [B]).

5.2. Proposition. *Let T be a closed, bounded, convex subset of the plane and let ℓ be a line. Then the number of points in $\ell \cap \partial T$ is either 0, 1, 2, or ∞.*

Proof. We may assume that T is not contained in a line, since the result is obvious in that case. Suppose $\ell \cap \partial T$ contains three points, p_1, p_2, and p_3, with p_2 between p_1 and p_3 on ℓ. We claim that $\ell \cap \partial T$ contains the entire segment $[p_1, p_3]$. Since T is convex, $[p_1, p_3] \subset$ T. If there were a point q of $[p_1, p_3]$ lying in the interior of T, then an open neghborhood of q would contain points x and y in T lying on either side of ℓ. But then the quadrilateral $p_1 x\, p_3 y$ would contain p_2 as an interior point of T, contradicting the fact that p_2 lies on the boundary of T.

5.3. Definition. Let T be a closed, convex subset of the plane. A line ℓ is a *support line* for T if $\ell \cap T$ is nonempty and T is entirely contained in one of the closed half-planes determined by ℓ.

From the previous argument, we see that a support line for T intersects ∂T in either a single point or a single closed interval. It is customary to write lines in the plane as solution sets to equations of the form

$$\cos(\theta)\, x + \sin(\theta)\, y = p.$$

Here, the vector $(\cos(\theta), \sin(\theta))$ is a unit normal to the line and p represents the directed distance from the origin to the line. This representation shows that the space whose points are lines of the plane is a Mobius strip: we may represent each line by a point in the set $\{(\theta, p) : 0 \leq \theta \leq \pi, -\infty < p < \infty\}$ with points $(0, p)$ and $(\pi, -p)$ representing the same line.

5.4. Proposition. *Let T be a closed, bounded, convex subset of the plane and let ℓ be a line. Then to each θ in $[0, \pi]$, there corresponds at least one and at most two support lines for T.*

Proof. Choose coordinates in the plane such that the origin lies in T. For each θ, consider the function f_θ that assigns to each point (a,b) of T its directed distance to the line having normal direction $(\cos(\theta), \sin(\theta))$ and passing through the origin. Since $f_\theta(a, b) = (a, b)\cdot(\cos(\theta), \sin(\theta))$ is continuous and T is compact, f_θ attains both a maximum and a minimum value on T. It is clear that the lines

$$x \cos(\theta) + y \sin(\theta) = \min f_\theta$$
$$x \cos(\theta) + y \sin(\theta) = \max f_\theta$$

are support lines for T. If T is contained in a line, the extreme values of f_θ may coincide (both being 0), but in any case we see that there is at least one support line corresponding to θ.

If the extreme values of f_θ are distinct, then there are distinct points p_{min} and p_{max} (necessarily on the boundary of T) with $f_\theta(p_{min}) = \min f_\theta$ and $f_\theta(p_{max}) = \max f_\theta$. Let v be a value of f_θ with $\min f_\theta < v < \max f_\theta$. By the intermediate value theorem, there is a (unique) point p_v of the segment $[p_{min}, p_{max}] \subset T$ with $f_\theta(p_v) = v$. Since $\min f_\theta \neq \max f_\theta$, the line $\cos(\theta)\, x + \sin(\theta)\, y = v$ must intersect $[p_{min}, p_{max}]$ transversally, so it cannot be a support line for T.

5.5. Corollary. *The set of support lines for T has measure zero.*

Proof. The union of the graphs $(\theta, \min f_\theta)$ and $(\theta, \max f_\theta)$ forms a single curve on the Mobius strip representing all lines in the plane. We call this curve the *support curve* of the convex set T (see Figure 2). The two-dimensional measure of a one-dimensional curve is zero.

6. Main Theorem and Comments

By the results of Section 5, if T is a closed, convex subset of H_n that is not contained in a great circle arc, then only the great circles meeting ∂T in exactly 2 points contribute to the Cauchy-Crofton integral for the length of ∂T. This is the key to the proof of our main result.

6.1. Theorem. *Let γ be a piecewise differentiable closed curve on S^2 having finite length, which is contained in some hemisphere H_n. Let $\gamma^{\tilde{}}$ be the convex hull of γ. Then*

1) $H(\gamma)$ *is an open and convex subset of each hemisphere H_x,* $x \in \gamma$*;*

2) $H(\gamma) = H(\gamma^{\tilde{}})$;

3) $\text{Area}(H(\gamma)) = 2\pi - \text{Perimeter}(\gamma^{\tilde{}})$.

Proof. That $H(\gamma)$ is contained in each H_x, $x \in \gamma$, follows from the condition $m \cdot x > 0$ that determines whether m is in $H(\gamma)$. Convexity of $H(\gamma)$ follows from the distributive property for dot products: the great circle arc joining m, m' $\in H(\gamma)$ may be written as $\{((1 - t)\, m + m'y) / \mid (1 - t)\, m + tm' \mid : 0 \leq t \leq 1\}$, showing that each point on this arc corresponds to a hemisphere containing γ. As in the proof of 3.1, a small rotation of a great circle not meeting γ also yields a great circle not meeting γ, showing that $H(\gamma)$ is open.

Since $\gamma \subset \gamma^{\tilde{}}$ any hemisphere containing $\gamma^{\tilde{}}$ also contains γ. Since γ is compact, we see that if $H_m \supset \gamma$ and g is the great circle orthogonal to m then $\text{dist}(\gamma, g)$ is bounded below by some positive number $\delta(m)$. Let p be a point in γ^*. Clearly, dist(p,g) is no less than the minimum, over all geodesic triangles containing p and having vertices on γ, of the distance from that triangle to g. But the distance function on each triangle assumes its minimum at a vertex of the triangle, so that $\text{dist}(p,g) \geq \text{dist}(\gamma, g) = \delta(m) > 0$. From this we see that γ^* is also contained in H_m. Thus $H(\gamma) = H(\gamma^{\tilde{}})$.

The convex hull of a closed set is closed, so the image of $\gamma^{\tilde{}}$ under the central projection $x \rightarrow x/n.x$ is a closed, bounded, convex set in the image plane. A reference is given in [S1] (p. 1) for the fact that the boundary of such a set is piecewise differentiable and has finite length. Since the inverse of the central projection map is differentiable and length decreasing, we see that $\partial \gamma^{\tilde{}}$ satisfies criteria that insure that the Cauchy-Crofton integral for its length converges. Applying Proposition 5.3 and Corollary 5.5 to evaluate this integral we have

$$
\begin{aligned}
2 \text{ length } (\partial \gamma^{\tilde{}}) &= 1/2 \iint_E N(n)\, dA, \text{ with } E = \{\text{normals to great circles intersecting } \partial \gamma^{\tilde{}}\}, \\
&= \iint_F dA,\ F = \{\text{normals to great circles intersecting} \partial \gamma^{\tilde{}} \text{ in 2 points}\}, \\
&= 4\pi - \iint_G dA, \text{ with } G = \{\text{normals to great circles not intersecting } \partial \gamma^{\tilde{}}\}.
\end{aligned}
$$

From this it follows that length $(\partial\gamma^{\sim}) = 2\pi$ - Area $(H(\gamma^{\sim}))$ because only one of each pair of normals in G corresponds to a hemisphere containing $\partial\,\gamma^{\sim}$. This completes the proof of the theorem.

In this theorem, γ may be replaced by any closed set K. An approximation of $K^{\sim}$ by a convex polygon yields an approximation of H(K) by the dual polygon $(K^{\sim})^*$. We conclude by noting that Theorem 6.1 has a probablistic interpretation. The probability that a random great circle fails to meet the set K is equal to 1 - Perimeter $(K^{\sim}) / 2\pi$. Further results of this type appear in [S1] (p. 318).

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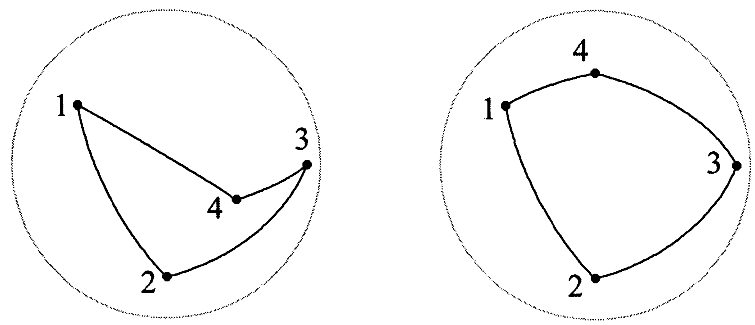

Figure 1. Only the spherical quadrilateral on the right satisfies the hypotheses of 4.3.

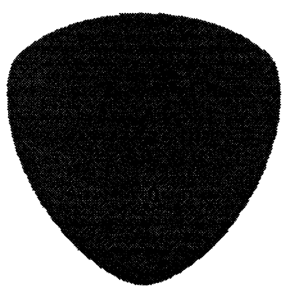

Figure 2. A convex set and its support curve on the Mobius strip of lines in the plane.