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Ordinary and Particial Differential Equations: An Introduction to Dynamical Systems

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Ordinary and Partial Differential Equations

An Introduction to Dynamical Systems

John W. Cain, Ph.D. and Angela M. Reynolds, Ph.D.

CHAPTER 1

Introduction

The mathematical sub-discipline of *differential equations and dynamical systems* is foundational in the study of applied mathematics. Differential equations arise in a variety of contexts, some purely theoretical and some of practical interest. As you read this textbook, you will find that the qualitative and quantitative study of differential equations incorporates an elegant blend of linear algebra and advanced calculus. For this reason, it is expected that the reader has already completed courses in (i) linear algebra; (ii) multivariable calculus; and (iii) introductory differential equations. Familiarity with the following topics is especially desirable:

☞ From basic differential equations: separable differential equations and separation of variables; and solving linear, constant-coefficient differential equations using characteristic equations.

☞ From linear algebra: solving systems of m algebraic equations with n unknowns; matrix inversion; linear independence; and eigenvalues/eigenvectors.

☞ From multivariable calculus: parametrized curves; partial derivatives and gradients; and approximating a surface using a tangent plane.

Some of these topics will be reviewed as we encounter them later—in this chapter, we will recall a few basic notions from an introductory course in differential equations. Readers are encouraged to supplement this book with the excellent textbooks of Hubbard and West [5], Meiss [7], Perko [8], Strauss [10], and Strogatz [11].

Question: Why study differential equations?

Answer: When scientists attempt to mathematically model various natural phenomena, they often invoke physical "laws" or biological "principles" which govern the *rates of change* of certain quantities of interest. Hence, the equations in mathematical models tend to include derivatives. For example, suppose that a hot cup of coffee is placed in a room of constant ambient temperature α. Newton's Law of Cooling states that the *rate of change* of the coffee temperature $T(t)$ is proportional to the difference between the coffee's temperature and the room temperature. Mathematically, this can be expressed as $\frac{dT}{dt} = k(T - \alpha)$, where k is a proportionality constant.

Solution techniques for differential equations (DEs) depend in part upon how many independent variables and dependent variables the system has.

Example 1.0.1. One independent variable and one independent variable. In writing the equation

$$\frac{d^2y}{dx^2} + \cos(xy) = 3,$$

it is understood that y is the dependent variable and x is the independent variable.

When a differential equation involves a single independent variable, we refer to the equation as an *ordinary differential equation (ODE).*

Example 1.0.2. If there are several dependent variables and a single independent variable, we might have equations such as

$$\frac{dy}{dx} = x^2y - xy^2 + z, \qquad \frac{dz}{dx} = z - y\cos x.$$

This is a *system* of two ODEs, and it is understood that x is the independent variable.

Example 1.0.3. One dependent variable, several independent variables. Consider the DE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

This equation involves three independent variables (x, y, and t) and one dependent variable, u. This is an example of a *partial differential equation (PDE).* If there are several independent variables and several dependent variables, one may have systems of PDEs.

Although these concepts are probably familiar to the reader, we give a more exact definition for what we mean by ODE. Suppose that x and y are independent and dependent variables, respectively, and let $y^{(k)}(x)$ denote the kth derivative of y with respect to x. (If $k \leq 3$, we will use primes.)

Definition 1.0.4. Any equation of the form $F(x, y, y', y'', \ldots, y^{(n)}) = 0$ is called an *ordinary differential equation*. If $y^{(n)}$ is the highest derivative appearing in the equation, we say that the ODE is of *order n*.

Example 1.0.5.

$$\left(\frac{d^3y}{dx^3}\right)^2 - (\cos x)\frac{dy}{dx} = y\frac{d^2y}{dx^2}$$

can be written as $(y''')^2 - yy'' - (\cos x)y' = 0$, so using the notation in the above Definition, we would have $F(x, y, y', y'', y''') = (y''')^2 - yy'' - (\cos x)y'$. This is a third-order ODE.

Definition 1.0.6. A *solution* of the ODE $F(x, y, y', y'', \ldots, y^{(n)}) = 0$ on an interval I is any function $y(x)$ which is n-times differentiable and satisfies the equation on I.

Example 1.0.7. For any choice of constant A, the function

$$y(x) = \frac{Ae^x}{1 + Ae^x}$$

is a solution of the first-order ODE $y' = y - y^2$ for all real x. To see why, we use the quotient rule to calculate

$$y' = \frac{Ae^x(1 + Ae^x) - (Ae^x)^2}{(1 + Ae^x)^2} = \frac{Ae^x}{(1 + Ae^x)^2}.$$

By comparison, we calculate that

$$y - y^2 = \frac{Ae^x}{(1 + Ae^x)} - \frac{(Ae^x)^2}{(1 + Ae^x)^2} = \frac{Ae^x}{(1 + Ae^x)^2}.$$

Therefore, $y' = y - y^2$, as claimed.

The definition of a solution of an ODE is easily extended to systems of ODEs (see below). In what follows, we will focus solely on systems of *first-order* ODEs. This may seem overly restrictive, until we make the following observation.

Observation. Any nth-order ODE can be written as a system of n first-order ODEs. The process of doing so is straightforward, as illustrated in the following example:

Example 1.0.8. Consider the second-order ODE $y'' + (\cos x)y' + y^2 = e^x$. To avoid using second derivatives, we introduce a new dependent variable $z = y'$ so that $z' = y''$. Our ODE can be re-written as $z' + (\cos x)z + y^2 = e^x$. Thus, we have obtained a system of two first-order ODEs:

$$\frac{dy}{dx} = z, \qquad \frac{dz}{dx} = -(\cos x)z - y^2 + e^x.$$

A *solution* of the above system of ODEs on an open interval I is any *vector* of differentiable functions $[y(x), z(x)]$ which simultaneously satisfy both ODEs when $x \in I$.

Example 1.0.9. Consider the system

$$\frac{dy}{dt} = z, \qquad \frac{dz}{dt} = -y.$$

We claim that for any choices of constants C_1 and C_2,

$$\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} C_1 \cos t + C_2 \sin t \\ -C_1 \sin t + C_2 \cos t \end{bmatrix}$$

is a solution of the system. To verify this, assume that y and z have this form. Differentiation reveals that $y' = -C_1 \sin t + C_2 \cos t$ and $z' = -C_1 \cos t - C_2 \sin t$. Thus, $y' = z$ and $z' = -y$, as required.

1.1. Initial and Boundary Value Problems

In the previous example, the solution of the system of ODEs contains arbitrary constants C_1 and C_2. Therefore, the system has infinitely many solutions. In practice, one often has additional information about the underlying system, allowing us to select a particular solution of practical interest. For example, suppose that a cup of coffee is cooling off and obeys Newton's Law of Cooling. In order to predict the coffee's temperature at future times, we would need to specify the temperature of the coffee at some reference time (usually considered to be the "initial" time). By specifying auxiliary conditions that solutions of an

ODE must satisfy, we may be able to single out a *particular* solution. There are two usual ways of specifying auxiliary conditions.

Initial conditions. Suppose $F(x, y, y', y'', \ldots, y^{(n)}) = 0$ is an nth order ODE which has a solution on an open interval I containing $x = x_0$. Recall from your course on basic differential equations that, under reasonable assumptions, we would expect the general solution of this ODE to contain n arbitrary constants. One way to eliminate these constants and single out one particular solution is to specify n *initial conditions*. To do so, we may specify values for

$$y(x_0),\ y'(x_0),\ y''(x_0),\ \ldots\ y^{(n-1)}(x_0).$$

We regard x_0 as representing some "initial time". An ODE together with its initial conditions (ICs) forms an *initial value problem* (IVP). Usually, initial conditions will be specified at $x_0 = 0$.

Example 1.1.1. Consider the second-order ODE $y''(x) + y(x) = 0$. You can check that the general solution is $y(x) = C_1 \cos x + C_2 \sin(x)$, where C_1 and C_2 are arbitrary constants. To single out a particular solution, we would need to specify two initial conditions. For example, if we require that $y(0) = 1$ and $y'(0) = 0$, we find that $C_1 = 1$ and $C_2 = 0$. Hence, we obtain a particular solution $y(x) = \cos x$.

If we have a system of n first-order ODEs, we will specify one initial condition for each independent variable. If the dependent variables are

$$y_1(x), y_2(x), \ldots y_n(x),$$

we typically specify the values of

$$y_1(0), y_2(0), \ldots, y_n(0).$$

Boundary conditions. Instead of specifying requirements that y and its derivatives must satisfy at *one* particular value of the independent variable x, we could instead impose requirements on y and its derivatives at *different* x values. The result is called a *boundary value problem* (BVP).

Example 1.1.2. Consider the boundary value problem $y'' + y = 0$ with boundary conditions $y(0) = 1$ and $y(\pi/2) = 0$. The general solution of the ODE is $y(x) = C_1 \cos x + C_2 \sin x$. Using the first boundary condition, we find that

$C_1 = 1$. Since $y'(x) = -C_1 \sin x + C_2 \cos x$, the second boundary condition tells us that $-C_1 = 0$. Notice that the two boundary conditions produce conflicting requirements on C_1. Consequently, the BVP has no solutions.

As the previous example suggests, boundary value problems can be a tricky matter. In the ODE portion of this text, we consider only initial value problems.

Exercises

1. Write the equation of the line that passes through the points $(-1, 2, 3)$ and $(4, 0, -1)$ in $\mathbb{R}^3$, three-dimensional Euclidean space.
2. Find the general solution of the differential equation
$$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = 0.$$
3. Find the general solution of the differential equation
$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0.$$
4. Solve the IVP
$$y'' - 3y' + 2y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$
5. Solve (if possible) the BVP
$$y'' - 3y' + 2y = 0, \quad y(0) = 0, \quad y(1) = e.$$
6. Solve the IVP
$$y^{(4)} - y'' = 0, \qquad \begin{matrix} y(0) = 1, & y'(0) = 0, \\ y''(0) = -1, & y'''(0) = 0. \end{matrix}$$
7. Solve the differential equation
$$\frac{dy}{dx} = (y+2)(y+1).$$

8. Solve the IVP

$$\frac{dy}{dx} = e^y \sin x, \qquad y(0) = 0.$$

9. Find the equations of the planes tangent to the surface

$$z = f(x, y) = x^2 - 2x + y^2 - 2y + 2$$

at the points $(x, y, z) = (1, 1, 0)$ and $(x, y, z) = (0, 2, 2)$.

10. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$$

and, for each eigenvalue, find a corresponding eigenvector.

11. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

and, for each eigenvalue, find a corresponding eigenvector.

12. Write the following differential equations as systems of first-order ODEs:

$$y'' - 5y' + 6y = 0$$

$$-y'' - 2y' = 7\cos(y')$$

$$y^{(4)} - y'' + 8y' + y^2 = e^x.$$