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A Superposed Log-Linear Failure Intensity Model for Repairable Artillery Systems

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This article investigates complex repairable artillery systems that include several failure modes. We derive a superposed process based on a mixture of nonhomogeneous Poisson processes in a minimal repair model. This allows for a bathtub-shaped failure intensity that models artillery data better than currently used methods. The method of maximum likelihood is used to estimate model parameters and construct confidence intervals for the cumulative intensity of the superposed process. Finally, we propose an optimal maintenance policy for repairable systems with bathtub-shaped intensity and apply it to the artillery-failure data.

Key Words: Maximum Likelihood, Minimal Repair; Nonhomogeneous Poisson Process; Optimal Preventive Maintenance; Proportional Age-Reduction Model; Superposed Process.

1. Introduction

MODERN artillery consists of numerous parts working together in a complex system, and artillery reliability depends on numerous potential failure modes. Due to the prohibitive cost of testing field artillery during the manufacturing phase, equipment is subject to a high risk of early failures. Unless the system is badly damaged, failures are followed by repairs that sometimes help eliminate future failures from the same failure mode.

Repair processes of this type can emulate a *minimal repair* model in which the repair or the substitution of a failed part tends to have a negligible effect on overall system reliability, restoring the system performance to the exact same condition as it was just before the failure. Because the system is restored to its current state (immediately preceding the most recent failure), the assumption of minimal repair reveals a failure pattern governed by a nonhomogeneous Poisson process (NHPP). By defining the expected number of failures in the time interval $(0, t]$ as $\Lambda(t)$, we have corresponding intensity function $\lambda(t) = d\Lambda(t)/dt$.

The NHPP has been widely used in modeling failure frequency for repairable systems (Ascher and Feingold (1984), Krivtsov (2007)). The most popular NHPPs include the power law process (PLP) (Crow (1974)), which has intensity function $\lambda(t) = (\beta/\alpha)(t/\alpha)^{\beta-1}$, and the log-linear process (Cox and Lewis (1966)), with intensity $\lambda(t) = \alpha e^{\beta t}$. Both the PLP and the LLP have been introduced to model the failure patterns of a repairable system having monotonic intensity, i.e., decreasing failure patterns with $\beta < 1$ for the PLP ($\beta < 0$ for the LLP) or increasing failure patterns with $\beta > 1$ for the PLP ($\beta > 0$ for the LLP).

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With the described artillery, however, a system is subject both to early (or *infant mortality*) failures due to the presence of defective parts or assembly defects that are not screened out completely through the burn-in process, as well as wear-out failures caused by deterioration phenomena. This causes a nonmonotonic trend in the failure data, in which the intensity function initially decreases (as defective parts are weeded out of the system), followed by a long period of constant or near-constant intensity until wear-out finally occurs, at which time the intensity function begins to increase. This is called the *bathtub-shaped* failure intensity, which is typical for large and complex equipment with a number of different failure modes (Nelson (1988)). The PLP and LLP are too simplistic to accommodate this bathtub characteristic of the failure process.

As an alternative, unions of several independent NHPPs called *superposed* Poisson processes (SPPs) have been developed to model this kind of nonmonotonic failure intensity. When any subsystem failure can independently cause the system to break down, the superposed model is a natural model for system failure. For an SPP based on J independent processes, let $N_j(t)$ be the number of failures in $(0, t]$ for the jth subsystem ($j = 1, 2, \ldots, J$) with the intensity function $\lambda_j(t) = dE[N_j(t)]/dt$. The number of failures in $(0, t]$ for the system in the SPP is characterized by $N(t) = \sum_{i=1}^{J} N_j(t)$. If $N_j(t)$, $j = 1, 2, \ldots, J$, are independent nonhomogeneous Poisson processes, then $N(t)$ is also a nonhomogeneous Poisson process with intensity function $\lambda(t) = \sum_{j=1}^{J} \lambda_j(t)$.

SPPs have found successful application in modeling software reliability, where early detection and removal of coding errors can sometimes lead to reliability growth. For example, the Musa–Okumoto (1984) process is based on $J = 2$ with $\lambda_1(t) = \alpha_1/(t + \alpha_2)$ and $\lambda_2(t) = \beta_1\beta_2 t^{\beta_1 - 1}$. More recently, Pulcini (2001a) proposed the superposition of two independent PLPs (called the "superposed power law process" (S-PLP)) to model the bathtub shaped-failure pattern of a repairable system with intensity function

$$\lambda(t) = \frac{\beta_1}{\alpha_1}\left(\frac{t}{\alpha_1}\right)^{\beta_1 - 1} + \frac{\beta_2}{\alpha_2}\left(\frac{t}{\alpha_2}\right)^{\beta_2 - 1},$$

with α_j and $\beta_j > 0$ for $j = 1, 2$. In Pulcini's model, the parameters β_1, β_2 determine the failure patterns of a repairable system. For example, $\beta_1 < 1$ models the failure pattern of a system improving over time, while $\beta_2 > 1$ models that of a system deteriorating over time. As a result, the S-PLP with $\beta_1 < 1$ and $\beta_2 > 1$ is able to model a repairable system with bathtub-shaped failure intensity.

Guida and Pulcini (2009) proposed the bathtub-bounded intensity process (BBIP) represented by the following superposed intensity function

$$\lambda(t) = ae^{-t/b} + \alpha\left(1 - e^{-t/\beta}\right), \qquad a, b, \alpha, \beta > 0,$$

where the first component represents a log-linear process with decreasing intensity function and the latter component is a bounded intensity process with increasing bounded intensity function (Pulcini (2001b)). Guida and Pulcini (2009) show that the BBIP is able to model the failure pattern of a repairable system subject both to early failures and to deterioration phenomena and features a finite asymptote as the system age increases.

In this paper, we consider superposed Poisson processes for modeling field artillery failure-time data that exhibit a bathtub-shaped failure intensity function. We will find that existing models, including the S-PLP and the BBIP, do not adequately capture the nonmonotonic trend in the failure process for this field artillery. Because of this, we propose a superposed model of two independent log-linear processes to analyze the reliability of repairable systems experiencing both early failures and wear-out failures.

While the Republic of Korea (ROK) Army was performing artillery exercises in the field, they collected unscheduled maintenance data over a fixed period of time. The unscheduled maintenance actions were caused by system failures or imminent failures. Figure 1 shows an event plot of eight artillery repair-data sets along with their respective observation periods and reported failure times. In general, many failures were observed during the early and final periods of data collection.

Because they allow for a bathtub-shaped failure intensity, both the S-PLP and BBIP are reasonable candidates to model the artillery failure data. However, there is strong evidence in Figure 2 to suggest both models are inadequate in this case. Figure 2 shows the cumulative failure plot for the first artillery failure-data set (ID-1) along with cumulative failure estimates for the S-PLP and BBIP based on the statistical estimation methods in Pulcini (2001a, 2001b). Both models fail to adequately characterize the failure intensity of the artillery data. The BBIP is especially inflexible in this case and fails to correctly model either early failures (from burn in) or later

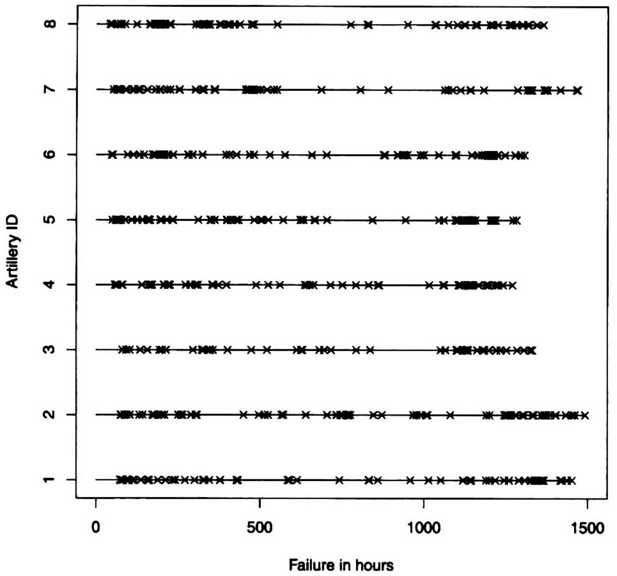


FIGURE 1. Event Plot Showing Failure Times for Eight Sets of ROK Artillery-Repair Data Across a 1500-Hour Period of Observation.

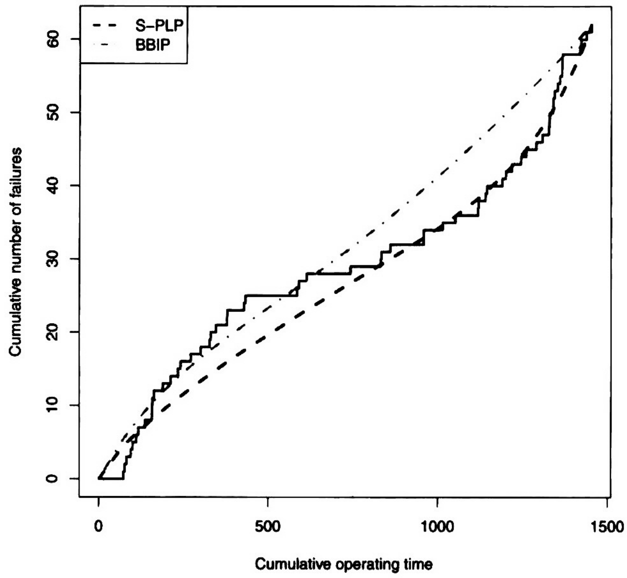


FIGURE 2. Plot of the Cumulative Number of Failures vs. Cumulative Operating Time for ROK Artillery ID-1, Along with the S-PLP and the BBIP Model Fits.

failures due to wear. These findings help motivate the derivation of a new model in the next section.

In the following section, we introduce a superposed log-linear process to model ROK Army artillery system failures and we derive the maximum-likelihood estimators (MLEs) for the model parameters, along with their confidence intervals. In the final part of the paper, we derive an optimal maintenance policy for a repairable system with bathtub-type intensity by minimizing the limiting expected cost per unit of time.

2. Superposed Log-linear Process for Bathtub-Shaped Intensity

Consider a repairable system with failures observed over the time interval $(0, T]$. Suppose that the failures are subject to two different failure modes and that each of the modes are modeled by an LLP with parameters α_j and β_j for $j = 1, 2$. We propose a *superposed log-linear process* (S-LLP) with intensity function

$$\lambda(t) = \alpha_1 e^{-\beta_1 t} + \alpha_2 e^{\beta_2 t}, \qquad \alpha_1, \alpha_2, \beta_1, \beta_2 > 0, \quad (1)$$

for $t \geq 0$. A key difference between the S-LLP and previously mentioned SPPs is evident in the parameters β_1 and β_2. By limiting them to be strictly nonnegative, the superposed process is a mixture of an increasing and a decreasing pair of intensity functions. Note that, if $\beta_1 = \beta_2 = 0$, the S-LLP is reduced to the homogeneous Poisson process (HPP) with a constant intensity $\lambda \equiv \alpha_1 + \alpha_2$. Unlike the S-PLP intensity function for $\beta_1 < 1$ or $\beta_2 < 1$, the S-LLP intensity function (1) is finite at $t = 0$.

The first derivative of intensity function $\lambda(t)$ with respect to t,

$$\lambda'(t) = -\alpha_1\beta_1 e^{-\beta_1 t} + \alpha_2\beta_2 e^{\beta_2 t},$$

is equal to $\alpha_2\beta_2 - \alpha_1\beta_1$ at $t = 0$; hence, $\lambda(t)$ is initially decreasing if and only if $\alpha_1\beta_1 > \alpha_2\beta_2$, and $\lambda'(t)$ is equal to 0 at $t = \tau$, where τ is given by

$$\tau = \frac{1}{\beta_1 + \beta_2} \ln\left(\frac{\alpha_1\beta_1}{\alpha_2\beta_2}\right).$$

The point with minimum intensity (τ) lies between 0 and T if $0 \leq \ln(\alpha_1\beta_1/\alpha_2\beta_2) \leq (\beta_1 + \beta_2)T$. The second derivative of the intensity function is

$$\lambda''(\tau) = \alpha_1\beta_1^2 e^{-\beta_1\tau} + \alpha_2\beta_2^2 e^{\beta_2\tau} > 0,$$

and τ represents a unique time-point having minimum intensity value

$$\lambda(\tau) = \alpha_1 e^{-\beta_1\tau}\left(\frac{\beta_1 + \beta_2}{\beta_2}\right).$$

That is, the intensity decreases until $t = \tau$, after which it increases from $t = \tau$ to $t = T$. Thus, the intensity function (1) reflects a *bathtub* behavior of sequential failures in a repairable system when the

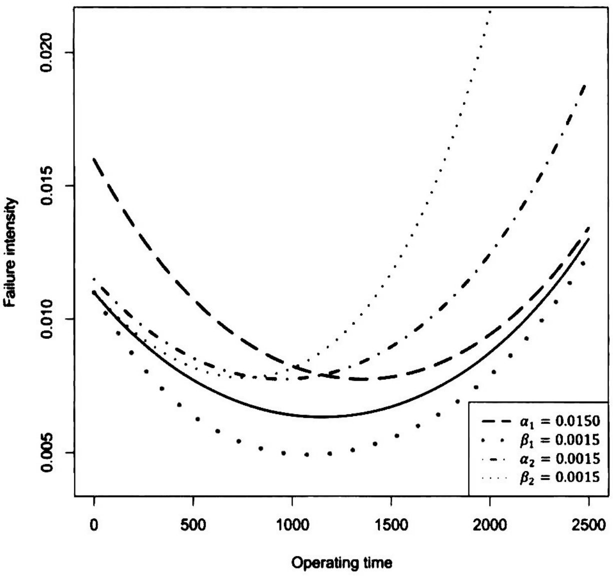


FIGURE 3. Behavior of the Intensity Function for the S-LLP. (The solid line represents $\lambda(t)$ with $\alpha_1 = 0.010$, $\beta_1 = 0.001$, $\alpha_2 = 0.001$, and $\beta_2 = 0.001$. The other curves represent the S-LLPs with different parameter values from the corresponding parameters in the solid line.)

system is subject both to early failures and to wear-out failures. The expected number of failures up to t is given by

$$\Lambda(t) = \int_0^t \lambda(u)\,\mathrm{d}u = \frac{\alpha_1}{\beta_1}\left(1 - e^{-\beta_1 t}\right) + \frac{\alpha_2}{\beta_2}\left(e^{\beta_2 t} - 1\right), \qquad t \geq 0.$$

Similar to the S-PLP, it is the sum of expected number of failures caused by each failure mode and it has an inflection point. Figure 3 illustrates the behavior of $\lambda(t)$ for selected values of the S-LLP parameters that satisfy the inequalities $\alpha_1\beta_1 > \alpha_2\beta_2$. Clearly, Figure 3 exhibits a bathtub-shaped intensity function for a variety of selected parameter values of $\boldsymbol{\theta} \equiv (\alpha_1, \alpha_2, \beta_1, \beta_2)^T$ for which $\alpha_1\beta_1 > \alpha_2\beta_2$.

2.1. Maximum-Likelihood Estimation

We consider the likelihood function for an NHPP with the first n failure times, $\boldsymbol{t} = t_1 < t_2 < \cdots < t_n$, that are observed until T. Under a failure-truncated sampling, the log-likelihood function of the S-LLP is

$$\ell\left(\alpha_1, \alpha_2, \beta_1, \beta_2; \boldsymbol{t}\right) = \sum_{i=1}^{n} \ln\left[\alpha_1 e^{-\beta_1 t_i} + \alpha_2 e^{\beta_2 t_i}\right] - \left[\frac{\alpha_1}{\beta_1}\left(1 - e^{-\beta_1 t_n}\right) + \frac{\alpha_2}{\beta_2}\left(e^{\beta_2 t_n} - 1\right)\right], \quad (2)$$

and t_n is replaced by T under a time-truncated sampling. The maximum-likelihood estimators (MLEs) of the parameters $\boldsymbol{\theta}$ can be found by solving the following likelihood equations:

$$\frac{\partial \ell}{\partial \alpha_1} = \sum_{i=1}^{n} \frac{e^{-\beta_1 t_i}}{\alpha_1 e^{-\beta_1 t_i} + \alpha_2 e^{\beta_2 t_i}} - \frac{1}{\beta_1}\left(1 - e^{-\beta_1 t_n}\right) = 0,$$

$$\frac{\partial \ell}{\partial \beta_1} = \sum_{i=1}^{n} \frac{-\alpha_1 t_i e^{-\beta_1 t_i}}{\alpha_1 e^{-\beta_1 t_i} + \alpha_2 e^{\beta_2 t_i}} + \left[\frac{\alpha_1}{\beta_1^2}\left(1 - e^{-\beta_1 t_n}\right) - \frac{\alpha_1 t_n}{\beta_1} e^{-\beta_1 t_n}\right] = 0,$$

$$\frac{\partial \ell}{\partial \alpha_2} = \sum_{i=1}^{n} \frac{e^{\beta_2 t_i}}{\alpha_1 e^{-\beta_1 t_i} + \alpha_2 e^{\beta_2 t_i}} - \frac{1}{\beta_2}\left(e^{\beta_2 t_n} - 1\right) = 0,$$

$$\frac{\partial \ell}{\partial \beta_2} = \sum_{i=1}^{n} \frac{\alpha_2 t_i e^{\beta_2 t_i}}{\alpha_1 e^{-\beta_1 t_i} + \alpha_2 e^{\beta_2 t_i}} + \left[\frac{\alpha_2}{\beta_2^2}\left(e^{\beta_2 t_n} - 1\right) - \frac{\alpha_2 t_n}{\beta_2} e^{\beta_2 t_n}\right] = 0. \quad (3)$$

Obviously, there is no closed-form solution to the MLEs in Equation (3), and these equations must be solved numerically. Even though $\ell(\alpha_1, \alpha_2, \beta_1, \beta_2; \boldsymbol{t})$ is an amalgamation of relatively well-behaved (generally concave) functions, a general search method such as Newton–Raphson is slow to work across four dimensions. In Appendix A, we describe a slightly more efficient numeric method based on a conditional-likelihood method used by Cox and Lewis (1966, p. 46).

Once the MLEs of the model parameters have been obtained, the MLEs of other quantities of interest, such as the expected number of failures up to a given time, $\Lambda(t)$, as well as the probability distribution of the number of failures occurring in a future time interval $\Pr\{N(T, T + \Delta) = k\}$ can be given as

$$\widehat{\Lambda}(t) = \frac{\hat{\alpha}_1}{\hat{\beta}_1}\left(1 - e^{-\hat{\beta}_1 t}\right) + \frac{\hat{\alpha}_2}{\hat{\beta}_2}\left(e^{\hat{\beta}_2 t} - 1\right) \quad (4)$$

and

$$\widehat{\Pr}\{N(T, T + \Delta) = k\} = \frac{\left[\widehat{\Lambda}(T + \Delta) - \widehat{\Lambda}(T)\right]^k}{k!} \cdot e^{-\left[\widehat{\Lambda}(T+\Delta) - \widehat{\Lambda}(T)\right]},$$

for $k = 0, 1, 2, \ldots$ and $\widehat{\Lambda}(T) = n$.

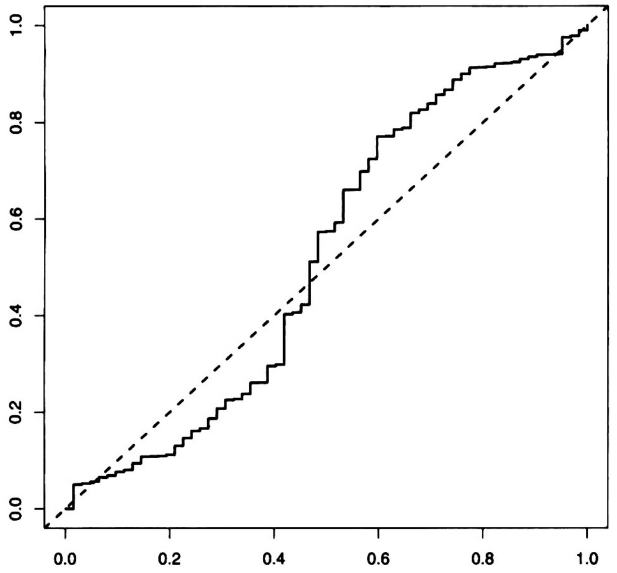


FIGURE 4. Total Time on Test (TTT) Plot for Failure Data of ROK Artillery ID-1.

We can construct confidence intervals for these and other functions based on standard errors derived from the (observed) Fisher information matrix. A large-sample approximation of estimated standard errors of the ML estimators is given by the estimated variance–covariance matrix $\widehat{\boldsymbol{\Sigma}}_{\hat{\boldsymbol{\theta}}}$ for $\hat{\boldsymbol{\theta}} \equiv (\hat{\alpha}_1, \hat{\beta}_1, \hat{\alpha}_2, \hat{\beta}_2)^{\mathrm{T}}$, where $\widehat{\boldsymbol{\Sigma}}_{\hat{\boldsymbol{\theta}}}$ is computed as the inverse of the estimated Fisher information matrix. We relegate computational details of these results to Appendix B.

We are primarily interested in constructing confidence intervals for $\Lambda(t)$ instead of the basic parameter set $\boldsymbol{\theta}$. We approximate the standard error for $\widehat{\Lambda}(t)$ using properties of $\widehat{\boldsymbol{\Sigma}}_{\hat{\boldsymbol{\theta}}}$ and by using the delta method on Equation (4). In general, for a differentiable real-valued function $g(\boldsymbol{\theta})$, the approximate standard error of $\hat{g} \equiv g(\hat{\boldsymbol{\theta}})$ can be obtained by using the delta method as

$$\widehat{\text{s.e.}}(\hat{g}) = \left(\sum_{i=1}^{4} \left(\frac{\partial g}{\partial \theta_i}\bigg|_{\hat{\boldsymbol{\theta}}} \right)^2 \widehat{\text{Var}}(\hat{\theta}_i) + \sum_{i=1}^{4} \sum_{j \neq i}^{4} \left(\frac{\partial g}{\partial \theta_i}\bigg|_{\hat{\boldsymbol{\theta}}} \right) \left(\frac{\partial g}{\partial \theta_j}\bigg|_{\hat{\boldsymbol{\theta}}} \right) \cdot \widehat{\text{Cov}}(\hat{\theta}_i, \hat{\theta}_j) \right)^{1/2}, \quad (5)$$

where $(\theta_1, \theta_2, \theta_3, \theta_4) \equiv (\alpha_1, \beta_1, \alpha_2, \beta_2)$.

When the function $g(\boldsymbol{\theta})$ is invertible, the approximate standard error (5) is exactly the same as that given by the estimated variance–covariance matrix relative to the log-likelihood function reparameterized in terms of g. On the other hand, when $g(\boldsymbol{\theta})$ is not invertible (as in the case of $\Lambda(t)$), the log-likelihood function cannot be reparameterized directly and the delta method seems to be the only available method that does not require resampling methods (Guida and Pulcini (2009)). The approximate $(1-\gamma)\%$ confidence interval for the function g results in either

$$\hat{g} \pm z_{\gamma/2} \cdot \widehat{\text{s.e.}}(\hat{g}) \quad \text{or} \quad \hat{g} \exp\{\pm z_{\gamma/2} \cdot \widehat{\text{s.e.}}(\hat{g})/\hat{g}\}$$

using the normal approximation or the lognormal approximation, respectively. Although the normal assumptions (based on asymptotic properties of the MLE) are not perfectly realized for $\widehat{\Lambda}(t)$ at small values of t, we do not consider transformations in this case because confidence intervals for $\Lambda(t)$ are of more interest at values of t not close to zero.

2.2. Analysis of Artillery-Repair Data

The proposed model was applied to field-repair data of eight sets of artillery systems. Each artillery system was subject to minimal repair at time of failure and all failure data for the eight artillery systems were treated as failure-truncated samples. As shown in Figure 1, due to a number of failures observed during the early and final periods of data collection, the bathtub-type failure intensity potentially seems to be appropriate to describe the failure pattern of the artillery systems.

In practice, decisions concerning failure patterns have been made using graphical techniques or statistical trend tests (Rigdon and Basu (2000)). The total time on test (TTT) plot (Barlow and Davis (1977)), for one, helps reveal failure patterns through curvature. A bathtub-shaped failure process can be observed in a TTT plot by an S-shaped function. For example, the first artillery data set (ID-1) consists of 62 failure times observed until $t_{62} = 1452$ hours and its TTT plot is contained in Figure 4, which shows a clear indication of a bathtub-shaped intensity function for failure data of the equipment. It was observed that the bathtub-shaped patterns of failures are also dominant in the other artilleries in the TTT plots (we do not repeat plots here).

As a test for nonmonotonic trends in recurrent failures, a large positive value of Vaurio's statistic

TABLE 1. Statistical Trend Tests for Eight Sets of ROK Artillery-Repair Data

ID	Vaurio's statistic (V)	p-Value
1	3.9790	<0.0001
2	1.9863	0.0235
3	2.5615	0.0052
4	2.5967	0.0047
5	3.3462	0.0004
6	3.6172	0.0001
7	3.5558	0.0002
8	4.3812	<0.0001

(Vaurio (1999))

$$V = \frac{\sum_{i=1}^{n} |t_i - t_n/2| - nt_n/4}{t_n\sqrt{n/48}}$$

indicates the presence of a bathtub behavior, while a large negative value indicates the presence of an inverse-bathtub behavior. In applying the Vaurio's trend test to eight sets of artillery-repair data, we summarized the test results in Table 1, along with their p-values. At significance level $\alpha = 0.05$, the test results provide statistical evidence of the bathtub behavior of failure intensity with respect to failure data of the eight artillery systems.

Based on the log-likelihood in Equation (2), MLEs for S-LLP model parameters were computed using the artillery data and the algorithm described in Appendix A. Estimates of the S-LLP model, along with their standard errors, are given in Table 2. To obtain the standard errors of $\hat{\boldsymbol{\theta}} \equiv (\hat{\alpha}_1, \hat{\beta}_1, \hat{\alpha}_2, \hat{\beta}_2)^{\mathrm{T}}$, the estimated variance–covariance matrix was computed as the inverse of the estimated Fisher information matrix. For artillery ID-1, for instance, the estimated variance–covariance matrix is

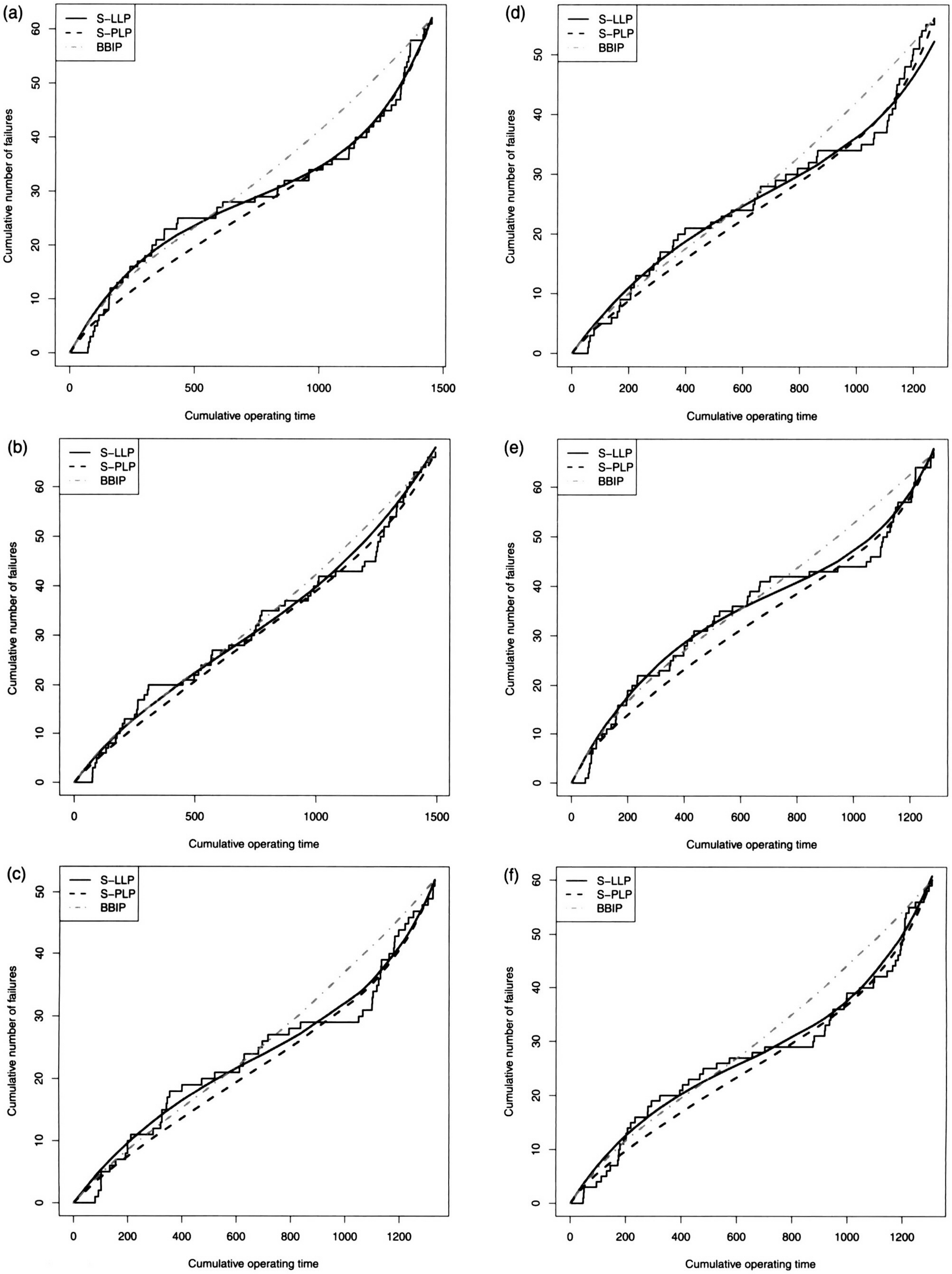
$$\widehat{\boldsymbol{\Sigma}}_{\hat{\boldsymbol{\theta}}} = \begin{bmatrix} 746.2495 & 24.5022 & 7.6580 & -10.1828 \\ & 1.4427 & 0.8064 & -1.1044 \\ & & 1.1620 & -1.6548 \\ & & & 2.3817 \end{bmatrix} \times 10^{-6},$$

and the standard errors of $\hat{\boldsymbol{\theta}}$ are the square roots of diagonal elements in $\widehat{\boldsymbol{\Sigma}}_{\hat{\boldsymbol{\theta}}}$. Approximate 95% confidence intervals for $\boldsymbol{\theta}$ can be constructed using Eq. (14) from Appendix B, using the log-normal approximation.

Using the MLEs for the S-LLP model parameters, we can obtain the MLE for the expected number of failures, $\widehat{\Lambda}(t)$, from Equation (4). Figure 5 depicts $\widehat{\Lambda}(t)$ under the S-PLP and the BBIP assumption, as well as $\widehat{\Lambda}(t)$ under the S-LLP assumption. The figure shows that the S-LLP provides the best representation for the whole data set of eight artillery systems. Admittedly, the S-LLP model as well as the S-PLP and the BBIP models fail to handle the early failure data. All of the artillery-repair data contain a time

TABLE 2. ML Estimates of the S-LLP Parameters and Their Standard Errors for Eight Sets of Artillery-Repair Data (corresponding approximate 95% confidence intervals under the log-normal approximation in parentheses)

ID	$\hat{\alpha}_1$	$\widehat{\text{s.e.}}(\hat{\alpha}_1)$	$\hat{\beta}_1$	$\widehat{\text{s.e.}}(\hat{\beta}_1)$	$\hat{\alpha}_2$	$\widehat{\text{s.e.}}(\hat{\alpha}_2)$	$\hat{\beta}_2$	$\widehat{\text{s.e.}}(\hat{\beta}_2)$
1	0.0850	0.0273	0.0028	0.0012	0.0005	0.0011	0.0037	0.0015
	(0.0453, 0.1596)		(0.0012, 0.0065)		(1.1×10^{-5}, 0.0273)		(0.0017, 0.0084)	
2	0.0599	0.0231	0.0021	0.0014	0.0055	0.0053	0.0017	0.0007
	(0.0282, 0.1275)		(0.0006, 0.0077)		(0.0008, 0.0359)		(0.0008, 0.0039)	
3	0.0568	0.0197	0.0018	0.0010	0.0004	0.0008	0.0042	0.0017
	(0.0288, 0.1120)		(0.0006, 0.0053)		(7.0×10^{-6}, 0.0200)		(0.0019, 0.0092)	
4	0.0630	0.0206	0.0020	0.0010	0.0006	0.0008	0.0041	0.0012
	(0.0322, 0.1196)		(0.0007, 0.0054)		(3.4×10^{-5}, 0.0098)		(0.0022, 0.0074)	
5	0.1136	0.0302	0.0027	0.0009	0.0008	0.0013	0.0037	0.0013
	(0.0675, 0.1912)		(0.0015, 0.0050)		(4.2×10^{-5}, 0.0162)		(0.0019, 0.0076)	
6	0.0793	0.0262	0.0026	0.0012	0.0006	0.0011	0.0040	0.0015
	(0.0415, 0.1515)		(0.0011, 0.0064)		(2.1×10^{-5}, 0.0192)		(0.0020, 0.0083)	
7	0.1216	0.0311	0.0030	0.0008	0.0005	0.0005	0.0032	0.0007
	(0.0736, 0.2008)		(0.0018, 0.0052)		(0.0001, 0.0032)		(0.0020, 0.0050)	
8	0.1334	0.0349	0.0034	0.0009	0.0004	0.0004	0.0039	0.0007
	(0.0799, 0.2229)		(0.0021, 0.0056)		(0.0001, 0.0023)		(0.0027, 0.0057)	


(a)
S-LLP
S-PLP
BBIP
Cumulative number of failures
0
10
20
30
40
50
60
0
500
1000
1500
Cumulative operating time
(d)
S-LLP
S-PLP
BBIP
Cumulative number of failures
0
10
20
30
40
50
0
200
400
600
800
1000
1200
Cumulative operating time
(b)
S-LLP
S-PLP
BBIP
Cumulative number of failures
0
10
20
30
40
50
60
0
500
1000
1500
Cumulative operating time
(e)
S-LLP
S-PLP
BBIP
Cumulative number of failures
0
10
20
30
40
50
60
0
200
400
600
800
1000
1200
Cumulative operating time
(c)
S-LLP
S-PLP
BBIP
Cumulative number of failures
0
10
20
30
40
50
0
200
400
600
800
1000
1200
Cumulative operating time
(f)
S-LLP
S-PLP
BBIP
Cumulative number of failures
0
10
20
30
40
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60
0
200
400
600
800
1000
1200
Cumulative operating time

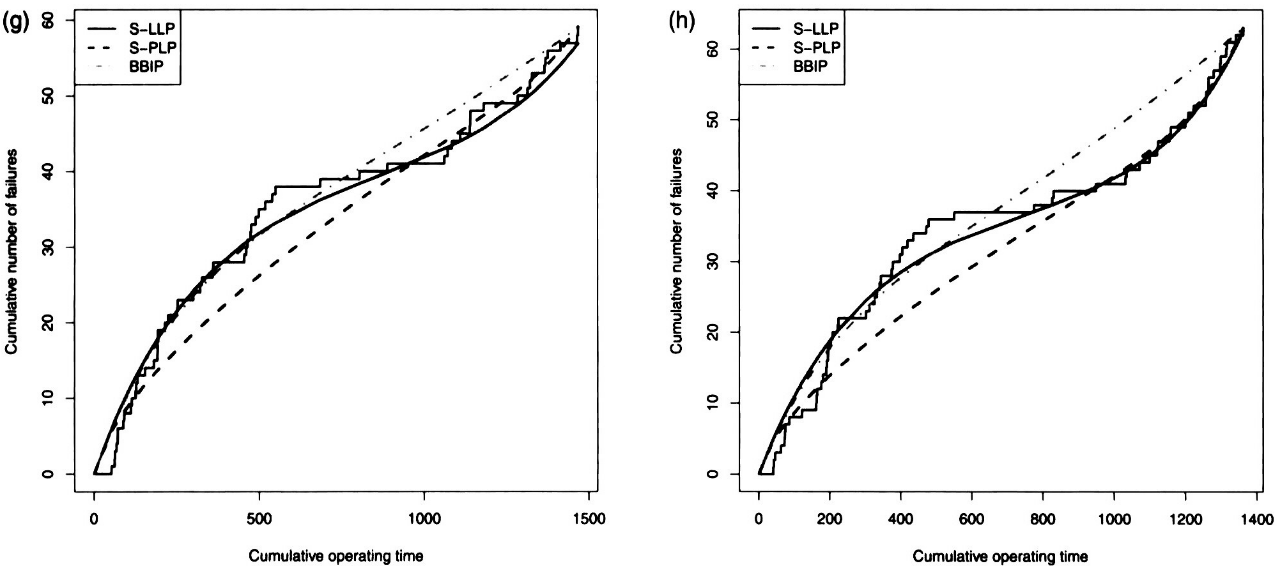


FIGURE 5. Observed Cumulative Number of Failures Along with the Expected Number of Failures $\Lambda(t)$ Under the S-PLP, the BBIP, and the S-LLP Assumption for Eight Sets of ROK Artillery-Repair Data. (a) ID-1, (b) ID-2, (c) ID-3, (d) ID-4, (e) ID-5, (f) ID-6, (g) ID-7, (h) ID-8.

lag to first failure (see Figure 1), and it is not easy for the superposed models to represent a bathtub-shaped failure intensity that can explicitly fit the time lag to first failure. More complex and highly parameterized models, for instance, that include the addition of a constant into the intensity functions of S-PLP, BBIP, and S-LLP, may be an alternative to capture the time lag, but it will greatly increase model complexity as well.

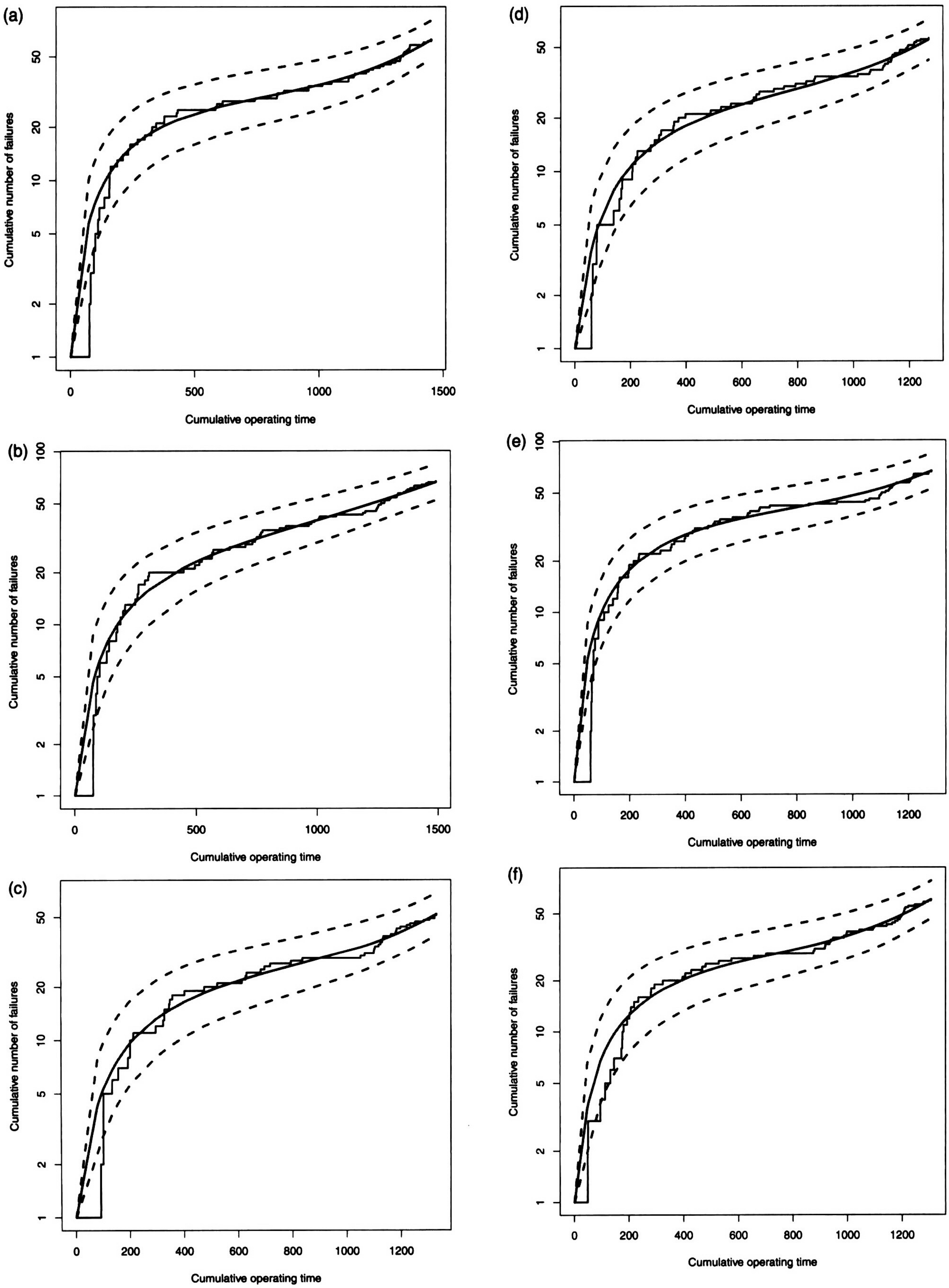
Under the S-LLP model, 95% (pointwise) confidence intervals for $\Lambda(t)$ are plotted for eight individual sets of artillery-repair data in Figure 6. For instance, the estimated minimum intensity point (τ) for ID-1 is $\hat{\tau} = 733.22$ hours (as shown in Table 3) and the 95% confidence interval for $\Lambda(t)$ at that time is $(21.23, 38.65)$.

3. Optimal Preventive Maintenance for the S-LLP

Like many expensive and complex systems, artillery must be constantly repaired, serviced, and adequately maintained during its lifetime of use. To determine when system replacement would be economical, an adequate maintenance strategy is crucial to save operational costs by improving the system reliability. In general, a maintenance action can be classified largely into perfect (preventive) maintenance, minimal repair, and imperfect (preventive) maintenance (Pham and Wang (1996)). In establishing an optimal maintenance policy for a complex system, the NHPP has played a key role in modeling the random occurrences of failures where the repair or the substitution of a failed part brings the system to the condition just prior to failure (or *minimal repair*). Gilardoni and Colosimo (2007) derived an optimal perfect maintenance strategy by minimizing expected cost per unit of time for the NHPP, particularly based on the PLP model. Tsai et al. (2011) developed a maintenance strategy for repairable systems by combining imperfect maintenance actions at prescheduled times and minimal repair actions for failures if they occur, using the PLP to model the recurrent failures.

Preventive maintenance (PM) is reasonable only when the failure intensity is increasing, and the PM should be done only when the failure intensity reaches a sufficiently high value. However, we may be curious to know whether the PM is needed for nonmonotonic failure intensity and, if necessary, how the PM strategy is best implemented. Even though the analysis of nonmonotonic trends in systems with recurrent failures are found in the current research literature, optimal maintenance policies for such systems are not yet fully studied. The purpose of this section is to derive an optimal PM policy based on the S-LLP model. The optimality is defined as the minimization of the expected cost per unit of time for the PM policy.

Consider a repairable system operated over the


(a)
(b)
(c)
(d)
(e)
(f)
Cumulative number of failures
Cumulative operating time
1
2
5
10
20
50
100
0
200
400
500
600
800
1000
1200
1500

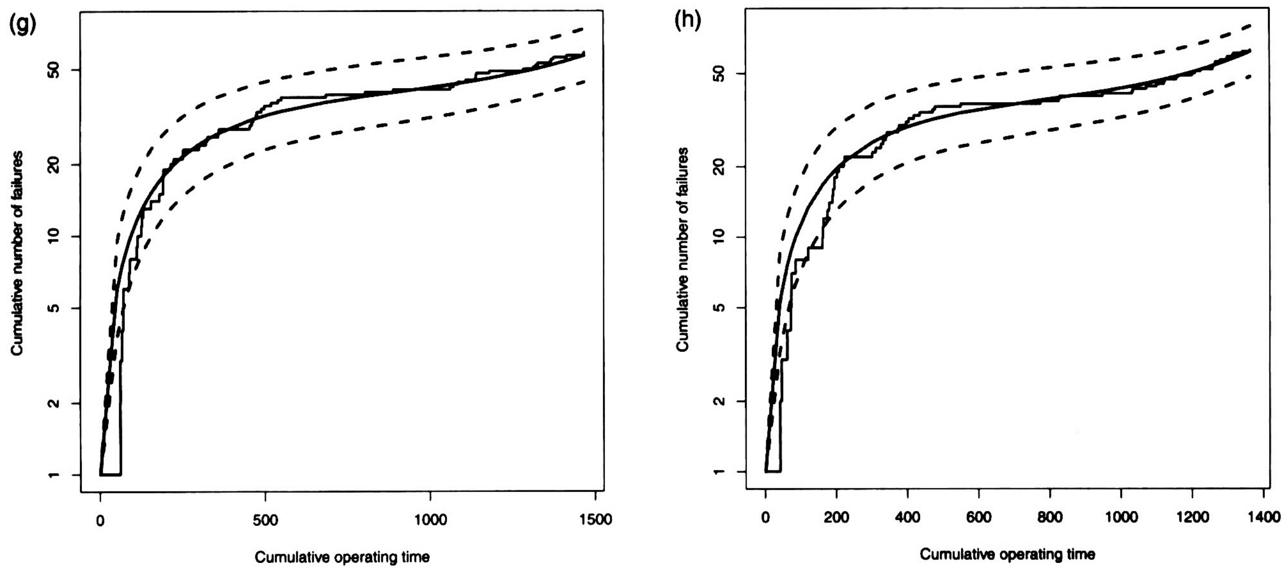


FIGURE 6. 95% Pointwise Confidence Intervals for $\Lambda(t)$ Under the S-LLP Model for Eight Sets of ROK Artillery-Repair Data. (The vertical axis is log scaled for better representation of the confidence intervals.) (a) ID-1, (b) ID-2, (c) ID-3, (d) ID-4, (e) ID-5, (f) ID-6, (g) ID-7, (h) ID-8.

time interval $(0, T]$ and assume the following conditions:

1. PM check points are scheduled after $\xi_1 < \xi_2 < \cdots < \xi_{m-1} < \xi_m$ units of time. Based on the PM epochs, we decompose the time interval $(0, T]$ as $(0, \xi_1] \cup (\xi_1, \xi_2] \cup \cdots \cup (\xi_{m-1}, \xi_m] \cup (\xi_m, T]$.
2. PM activity is performed when failure intensity reaches an unacceptably high level. A repair (or replacement) action with negligible repair time is executed at each PM check point, causing a fixed cost C_{PM}. The PM activity improves the condition of the system noticeably, but it does not bring the system condition to as good as new (imperfect PM).
3. Between successive PM check points, a minimal repair is carried out after each failure and the expected cost for each repair is C_{MR}.
4. Repair costs are independent of failure times.

For the S-LLP model, initial decreasing intensity is mainly due to manufacturing defects that are not screened out completely during the burn-in process. The defects can be mostly eliminated through the PM (or overhaul) activity. As a result, the decreasing intensity stage will disappear after the first PM and the intensity function after the PM activity is not bathtub-shaped any more. For this reason, perfect PM that restores the system condition to as good as new is not realistic for the S-LLP model.

We introduce the proportional age reduction (PAR) model (Malik, (1979)) to accommodate the flexibility for failure process after the PM activity. The PAR model assumes that the PM activities change the system time-of-failure-rate function to some newer time but not all the way to zero. Each PM activity reduces the age of the equipment by a quantity proportional to the operating time elapsed from the most recent PM. The PAR model has been frequently adopted in developing optimal maintenance policies due to its flexibility (Shin et al. (1996), Chan and Shaw (1993), and Pulcini (2001c)). The combined model of the bathtub-shaped failure process and the PM activity, which is called the PAR-S-LLP model herein, assumes that the failure process between two successive PM activities follows the S-LLP model and that, when the ith PM is performed, the age t of the system is reduced to a fraction ρ $(0 \le \rho \le 1)$ of the epoch ξ_i of the PM activity for $i = 2, 3, \ldots, m$, where ρ is the improvement parameter.

Based on the PAR-S-LLP model, the initial failure intensity, i.e., the intensity function up to the first PM point ξ_1, is

$$\lambda_1(t) = \alpha_1 e^{-\beta_1 t} + \alpha_2 e^{\beta_2 t}, \qquad t \le \xi_1,$$

and the conditional intensity function at generic time t in the interval $(\xi_{i-1}, \xi_i]$, given that the last PM has

TABLE 3. First PM Check Points (ξ_1's) Under a Variety of PM and Repair Costs for the Eight Artillery Systems

		C_{PM}/C_{MR}						
ID	ρ	0.1	0.5	1.0	5.0	10	20	τ
1	1.0	845.69	922.23	968.42	1116.06	1199.39	1295.02	
	0.9	881.94	956.41	1001.45	1145.60	1226.99	1320.47	733.22
	0.8	920.95	1001.53	1046.44	1187.63	1266.84	1357.69	
2	1.0	794.75	890.32	950.71	1155.69	1279.73	1429.55	
	0.9	817.16	915.23	976.29	1182.18	1306.13	1455.41	626.82
	0.8	854.10	954.39	1016.38	1224.33	1348.82	1498.25	
3	1.0	811.29	887.20	932.19	1072.84	1150.55	1238.78	
	0.9	844.75	918.25	961.98	1099.04	1174.94	1261.17	696.20
	0.8	882.91	959.91	1003.09	1136.61	1210.31	1294.01	
4	1.0	788.56	859.11	901.55	1036.02	1111.30	1197.40	
	0.9	828.12	895.33	935.92	1065.60	1138.59	1222.32	698.80
	0.8	873.50	939.07	979.77	1106.22	1176.93	1258.02	
5	1.0	826.77	894.14	935.16	1067.77	1143.66	1231.75	
	0.9	867.47	932.25	971.73	1100.12	1173.86	1259.60	738.92
	0.8	923.67	977.97	1018.31	1144.53	1216.25	1299.41	
6	1.0	754.45	824.86	867.45	1003.44	1080.15	1168.23	
	0.9	781.48	851.46	893.51	1027.43	1102.82	1189.35	636.99
	0.8	816.98	889.37	931.14	1062.72	1136.44	1220.86	
7	1.0	988.00	1071.94	1123.05	1288.49	1382.90	1491.94	
	0.9	1031.59	1113.40	1163.26	1324.68	1416.83	1523.24	866.77
	0.8	1083.46	1166.98	1217.18	1375.70	1465.35	1568.62	
8	1.0	863.27	935.49	979.25	1119.69	1199.15	1290.39	
	0.9	900.91	971.18	1013.77	1150.54	1227.96	1316.88	757.48
	0.8	946.87	1017.39	1060.17	1194.03	1269.12	1355.22	

been performed at ξ_{i-1}, is given by

$$\begin{aligned}\lambda_i(t|\xi_{i-1}) &\equiv \lambda_1(t+\tau-\rho\xi_{i-1}) \\ &= \alpha_1 e^{-\beta_1(t+\tau-\rho\xi_{i-1})} + \alpha_2 e^{\beta_2(t+\tau-\rho\xi_{i-1})} \quad (6)\end{aligned}$$

for $i = 2, 3, \ldots, m$. By shifting to the amount of time τ after each PM, the PAR-S-LLP model (6) does not take the initial stage of decreasing intensity into account. Under the PAR-S-LLP model, the factor ρ measures the degree of improvement introduced into the system by the PM activities. When $\rho = 0$, the PM effectiveness is null; hence, such a case is not considered here. When $\rho = 1$, the age reduction is complete and the PM activities restore the system condition to the minimum intensity at time τ rather than the system condition at $t = 0$.

For the PAR-S-LLP model, the total expected cost is given by

$$\begin{aligned}&TC_{(0,T]}(\boldsymbol{\xi}) \\ &= C_{MR}\left\{\int_0^{\xi_1}\lambda_1(u)\,du + \sum_{i=2}^{m+1}\int_{\xi_{i-1}}^{\xi_i}\lambda_i(u \mid \xi_{i-1})\,du\right\} \\ &\quad + mC_{PM} \quad (7)\end{aligned}$$

for the set of PM epochs $\boldsymbol{\xi} \equiv (\xi_1, \ldots, \xi_m)$. Here, ξ_{m+1} denotes T. Because $TC_{(0,T]}(\boldsymbol{\xi})$ is an increasing func-

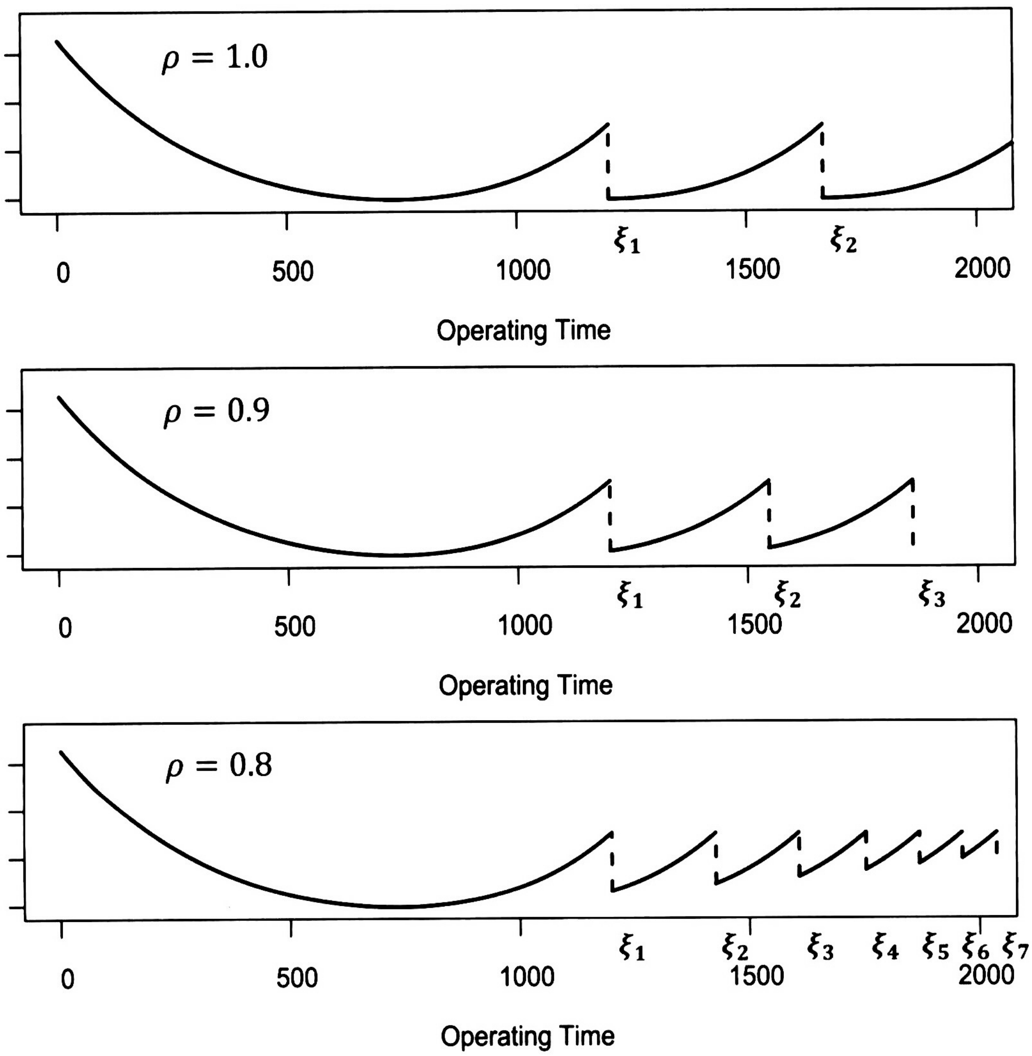


FIGURE 7.Plot of the PM Check Points for the PAR-S-LLP Model (ξ_1 = 1,300, τ = 733.22).

tion of T, we work with the expected cost per unit of time

$$
\begin{aligned}
LC(\boldsymbol{\xi}) &= \frac{TC_{(0,T]}(\boldsymbol{\xi})}{T} \\
&= \frac{1}{T}\Bigg[C_{\mathrm{MR}}\Bigg\{\int_0^{\xi_1} \lambda_1(u)\,\mathrm{d}u \\
&\qquad + \sum_{i=2}^{m+1}\int_{\xi_{i-1}}^{\xi_i} \lambda_i(u|\xi_{i-1})\,\mathrm{d}u\Bigg\} \\
&\qquad + mC_{\mathrm{PM}}\Bigg]. \qquad (8)
\end{aligned}
$$

After the first PM point, ξ_1 is obtained by solving the Equation (8) using some numerical search algorithms, such as Newton's method; the subsequent PM point ξ_i for $i = 2, 3, \ldots, m$ is obtained as

$$
\xi_i = \begin{cases} \left(\dfrac{1-\rho^i}{1-\rho}\right)\xi_1 - \left(\dfrac{1-\rho^{i-1}}{1-\rho}\right)\tau & \text{for } \rho \neq 1 \\ i\xi_1 - (i-1)\tau & \text{for } \rho = 1. \end{cases}
$$

The details of these results are relegated to Appendix C.

For example, Figure 7 plots the PM check points with various ρ values under the PAR-S-LLP model. The ith PM activity is assumed to be performed whenever failure intensity reaches the intensity value at first PM point ξ_1 for $i = 2, 3, \ldots, m$. As the ρ value decreases, the improvement effect of the PM activity is smaller; thus, more PM activities are required in the interval $(0, T]$ and starting failure intensities increase after more PM activities are carried out.

TABLE 4. PM Check Points (ξ_1's) Under a Variety of PM and Repair Costs for ID-1 ($m = 5$)

		C_{PM}/C_{MR}					
ρ	ξ_i	0.1	0.5	1.0	5	10	20
1.0	ξ_1	845.69	922.23	968.42	1116.06	1199.39	1295.02
	ξ_2	958.16	1111.23	1203.63	1498.91	1665.56	1856.83
	ξ_3	1070.63	1300.24	1438.83	1881.75	2131.73	2418.63
	ξ_4	1183.10	1489.24	1674.04	2264.59	2597.89	2980.43
	ξ_5	1295.56	1678.25	1909.24	2647.43	3064.06	3542.23
0.9	ξ_1	881.94	956.41	1001.45	1145.60	1226.99	1320.47
	ξ_2	942.46	1083.96	1169.54	1443.42	1598.06	1775.66
	ξ_3	996.93	1198.76	1320.81	1711.46	1932.02	2185.34
	ξ_4	1045.96	1302.07	1456.96	1952.69	2232.59	2554.06
	ξ_5	1090.08	1395.06	1579.50	2169.80	2503.10	2885.90
0.8	ξ_1	920.95	1001.53	1046.44	1187.63	1266.84	1357.69
	ξ_2	924.48	1069.54	1150.37	1404.51	1547.10	1710.61
	ξ_3	927.31	1123.94	1233.51	1578.01	1771.30	1992.96
	ξ_4	929.57	1167.47	1300.02	1716.81	1950.67	2218.83
	ξ_5	931.38	1202.29	1353.24	1827.86	2094.16	2399.53

3.1. Optimal Preventive Maintenance for Artillery-Repair Data

Based on the MLEs from eight sets of the artillery-repair data in Table 2, optimum PM policies can be studied under a variety of cost variables. Table 3 presents the estimated optimum first PM points (ξ_1's) according to the ratio of PM cost to repair cost, C_{PM}/C_{MR}, along with minimal intensity points (τ's) for comparison. Table 3 shows that the ξ_1 solutions are consistently larger than τ for all eight sets of the artillery-repair data because we need not perform the PM during decreasing failure intensity. Note that the first PM points are delayed at smaller ρ values. Table 4, which presents the PM check points (ξ_i's) for ID-1 when five PM activities ($m = 5$) performed, shows that the improvement effect of the PM activity is less at smaller ρ values; hence, more frequent PM must be done during a given time interval. In general, because the repair cost after failure is much higher than the PM cost, the PM activity should be executed as frequently as possible in order to improve the reliability of repairable systems, but it may cause unnecessary costs. If the earlier part of infant mortality can be eliminated through design changes or the burn-in process, general PM policies in the literature can be applied to the artillery systems by assuming the PLP with increasing failure intensity with only wear-out failures.

In our artillery example, the actual artillery systems operated without any scheduled preventive maintenance. Instead, the military employed a fixed block-maintenance schedule in which the overhaul is scheduled every 5 years. The overhaul cost is much higher than the repair cost. At such condition, that is, $C_{PM}/C_{MR} \gg 1$, the PM activities must be delayed enough to save the overhaul costs. Table 4 also shows that time to the first PM point is delayed at higher cost ratios of C_{PM}/C_{MR}.

4. Conclusion

The repairable ROK artillery systems featured in this paper have exhibited a bathtub-shaped failure intensity that can be modeled with a superposed process based on a mixture of nonhomogeneous Poisson processes in a minimal repair model. The derived S-LLP model is shown to be much better at fitting the repair data than previous models that have been derived for bathtub-shaped failure intensities. Although the estimation problem is computationally cumbersome, the MLEs are straightforward and can be used to construct approximate confidence bounds for cumulative failure intensity. In applying the optimal

PM policy to eight sets of artillery repair data possessing a bathtub-shaped failure intensity, we investigate the effects of the preventive (or overhaul) maintenance on the reliability of such systems with non-monotonic failure intensity.

Recently, Arab et al. (2012) provided a Bayesian inference for the piecewise exponential model to predict the reliability of multiple repairable systems. By formally incorporating the prior information about system performance into hierarchical Bayesian models and empirical Bayesian models, they sought to account for a similarity of multiple systems, mainly via common prior distributions for the parameters of the piecewise exponential model. Future research directions include a hierarchical Bayesian model or mixedeffects model for multiple repairable systems with the bathtub-shaped failure intensity to simultaneously analyze the artillery data at one time. The hierarchical Bayesian model (mixed effects model) is expected to capture a similarity (system-to-system variation) in failure intensity of multiple repairable systems. Tan et al. (2007) proposed a generalized linear mixed model for the PLP to analyze failure data from multicopy repairable generator systems. Because the mean function in the S-LLP is nonlinear while the mean (link) function in the PLP is linear, the inference procedure for the parameters in the mixedeffects model is more challenging with the S-LLP model.

Appendix A
ML Method Under Modified Log-Likelihood Function

For the log-linear process, Cox and Lewis (1966, p. 46) derived MLEs based on conditional likelihood by conditioning on some aspect of the observed data to remove nuisance parameters. We introduce an approach similar to that in Pulcini (2001a) by multiplying both sides of the first and the third equations in Equation (3) by α_1 and α_2, respectively, to obtain

$$\frac{\alpha_1}{\beta_1}\left(1-e^{-\beta_1 t_n}\right)+\frac{\alpha_2}{\beta_2}\left(e^{\beta_2 t_n}-1\right)=n. \qquad (9)$$

Let n_1 and n_2 be the expected number of failures up to time t_n for the two pieces of the intensity function. Then n_1 and n_2 satisfy

$$n_1=\frac{\alpha_1}{\beta_1}\left(1-e^{-\beta_1 t_n}\right),\quad n_2=\frac{\alpha_2}{\beta_2}\left(e^{\beta_2 t_n}-1\right), \qquad (10)$$

with $n = n_1 + n_2$. Equation (10) can be rewritten with respect to α_1 and α_2, respectively

$$\alpha_1=\frac{n_1\beta_1}{1-e^{-\beta_1 t_n}},\quad \alpha_2=\frac{n_2\beta_2}{e^{\beta_2 t_n}-1}, \qquad (11)$$

and by substituting Equation (11) and $n_1 = n - n_2$ into Equation (2), the log-likelihood function is modified as

$$\begin{aligned}\ell(\beta_1,\beta_2,n_2;\boldsymbol{t}) &= \sum_{i=1}^{n}\ln\left[\frac{(n-n_2)\beta_1}{1-e^{-\beta_1 t_n}}e^{-\beta_1 t_i}+\frac{n_2\beta_2}{e^{\beta_2 t_n}-1}e^{\beta_2 t_i}\right] \\ &\quad - n. \end{aligned} \qquad (12)$$

Thus, the ML estimators $\hat{\beta}_1$, $\hat{\beta}_2$ and $\hat{n}_2$ can be obtained by maximizing Equation (12) subject to the nonlinear constraint $e^{\beta_2 t_n} < (\alpha_2/\beta_2)n + 1$. After $\hat{\beta}_1$, $\hat{\beta}_2$, $\hat{n}_2$ have been obtained, $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are easily derived by plugging in the MLEs into Equation (11). Because a closed-form solution for the ML estimators $\hat{\beta}_1$, $\hat{\beta}_2$, and $\hat{n}_2$ does not exist, numerical search methods such as the Newton–Raphson algorithm can be used.

Appendix B
Approximate Confidence Intervals for $\boldsymbol{\theta}$

The estimated variance–covariance matrix is computed as the inverse of the estimated Fisher information matrix, namely,

$$\hat{\boldsymbol{\Sigma}}_{\hat{\boldsymbol{\theta}}} \equiv \mathcal{I}_{\hat{\boldsymbol{\theta}}}^{-1} = \begin{bmatrix} -\frac{\partial^2\ell}{\partial\alpha_1^2} & -\frac{\partial^2\ell}{\partial\alpha_1\partial\beta_1} & -\frac{\partial^2\ell}{\partial\alpha_1\partial\alpha_2} & -\frac{\partial^2\ell}{\partial\alpha_1\partial\beta_2} \\ & -\frac{\partial^2\ell}{\partial\beta_1^2} & -\frac{\partial^2\ell}{\partial\beta_1\partial\alpha_2} & -\frac{\partial^2\ell}{\partial\beta_1\partial\beta_2} \\ & & -\frac{\partial^2\ell}{\partial\alpha_2^2} & -\frac{\partial^2\ell}{\partial\alpha_2\partial\beta_2} \\ & & & -\frac{\partial^2\ell}{\partial\beta_2^2} \end{bmatrix}^{-1},$$

evaluated at $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}$. The entries in the Fisher information matrix $\mathcal{I}_{\hat{\boldsymbol{\theta}}}$ are

$$-\frac{\partial^2\ell}{\partial\alpha_1^2}\bigg|_{\hat{\boldsymbol{\theta}}}=\sum_{i=1}^{n}\frac{e^{-\hat{\beta}_1 t_i}}{(\hat{\alpha}_1 e^{-\hat{\beta}_1 t_i}+\hat{\alpha}_2 e^{\hat{\beta}_2 t_i})^2},$$

$$\begin{aligned}-\frac{\partial^2\ell}{\partial\alpha_1\partial\beta_1}\bigg|_{\hat{\boldsymbol{\theta}}} &= \sum_{i=1}^{n}\frac{\hat{\alpha}_2 t_i e^{(\hat{\beta}_2-\hat{\beta}_1)t_i}}{(\hat{\alpha}_1 e^{-\hat{\beta}_1 t_i}+\hat{\alpha}_2 e^{\hat{\beta}_2 t_i})^2} \\ &\quad +\frac{t_n e^{-\hat{\beta}_1 t_n}}{\hat{\beta}_1}-\frac{1}{\hat{\beta}_1^2}(1-e^{-\hat{\beta}_1 t_n}),\end{aligned}$$

$$-\frac{\partial^2\ell}{\partial\alpha_1\partial\alpha_2}\bigg|_{\hat{\boldsymbol{\theta}}}=\sum_{i=1}^{n}\frac{e^{(\hat{\beta}_2-\hat{\beta}_1)t_i}}{(\hat{\alpha}_1 e^{-\hat{\beta}_1 t_i}+\hat{\alpha}_2 e^{\hat{\beta}_2 t_i})^2},$$

$$-\frac{\partial^2\ell}{\partial\alpha_1\partial\beta_2}\bigg|_{\hat{\boldsymbol{\theta}}} = \sum_{i=1}^{n} \frac{\hat{\alpha}_2 t_i e^{(\hat{\beta}_2-\hat{\beta}_1)t_i}}{(\hat{\alpha}_1 e^{-\hat{\beta}_1 t_i} + \hat{\alpha}_2 e^{\hat{\beta}_2 t_i})^2},$$

$$-\frac{\partial^2\ell}{\partial\beta_1^2}\bigg|_{\hat{\boldsymbol{\theta}}} = \sum_{i=1}^{n} \frac{-\hat{\alpha}_1\hat{\alpha}_2 t_i^2 e^{(\hat{\beta}_2-\hat{\beta}_1)t_i}}{\left(\hat{\alpha}_1 e^{-\hat{\beta}_1 t_i} + \hat{\alpha}_2 e^{\hat{\beta}_2 t_i}\right)^2} + \frac{2\hat{\alpha}_1}{\hat{\beta}_1^3}\left(1 - e^{-\hat{\beta}_1 t_n}\right) - \frac{2\hat{\alpha}_1 t_n e^{-\hat{\beta}_1 t_n}}{\hat{\beta}_1^2} - \frac{\hat{\alpha}_1 t_n^2 e^{-\hat{\beta}_1 t_n}}{\hat{\beta}_1},$$

$$-\frac{\partial^2\ell}{\partial\beta_1\partial\alpha_2}\bigg|_{\hat{\boldsymbol{\theta}}} = \sum_{i=1}^{n} \frac{-\hat{\alpha}_1 t_i e^{(\hat{\beta}_2-\hat{\beta}_1)t_i}}{(\hat{\alpha}_1 e^{-\hat{\beta}_1 t_i} + \hat{\alpha}_2 e^{\hat{\beta}_2 t_i})^2},$$

$$-\frac{\partial^2\ell}{\partial\beta_1\partial\beta_2}\bigg|_{\hat{\boldsymbol{\theta}}} = \sum_{i=1}^{n} \frac{-\hat{\alpha}_1\hat{\alpha}_2 t_i^2 e^{(\hat{\beta}_2-\hat{\beta}_1)t_i}}{(\hat{\alpha}_1 e^{-\hat{\beta}_1 t_i} + \hat{\alpha}_2 e^{\hat{\beta}_2 t_i})^2},$$

$$-\frac{\partial^2\ell}{\partial\alpha_2^2}\bigg|_{\hat{\boldsymbol{\theta}}} = \sum_{i=1}^{n} \frac{e^{2\hat{\beta}_2 t_i}}{(\hat{\alpha}_1 e^{-\hat{\beta}_1 t_i} + \hat{\alpha}_2 e^{\hat{\beta}_2 t_i})^2},$$

$$-\frac{\partial^2\ell}{\partial\alpha_2\partial\beta_2}\bigg|_{\hat{\boldsymbol{\theta}}} = \sum_{i=1}^{n} \frac{-\hat{\alpha}_1 t_i e^{(\hat{\beta}_2-\hat{\beta}_1)t_i}}{(\hat{\alpha}_1 e^{-\hat{\beta}_1 t_i} + \hat{\alpha}_2 e^{\hat{\beta}_2 t_i})^2} - \frac{1}{\hat{\beta}_2^2}\left(e^{\hat{\beta}_1 t_n} - 1\right),$$

$$-\frac{\partial^2\ell}{\partial\beta_2^2}\bigg|_{\hat{\boldsymbol{\theta}}} = \sum_{i=1}^{n} \frac{-\hat{\alpha}_1\hat{\alpha}_2 t_i^2 e^{(\hat{\beta}_2-\hat{\beta}_1)t_i}}{(\hat{\alpha}_1 e^{-\hat{\beta}_1 t_i} + \hat{\alpha}_2 e^{\hat{\beta}_2 t_i})^2} + \frac{2\hat{\alpha}_2}{\hat{\beta}_2^3}\left(e^{\hat{\beta}_2 t_n} - 1\right) - \frac{2\hat{\alpha}_2 t_n e^{\hat{\beta}_2 t_n}}{\hat{\beta}_2^2} + \frac{\hat{\alpha}_2 t_n^2 e^{\hat{\beta}_2 t_n}}{\hat{\beta}_2}.$$

The estimated standard errors of the parameters are

$$\widehat{\text{s.e.}}(\hat{\alpha}_1) = \sqrt{(1,1) \text{ entry in } \widehat{\boldsymbol{\Sigma}}_{\hat{\boldsymbol{\theta}}}},$$
$$\widehat{\text{s.e.}}(\hat{\beta}_1) = \sqrt{(2,2) \text{ entry in } \widehat{\boldsymbol{\Sigma}}_{\hat{\boldsymbol{\theta}}}},$$
$$\widehat{\text{s.e.}}(\hat{\alpha}_2) = \sqrt{(3,3) \text{ entry in } \widehat{\boldsymbol{\Sigma}}_{\hat{\boldsymbol{\theta}}}},$$
$$\widehat{\text{s.e.}}(\hat{\beta}_2) = \sqrt{(4,4) \text{ entry in } \widehat{\boldsymbol{\Sigma}}_{\hat{\boldsymbol{\theta}}}}.$$

Either a normal or a log-normal distribution can be used to approximate the exact distribution of the ML estimators. Because the asymptotic distribution of the MLEs $\hat{\boldsymbol{\theta}}$ is multivariate normal, approximate $(1-\gamma)\%$ confidence intervals for the parameters are, respectively,

$$\hat{\alpha}_1 \pm z_{\gamma/2} \cdot \widehat{\text{s.e.}}(\hat{\alpha}_1),$$
$$\hat{\beta}_1 \pm z_{\gamma/2} \cdot \widehat{\text{s.e.}}(\hat{\beta}_1),$$
$$\hat{\alpha}_2 \pm z_{\gamma/2} \cdot \widehat{\text{s.e.}}(\hat{\alpha}_2),$$
$$\hat{\beta}_2 \pm z_{\gamma/2} \cdot \widehat{\text{s.e.}}(\hat{\beta}_2), \tag{13}$$

where $z_{\gamma/2}$ is the $(1-\gamma/2)$ quantile of the standard normal distribution.

If the parameter is constrained to be nonnegative, the normal approximation is sometimes unsatisfactory because the distribution of the MLE can be highly skewed. In these cases, the asymptotic normal distribution for $\ln\hat{\boldsymbol{\theta}}$ rather than for just $\hat{\boldsymbol{\theta}}$ can be used. The log-normal approximation generally works better because it avoids negative lower confidence limits, as often occurs under the normal approximation for small or moderate sample sizes (Black and Rigdon (1996), Guida and Pulcini (2009)). Using the log-normal approximation of the MLEs, the approximate $(1-\gamma)\%$ confidence interval of the parameters are, respectively (Pulcini (2001a)),

$$\hat{\alpha}_1 \exp\left\{\pm z_{\gamma/2} \cdot \widehat{\text{s.e.}}(\hat{\alpha}_1)/\hat{\alpha}_1\right\},$$
$$\hat{\beta}_1 \exp\left\{\pm z_{\gamma/2} \cdot \widehat{\text{s.e.}}(\hat{\beta}_1)/\hat{\beta}_1\right\},$$
$$\hat{\alpha}_2 \exp\left\{\pm z_{\gamma/2} \cdot \widehat{\text{s.e.}}(\hat{\alpha}_2)/\hat{\alpha}_2\right\},$$
$$\hat{\beta}_2 \exp\left\{\pm z_{\gamma/2} \cdot \widehat{\text{s.e.}}(\hat{\beta}_2)/\hat{\beta}_2\right\}. \tag{14}$$

Appendix C
PM Points for the PAR-S-LLP Model

We assume that the ith PM activity is performed whenever failure intensity reaches the intensity value at first PM point ξ_1 for $i = 2, 3, \ldots, m$; that is, $\lambda_1(\xi_1) = \lambda_2(\xi_2|\cdot) = \cdots = \lambda_m(\xi_m|\cdot)$. The second PM check point ξ_2 is obtained by solving the following equation:

$$f(\xi_2) = \alpha_1 e^{-\beta_1(\xi_2+\tau-\rho\xi_1)} + \alpha_2 e^{\beta_2(\xi_2+\tau-\rho\xi_1)} - \left(\alpha_1 e^{-\beta_1\xi_1} + \alpha_2 e^{\beta_2\xi_1}\right) = 0.$$

$f(\xi_2)$ is a strictly increasing function of ξ_2 and a unique solution exists such that

$$\xi_2 = \xi_1 - \tau + \rho\xi_1 = (1+\rho)\xi_1 - \tau.$$

Similarly,

$$\xi_3 = \xi_1 - \tau + \rho\xi_2 = (1+\rho+\rho^2)\xi_1 - (1+\rho)\tau,$$
$$\vdots$$
$$\begin{aligned}\xi_{m-1} &= \xi_1 - \tau + \rho\xi_{m-2} \\ &= (1+\rho+\cdots+\rho^{m-2})\xi_1 - (1+\rho+\cdots+\rho^{m-3})\tau, \\ \xi_m &= \xi_1 - \tau + \rho\xi_{m-1} \\ &= (1+\rho+\cdots+\rho^{m-1})\xi_1 - (1+\rho+\cdots+\rho^{m-2})\tau.\end{aligned}$$

The ith PM check point ξ_i can be written as

$$\xi_i = \begin{cases} \left(\frac{1-\rho^i}{1-\rho}\right)\xi_1 - \left(\frac{1-\rho^{i-1}}{1-\rho}\right)\tau, & \rho \neq 1 \\ i\xi_1 - (i-1)\tau, & \rho = 1 \end{cases}$$

for $i = 2, 3, \ldots, m$.

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