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1. Introduction

In this paper, we discuss the *nonlinear* extremal problem

$$\Gamma(\psi) := \sup_{\substack{f \in H^2 \\ \|f\|=1}} \left| \frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \psi f^2 \, dz \right| \tag{1}$$

and its relationship to the classical *linear* extremal problem

$$\Lambda(\psi) := \sup_{\substack{F \in H^1 \\ \|F\|_1=1}} \left| \frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \psi F \, dz \right| \tag{2}$$

and the compression of Toeplitz operators to the model spaces $H^2 \ominus \Theta H^2$. In the above, $\mathbb{D}$ denotes the open unit disk, $\partial\mathbb{D}$ denotes the unit circle, ψ is a rational function whose poles do not lie on $\partial\mathbb{D}$, H^p is the classical Hardy space [11, 19, 24], and Θ is a finite Blaschke product. We refer to ψ as the *kernel* of the extremal problems (1) and (2). Since the focus of this paper is primarily on H^2, we let $\|\cdot\|$ (with no subscript) denote the norm on H^2 while $\|\cdot\|_1$ denotes the norm on H^1. Despite the nonlinear nature of (1), our investigation will involve the theory of *linear* operators, specifically the relatively new field of truncated Toeplitz operators and complex symmetric operators [17, 18, 39].

The linear extremal problem (2) and its natural generalization to H^p $(1 \leq p \leq \infty)$ have a long and storied history dating back to the early twentieth century [11, 19, 20, 23, 27, 34]. For the moment, we single out several notable classical results which motivate this paper.

We first mention a theorem of Macintyre and Rogosinski [27, 34] (see also [23, p. 33]) which asserts that for any given rational kernel ψ having no poles on $\partial\mathbb{D}$ there exists an F_e in the unit ball of H^1 for which

$$\Lambda(\psi) = \frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \psi F_e \, dz.$$

This function F_e is called an *extremal function* for (2). In general, F_e is not necessarily unique. Indeed, one need only consider the extremal problem

$$\Lambda\left(\frac{1}{z^2}\right) = \sup_{\substack{F \in H^1 \\ \|F\|_1=1}} |F'(0)| = 1$$

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to see that the functions
$$\frac{(z-\alpha)(1-\overline{\alpha}z)}{1+|\alpha|^2}, \qquad \alpha \in \mathbb{C}, \tag{3}$$
all serve as extremal functions. (On the other hand, the analogous extremal problem for H^p with $p > 1$ always has a unique solution [11, Thm. 8.1]). Nevertheless, for general rational ψ we can always choose an extremal function F_e which is the square of an H^2 function [23, p. 33], from which it follows that
$$\Lambda(\psi) = \Gamma(\psi).$$
Using the theory of truncated Toeplitz operators and the structure of the underlying Jordan model spaces, we give a new proof of this fact in Proposition 1.

A second result worth mentioning here is due to Fejér [13], who showed that for any complex numbers $c_0, c_1, \ldots, c_n$ one has
$$\Lambda\left(\frac{c_0}{z} + \cdots + \frac{c_n}{z^{n+1}}\right) = \|H\|,$$
where H is the Hankel matrix (blank entries to be treated as zeros)
$$H = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_n & \\ c_2 & \cdots & c_n & & \\ \vdots & \iddots & & & \\ c_n & & & & \end{pmatrix}$$
and $\|H\|$ denotes the operator norm of H (i.e., the largest singular value of H). In the special case when $c_0 = c_1 = \cdots = c_n = 1$, Egerváry [12] provided the explicit formula
$$\Lambda\left(\frac{1}{z} + \frac{1}{z^2} + \cdots + \frac{1}{z^{n+1}}\right) = \frac{1}{2}\sec\frac{(n+1)\pi}{2n+3} \tag{4}$$
and showed that the polynomial
$$\frac{4}{2n+3}\left[\sin\frac{(n+1)\pi}{2n+3} + z\sin\frac{n\pi}{2n+3} + \cdots + z^n\sin\frac{\pi}{2n+3}\right]^2 \tag{5}$$
is an extremal function (In particular, note that this is the square of an H^2 function). Golusin in [21] finds an extremal function for the original (general) Fejér extremal problem as well as some others [20]. We refer the reader to [11, 19, 23] for further references.

Fejér's result can also be phrased in terms of Toeplitz matrices, which turn out to represent the simplest type of *truncated Toeplitz operator* – a class of operators which figures prominently in our approach to (1) (see Subsection 2.4). Indeed, if T is the following lower-triangular Toeplitz matrix
$$T = \begin{pmatrix} c_n & & & & \\ \vdots & \ddots & & & \\ c_2 & \cdots & c_n & & \\ c_1 & c_2 & \cdots & c_n & \\ c_0 & c_1 & c_2 & \cdots & c_n \end{pmatrix},$$
then $UT = H$, where U is the permutation matrix
$$U = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix},$$
and thus $\|T\| = \|H\|$.

In Theorem 1 and Corollary 2 we extend Fejér's result and show that $\Lambda(\psi) = \Gamma(\psi)$ is equal to the norm of a certain truncated Toeplitz operator on a canonically associated *model space* $K_\Theta = H^2 \ominus \Theta H^2$, where Θ is a certain finite Blaschke product associated with ψ. To relate $\Gamma(\psi)$ to the norm of this associated truncated Toeplitz operator, we develop a new variational characterization of the norm of a complex symmetric operator, something interesting in its own right.

For a given rational ψ, the discussion above tells us that there is an extremal function $f_e \in \mathrm{ball}(H^2)$. In other words, the function f_e satisfies

$$\Gamma(\psi) = \frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \psi f_e^2 \, dz.$$

Using the theory of complex symmetric operators, we will show in Corollary 2 that f_e belongs to a certain model space K_Θ associated with $\Gamma(\psi)$. More importantly, we establish a procedure to compute f_e along with necessary and sufficient conditions, in terms of truncated Toeplitz operators, which determine when the extremal function f_e is unique (up to a sign).

Next, we consider the obvious estimate

$$\Lambda(\psi) = \Gamma(\psi) \leq \max_{\zeta \in \partial\mathbb{D}} |\psi(\zeta)|.$$

Using some results from [37], we will show in Corollary 5 that equality holds if and only if $\psi = \lambda \frac{B_1}{B_2}$, where $\lambda \in \mathbb{C}$ and B_1, B_2 are finite Blaschke products with no common zeros and such that $\deg B_1 < \deg B_2$. In particular, this demonstrates the utility of relating $\Gamma(\psi)$ to the norm of a truncated Toeplitz operator.

Let us now suggest another application. Since, for $n = 0, 1, 2, \ldots$ and $\lambda \in \mathbb{D}$,

$$c_{n,\lambda} = \Lambda\left(\frac{n!}{(z-\lambda)^{n+1}}\right)$$

is the best constant in the inequality

$$|F^{(n)}(\lambda)| \leq c_{n,\lambda} \, \|F\|_1, \qquad F \in H^1,$$

we can use operator theoretic techniques to explicitly determine the constants $c_{n,\lambda}$ as well as the associated extremal functions. Although this problem has been well-studied [12, 20, 26, 27], we obtain the same results (see Theorem 3) using the new language of truncated Toeplitz operators.

In Section 9 we briefly explore how our results can be extended to handle certain kernels $\psi \in L^\infty(\partial\mathbb{D})$ which are not necessarily rational.

It should be remarked that (1) is not the first nonlinear extremal problem on the Hardy space to be explored. The reader is invited to consult [1, 3, 5, 14, 28, 41] for other examples of nonlinear extremal problems on H^p as well as other spaces of analytic functions. We thank D. Khavinson for pointing out these papers to us and for several enlightening conversations. In addition, we thank David Sherman for pointing out the identity in (62).

2. Preliminaries

As mentioned in the introduction, we will relate the extremal problems (1) and (2) to the emerging study of truncated Toeplitz operators and complex symmetric operators. This section lays out all of the appropriate definitions and basic results.

2.1. Model spaces. Given a rational function ψ whose poles do not lie on $\partial\mathbb{D}$, consider the associated finite Blaschke product

$$\Theta(z) = \prod_{j=1}^{n} \frac{z-\lambda_j}{1-\overline{\lambda_j} z} \tag{6}$$

whose zeros $\lambda_1, \lambda_2, \ldots, \lambda_n$, repeated according to multiplicity, are precisely the poles of ψ *which lie in* $\mathbb{D}$. The *degree*, $\deg\Theta$, of the Blaschke product Θ in (6) is defined to be n, the total number of zeros counted according to multiplicity. Note that ψ might have other poles which lie *outside* of the closed unit disk $\mathbb{D}^-$. However, these poles are *not* counted amongst the λ_j's. We also assume that ψ has at least one pole in $\mathbb{D}$, since otherwise $\Lambda(\psi) = \Gamma(\psi) = 0$, which is of no interest.

With the above Θ, we form the corresponding *model space*

$$K_\Theta := H^2 \ominus \Theta H^2.$$

More precisely, K_Θ is the closed subspace of H^2 defined by

$$H^2 \ominus \Theta H^2 := H^2 \cap (\Theta H^2)^\perp \tag{7}$$

where the implicit inner product is the standard L^2 inner product:

$$\langle u, v\rangle := \int_{\partial\mathbb{D}} u(\zeta)\overline{v(\zeta)}\frac{|d\zeta|}{2\pi}, \qquad u, v \in L^2 := L^2(\partial\mathbb{D}, \tfrac{|d\zeta|}{2\pi}). \tag{8}$$

The term 'model space' stems from the important role that K_Θ plays in the *model theory* for Hilbert space contractions – see [31, Part C]. Let us briefly recall several important facts about K_Θ. First note that Beurling's theorem [11, p. 114] asserts that ΘH^2 is a typical proper invariant subspace of finite codimension (since Θ is a finite Blaschke product) for the unilateral shift operator

$$[Sf](z) = zf(z), \quad f \in H^2.$$

It follows from a standard duality argument that K_Θ is a typical nontrivial, finite-dimensional invariant subspace for the backward shift operator

$$[S^*f](z) = \frac{f(z)-f(0)}{z}, \quad f \in H^2.$$

For further information on various function-theoretic aspects of the backward shift operator, we refer the reader to the seminal paper [10] and the recent texts [7, 35].

When the zeros $\lambda_1, \lambda_2, \ldots, \lambda_n$ of Θ are distinct, it is easy to verify that K_Θ is spanned by the *Cauchy kernels*

$$\frac{1}{1-\overline{\lambda_j}z}, \quad j = 1, 2, \ldots, n. \tag{9}$$

Thus each function f in K_Θ can be written uniquely as

$$f(z) = \frac{p(z)}{\prod_{j=1}^{n}(1-\overline{\lambda_j}z)}, \tag{10}$$

where $p(z)$ is an analytic polynomial of degree at most $n-1$. Conversely, by partial fractions, every such function belongs to K_Θ. In particular, each function in K_Θ is analytic in a neighborhood of $\mathbb{D}^-$. It is also not hard to see that the preceding discussion is still valid even if the zeros of Θ are not all distinct, so long as the terms in the denominator of (10) are repeated according to multiplicity and the Cauchy kernels in (9) are replaced by

$$\frac{1}{(1-\overline{\lambda_j}z)^m}, \quad 1 \le j \le n, \quad 1 \le m \le m_j, \tag{11}$$

where m_j denotes the multiplicity of λ_j as a zero of Θ.

From the representation (7) we find that K_Θ carries a natural isometric, conjugate-linear involution $C : K_\Theta \to K_\Theta$ defined in terms of boundary functions by

$$Cf := \overline{fz}\Theta. \tag{12}$$

Although, at first glance, the expression $\overline{fz}\Theta$ in (12) does not appear to correspond to the boundary values of an H^2 function, let alone one in K_Θ, a short computation using (7) reveals that $\langle Cf, \Theta h\rangle$ and $\langle Cf, \overline{zh}\rangle$ both vanish for all $h \in H^2$ and $f \in K_\Theta$, whence Cf indeed belongs to K_Θ.

An important aspect of the spaces K_Θ involves the *reproducing kernels*

$$k_w(z) = \frac{1 - \overline{\Theta(w)}\Theta(z)}{1 - \overline{w}z}, \quad z, w \in \mathbb{D}^-, \tag{13}$$

which enjoy the so-called *reproducing property*

$$\langle f, k_w\rangle = f(w), \quad w \in \mathbb{D}^-, \, f \in K_\Theta. \tag{14}$$

In light of the fact that Θ is a finite Blaschke product, note that k_w belongs to K_Θ for all w in $\mathbb{D}^-$ and that the analyticity of Θ on $\partial\mathbb{D}$ ensures that (13) is well-defined when both z and w belong to $\partial\mathbb{D}$. That (14) holds is a straightforward consequence of the Cauchy Integral Formula and the fact that $\langle f, z^n\Theta\rangle = 0$ for all $n \geq 0$. Letting P_Θ denote the orthogonal projection of L^2 onto K_Θ, we also observe that

$$[P_\Theta f](w) = \langle f, k_w\rangle, \quad f \in L^2, w \in \mathbb{D}^-. \tag{15}$$

Finally, another short computation reveals that

$$(Ck_w)(z) = \frac{\Theta(z) - \Theta(w)}{z - w}, \quad z, w \in \mathbb{D}^-. \tag{16}$$

2.2. Takenaka-Malmquist-Walsh bases. Suppose that the finite Blaschke product (6) has n zeros $\lambda_1, \lambda_2, \ldots, \lambda_n$ repeated according to their multiplicity. Although the functions from (11) form a basis for K_Θ, they are not orthonormal. However, there is a well-known orthonormal basis that can be constructed from these functions. Indeed, for $w \in \mathbb{D}$ we let B_w denote the disk automorphism

$$B_w(z) = \frac{z - w}{1 - \overline{w}z}$$

and consider the unit vectors

$$v_k(z) := \begin{cases} \dfrac{\sqrt{1 - |\lambda_1|^2}}{1 - \overline{\lambda_1}z} & \text{if } k = 1, \\ \left(\prod_{i=1}^{k-1} B_{\lambda_i}\right) \dfrac{\sqrt{1 - |\lambda_k|^2}}{1 - \overline{\lambda_k}z} & \text{if } 2 \leq k \leq n. \end{cases} \tag{17}$$

By (10), $v_k \in K_\Theta$ for all $1 \leq k \leq n$. A short computation based on the reproducing property and the fact that $B_w\overline{B_w} = 1$ on $\partial\mathbb{D}$ shows that for $j < k$ we have

$$\langle v_k, v_j\rangle = \left\langle \left(\prod_{i=j}^{k-1} B_{\lambda_i}\right) \frac{\sqrt{1 - |\lambda_k|^2}}{1 - \overline{\lambda_k}z}, \frac{\sqrt{1 - |\lambda_j|^2}}{1 - \overline{\lambda_j}z} \right\rangle = 0$$

(since $B_{\lambda_j}(\lambda_j) = 0$) while for $j = k$ we have $\langle v_j, v_j\rangle = 1$. Thus $\{v_1, v_2, \ldots, v_n\}$ constitutes an orthonormal basis for the model space K_Θ. A standard text refers to this observation as the Malmquist-Walsh Lemma [29, V.1] but these functions appeared as early as 1925 in a paper of Takenaka [40]. In light of this, we shall call $\{v_1, v_2, \ldots, v_n\}$ the *Takenaka-Malmquist-Walsh basis* for K_Θ.

We do wish to mention a particular important case that will be used in Section 8. Suppose that $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \lambda$ so that $\Theta = B_\lambda^n$. In this case, the functions

$$v_k(z) = \frac{\sqrt{1-|\lambda|^2}}{(1-\overline{\lambda}z)^k}(z-\lambda)^{k-1}, \quad 1 \le k \le n \tag{18}$$

form an orthonormal basis for K_Θ. Note that these vectors are simply an orthonormalization of the functions in (11).

2.3. Aleksandrov-Clark bases. It turns out that each model space K_Θ comes equipped with another natural family of orthonormal bases. Let us briefly describe their construction. Note that $|\Theta| = 1$ on $\partial\mathbb{D}$. Also note that $|\Theta'| > 0$ on $\partial\mathbb{D}$: write

$$\Theta(z) = \prod_{j=1}^{n} B_{\lambda_j}(z)$$

and use formal logarithmic differentiation and the fact that

$$B'_{\lambda_j}(z) = \frac{1-|\lambda_j|^2}{(1-\overline{\lambda_j}z)^2}$$

to get the identity

$$\Theta'(z) = \Theta(z)\sum_{j=1}^{n} \frac{1-|\lambda_j|^2}{(1-\overline{\lambda_j}z)(z-\lambda_j)}.$$

When $z = \zeta \in \partial\mathbb{D}$, we get

$$\Theta'(\zeta) = \Theta(\zeta)\overline{\zeta}\sum_{j=1}^{n} \frac{1-|\lambda_j|^2}{|1-\overline{\lambda_j}\zeta|^2}.$$

Since $|\Theta(\zeta)| = 1$ on $\partial\mathbb{D}$, the result follows.

Fix some $\beta \in \partial\mathbb{D}$ and note that since $|\Theta| = 1$ and $|\Theta'| > 0$ on $\partial\mathbb{D}$, the equation $\Theta(\zeta) = \beta$ has precisely n distinct solutions $\zeta_1, \zeta_2, \ldots, \zeta_n$ on $\partial\mathbb{D}$. Here, as before, n denotes the degree of the Blaschke product (6).

A short computation now shows that

$$\|k_{\zeta_j}\| = \sqrt{|\Theta'(\zeta_j)|}, \quad 1 \le j \le n.$$

Next we define unimodular constants $\omega_1, \omega_2, \ldots, \omega_n$ by

$$\omega_j := e^{\frac{i}{2}(\arg\beta - \arg\zeta_j)}, \quad 1 \le j \le n. \tag{19}$$

However the arguments of β and $\zeta_1, \zeta_2, \ldots, \zeta_n$ are selected, they are to remain consistent throughout the following. Now consider the functions

$$\begin{aligned} v_j(z) &:= \omega_j \frac{k_{\zeta_j}(z)}{\|k_{\zeta_j}\|} \\ &= \frac{\omega_j}{\sqrt{|\Theta'(\zeta_j)|}} \cdot \frac{1-\overline{\beta}\Theta(z)}{1-\overline{\zeta_j}z}. \end{aligned} \tag{20}$$

It is readily verified that $\langle v_j, v_k\rangle = \delta_{j,k}$ and, moreover, that

$$Cv_j = v_j, \quad 1 \le j \le n. \tag{21}$$

We refer to the basis in (20) as a *modified Aleksandrov-Clark basis* for K_Θ. The terminology stems from the fact that the v_j are the eigenvectors of a certain Aleksandrov-Clark operator on K_Θ [6, 8, 32, 38]. Such bases are discussed in further detail in [15, Sect. 8.1].

One notable advantage of using Aleksandrov-Clark bases over the Takenaka-Malmquist-Walsh basis is the fact that all Aleksandrov-Clark bases are *C-real* in the sense that (21) holds.

2.4. Truncated Toeplitz operators. As mentioned in the introduction, we will show in Corollary 2 that the extremum $\Gamma(\psi)$ in the nonlinear problem (1) coincides with the operator norm of a certain truncated Toeplitz operator on an associated model space. Let us now consider these operators.

Definition. For $\varphi \in L^\infty = L^\infty(\partial\mathbb{D}, \frac{|d\zeta|}{2\pi})$ and an inner function Θ, the operator $A_\varphi : K_\Theta \to K_\Theta$ defined by

$$A_\varphi f = P_\Theta(\varphi f) \tag{22}$$

is called the *truncated Toeplitz operator* on K_Θ with *symbol* φ.

Truncated Toeplitz operators are sometimes referred to as *compressed Toeplitz operators* since A_φ is simply the compression of the standard Toeplitz operator T_φ on H^2 to the model space K_Θ. Recall that if $P : L^2 \to H^2$ denotes the orthogonal projection from L^2 onto H^2 (called the Riesz or Szegö projection), then $T_\varphi f = P(\varphi f)$.

The operator norm of a truncated Toeplitz operator A_φ on K_Θ will be denoted by $\|A_\varphi\|_{K_\Theta \to K_\Theta}$ or, when the context is clear, simply by $\|A_\varphi\|$. It is easy to see from the definition (22) that A_φ is a bounded operator on K_Θ satisfying $\|A_\varphi\| \leq \|\varphi\|_\infty$. For further information, we direct the reader to the recent article [39], which appears destined to become the standard reference for truncated Toeplitz operators.

Maintaining the same notation as in (6), we observe that if the symbol φ of a truncated Toeplitz operator $A_\varphi : K_\Theta \to K_\Theta$ belongs to H^∞, then its eigenvalues are precisely the numbers $\varphi(\lambda_1), \varphi(\lambda_2), \ldots, \varphi(\lambda_n)$. It is perhaps easier to prove the corresponding statement for the adjoint $A_\varphi^* = A_{\overline{\varphi}}$. To see this, note that since $\Theta(\lambda_i) = 0$, then $k_{\lambda_i}(z) = (1 - \overline{\lambda_i} z)^{-1}$ and so, by the Cauchy Integral Formula,

$$\langle \varphi f, k_{\lambda_i} \rangle = \varphi(\lambda_i) f(\lambda_i) \quad \forall f \in K_\Theta.$$

Thus

$$\langle A_{\overline{\varphi}} k_{\lambda_i}, f \rangle = \langle P_\Theta(\overline{\varphi} k_{\lambda_i}), f \rangle = \langle k_{\lambda_i}, \varphi f \rangle = \overline{\varphi(\lambda_i) f(\lambda_i)} = \left\langle \overline{\varphi(\lambda_i)} k_{\lambda_i}, f \right\rangle$$

holds for all f in K_Θ whence

$$A_{\overline{\varphi}} k_{\lambda_i} = \overline{\varphi(\lambda_i)} k_{\lambda_i}, \quad i = 1, 2, \ldots, n. \tag{23}$$

We approach our nonlinear extremal problem $\Gamma(\psi)$ by considering a related system of "approximate anti-linear eigenvalue problems" (see Lemma 3) corresponding to certain truncated Toeplitz operators. This technique, introduced in [16, Thm. 2], can be thought of as a complex symmetric adaptation of Weyl's criterion [33, Thm. VII.12].

To make this more precise, we require a brief discussion of complex symmetric operators. Throughout the following, $\mathcal{H}$ will denote a separable complex Hilbert space and $\sigma(T)$ will denote the spectrum of a bounded linear operator $T : \mathcal{H} \to \mathcal{H}$.

Definition. A *conjugation* on $\mathcal{H}$ is a conjugate-linear operator $C : \mathcal{H} \to \mathcal{H}$ that is both *involutive* (i.e., $C^2 = I$) and *isometric* (i.e., $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$).

The canonical example of a conjugation is simply entry-by-entry complex conjugation on a l^2-space. In fact, each conjugation is unitarily equivalent to the canonical conjugation on a l^2-space of the appropriate dimension [17, Lem. 1]. From our perspective, the most pertinent example is the conjugation (12) on the model space K_Θ, which in light of (21), is easily seen to be entry-by-entry complex conjugation with respect to any of the modified Aleksandrov-Clark bases $\{v_1, v_2, \ldots, v_n\}$ defined by (20).

Definition. A bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ is C-*symmetric* if $T = CT^*C$ and *complex symmetric* if there exists a conjugation C with respect to which T is C-symmetric.

It turns out that T is a complex symmetric operator if and only if T is unitarily equivalent to a symmetric matrix with complex entries, regarded as an operator acting on an l^2-space of the appropriate dimension (see [15, Sect. 2.4] or [17, Prop. 2]). In fact, if C denotes the conjugation (12) on K_Θ, then any C-symmetric operator on K_Θ has a symmetric matrix representation with respect to any modified Aleksandrov-Clark basis. For further details, we refer the reader to [15, 17, 18].

The connection between complex symmetric operators and our nonlinear extremal problem $\Gamma(\psi)$ is furnished by Theorem 1 and Corollary 2 below. The proofs will depend on the following three lemmas.

Lemma 1. *For $\varphi \in L^\infty$ and Θ inner, the truncated Toeplitz operator A_φ on K_Θ is C-symmetric with respect to the conjugation $Cf = \overline{fz}\Theta$ on K_Θ.*

The proof, which is a straightforward computation, can be found in [17, Prop. 3] or [15, Thm. 5.1]. In particular, the preceding remarks ensure that a truncated Toeplitz operator A_φ has a complex symmetric (i.e., self-transpose) matrix representation with respect to any modified Aleksandrov-Clark basis (20). This property of truncated Toeplitz operators will be apparent when we consider several numerical examples later on. We also mention that the identity $CA_\varphi C = A_\varphi^*$ along with (23) will show that if $\varphi \in H^\infty$, then

$$A_\varphi Ck_{\lambda_i} = \varphi(\lambda_i)Ck_{\lambda_i},$$

where λ_i are the zeros of Θ.

2.5. The norm of a complex symmetric operator. In order to compute the quantity $\|A_\varphi\|_{K_\Theta \to K_\Theta}$, we require a few words concerning the polar decomposition of a complex symmetric operator. Recall that the *polar decomposition* $T = U|T|$ of a bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ expresses $T = U|T|$ uniquely as the product of a positive operator $|T| = \sqrt{T^*T}$ and a partial isometry U which satisfies $\ker U = \ker |T|$ and maps $\overline{\operatorname{ran}}\, |T|$ (the closure of the range of $|T|$) onto $\operatorname{ran} T$ [9, p. 248]. If T is a C-symmetric operator, then we can decompose the partial isometry U as the product of C with a *partial conjugation*. We say that an conjugate-linear operator J is a partial conjugation if J restricts to a conjugation on $(\ker J)^\perp$ (with values in the same space). In particular, the *linear* operator J^2 is the orthogonal projection onto the closed subspace $\operatorname{ran} J = (\ker J)^\perp$. The following lemma is [18, Thm. 2].

Lemma 2. *If $T : \mathcal{H} \to \mathcal{H}$ is a bounded C-symmetric operator, then $T = CJ|T|$ where J is a partial conjugation, supported on $\overline{\operatorname{ran}}\, |T|$, which commutes with $|T| = \sqrt{T^*T}$.*

Now recall that Weyl's criterion [33, Thm. VII.12] from the spectral theory of selfadjoint operators states that if A is a bounded selfadjoint operator, then λ belongs to $\sigma(A)$ if and only if there exists a sequence u_n of unit vectors so that

$$\lim_{n\to\infty} \|(A - \lambda I)u_n\| = 0.$$

The following result [16, Thm. 2] characterizes $\sigma(|T|)$ in terms of what one might call an *approximate anti-linear eigenvalue problem.*

Lemma 3. *If T is a bounded C-symmetric operator and $\lambda \neq 0$, then*

(i) $|\lambda| \in \sigma(|T|)$ *if and only if there exists a sequence of unit vectors u_n which satisfy the approximate anti-linear eigenvalue problem*

$$\lim_{n\to\infty} \|(T - \lambda C)u_n\| = 0. \tag{24}$$

Moreover, the u_n can be chosen so that $Ju_n = u_n$ for all n.

(ii) $|\lambda|$ *is an eigenvalue of* $|T|$ *(i.e., a singular value of* T*) if and only if the anti-linear eigenvalue problem* $Tu = \lambda Cu$ *has a nonzero solution* u.

To compute the norm of a truncated Toeplitz operator, and ultimately $\Gamma(\psi)$, we require the following general formula for the norm of a complex symmetric operator.

Theorem 1. *If* $T : \mathcal{H} \to \mathcal{H}$ *is a bounded* C*-symmetric operator, then*

(i) $\|T\| = \sup_{\|x\|=1} |\langle Tx, Cx\rangle|$.

(ii) *If* $\|x\| = 1$*, then* $\|T\| = |\langle Tx, Cx\rangle|$ *if and only if* $Tx = \omega\|T\|Cx$ *for some unimodular constant* ω.

(iii) *If* T *is compact, then the equation* $Tx = \|T\|Cx$ *has a unit vector solution. Furthermore, this unit vector solution is unique, up to a sign, if and only if the kernel of the operator* $|T| - \|T\|I$ *is one-dimensional.*

Proof. To prove (i) observe that since $|\langle Tx, Cx\rangle| \leq \|Tx\|\,\|Cx\| \leq \|T\|$ whenever $\|x\| \leq 1$, it suffices to prove that $\|T\| \leq \sup_{\|x\|=1} |\langle Tx, Cx\rangle|$. Let $\lambda := \|T\|$ (which belongs to $\sigma(|T|)$) and note that by Lemma 3, there exists a sequence u_n of unit vectors such that $Ju_n = u_n$ for all n and such that $\lim_{n\to\infty} \|Tu_n - \lambda Cu_n\| = 0$. Thus we have

$$\begin{aligned}
\|T\| &= \lambda\,\langle u_n, u_n\rangle && (\|u_n\| = 1)\\
&= \langle |T|u_n, u_n\rangle - \langle |T|u_n - \lambda u_n, u_n\rangle\\
&= \langle |T|Ju_n, u_n\rangle - \langle |T|Ju_n - \lambda u_n, u_n\rangle && (Ju_n = u_n)\\
&= \langle J|T|u_n, u_n\rangle - \langle J|T|u_n - \lambda u_n, u_n\rangle && (J|T| = |T|J, \text{ Lemma 2})\\
&\leq |\langle J|T|u_n, u_n\rangle| + |\langle J|T|u_n - \lambda u_n, u_n\rangle|\\
&= |\langle CJ|T|u_n, Cu_n\rangle| + |\langle CJ|T|u_n - \lambda Cu_n, Cu_n\rangle| && (C \text{ is isometric, } \lambda \geq 0)\\
&= |\langle Tu_n, Cu_n\rangle| + |\langle Tu_n - \lambda Cu_n, Cu_n\rangle| && (CJ|T| = T, \text{ Lemma 2})\\
&\leq \sup_{\|x\|=1} |\langle Tx, Cx\rangle| + \|Tu_n - \lambda Cu_n\| && (\|Cu_n\| = \|u_n\| = 1).
\end{aligned}$$

Since the second term in the previous line tends to zero as $n \to \infty$, the desired inequality follows. This proves (i).

Let us now consider (ii). First observe that the ($\Leftarrow$) implication is obvious. For the ($\Rightarrow$) implication, suppose that $\|x\| = 1$ and $\|T\| = |\langle Tx, Cx\rangle|$. By the CSB inequality we have

$$\|T\| = |\langle Tx, Cx\rangle| \leq \|Tx\|\|Cx\| \leq \|T\|\|Cx\| \leq \|T\|\|x\| = \|T\|$$

whence $|\langle Tx, Cx\rangle| = \|Tx\|\|Cx\|$. By the condition for equality in the CSB inequality, we find that $Tx = \langle Tx, Cx\rangle\, Cx$. In other words, $Tx = \omega\|T\|Cx$ for some $|\omega| = 1$, concluding the proof of (ii).

For (iii), first note that T is compact and thus $\|T\|$ is an eigenvalue of $|T|$ whence, in light of Lemma 3, the equation $Tx = \|T\|Cx$ has a unit vector solution. We now show that this solution is unique, up to sign, if and only if the kernel of the operator $|T| - \|T\|I$ is one-dimensional. For one direction, we employ the following representation of compact C-symmetric operators from [18, Thm. 3]:

$$T = \sum_{n\geq 0} \lambda_n \sum_{k=1}^{d_n} Cu_{n,k} \otimes u_{n,k},$$

where λ_n are the distinct eigenvalues of the selfadjoint operator $|T|$ (which, since T is compact, form a countable set whose only possible limit point is zero) and $\{u_{n,1}, \ldots, u_{n,d_n}\}$ is a certain orthonormal basis for the eigenspace of $|T|$ corresponding to λ_n which also satisfies the auxiliary condition $Tu_{n,k} = \lambda_n C u_{n,k}$. Thus if the unit vector solution to $Tx = \|T\| Cx$ is unique up to sign, then the kernel of $|T| - \|T\| I$ is one-dimensional.

On the other hand, if $Tx = \|T\|\, Cx$, then

$$\begin{aligned} Tx = \|T\|\, Cx \quad &\Rightarrow \quad CTx = \|T\|\, x && (\|T\| \in \mathbb{R},\, C^2 = I) \\ &\Rightarrow \quad CTCTx = \|T\|^2\, x && \\ &\Rightarrow \quad T^*Tx = \|T\|^2\, x && (CTC = T^*) \\ &\Rightarrow \quad |T|x = \|T\|\, x && \end{aligned}$$

since $|T| = \sqrt{T^*T}$. Suppose now that x, y are two unit vectors such that $Tx = \|T\|\, Cx$, $Ty = \|T\|\, Cy$, and $x \neq \pm y$. In this case, the conjugate-linearity of C, along with the facts that $\|x\| = \|y\| = 1$ and $x \neq \pm y$, ensures that x and y are not scalar multiples of each other. In particular, x and y are linearly independent and satisfy $|T|x = \|T\|\, x$ and $|T|y = \|T\|\, y$. This implies that the kernel of the operator $|T| - \|T\| I$ has dimension at least two, as claimed. ■

3. A reduction

As before, let Θ denote the finite Blaschke product (6) whose zeros, repeated according to multiplicity, are precisely the poles of the rational function ψ which lie in $\mathbb{D}$. Writing f as $f = P_\Theta f + \Theta h$ where $h \in H^2$, one can show that our original nonlinear extremal problem $\Gamma(\psi)$ in (1) can be reduced to the new problem

$$\sup_{\substack{f \in K_\Theta \\ \|f\|=1}} \left| \frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \psi f^2\, dz \right| \tag{25}$$

posed on the model space K_Θ. Indeed, for $f \in H^2$ we have

$$\begin{aligned} \frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \psi f^2\, dz &= \frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \big(\psi(P_\Theta f)^2 + 2\psi(P_\Theta f)\Theta h + \psi \Theta^2 h^2\big)\, dz \\ &= \frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \psi(P_\Theta f)^2\, dz. \end{aligned} \tag{26}$$

We obtain (26) by noting that the function $\psi\Theta$ belongs to H^∞ whence the second and third terms in the preceding line vanish by Cauchy's Theorem.

We gather from this reduction two important observations. The first is that an extremal function for the problem $\Gamma(\psi)$ exists since the nonlinear functional

$$f \mapsto \left| \frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \psi f^2\, dz \right|$$

is a continuous map and the supremum in (25) is over the (compact) unit ball of the *finite* dimensional space K_Θ. The second is that an extremal function for $\Gamma(\psi)$ lies in K_Θ and thus from the representation (10) has the form

$$f(z) = \frac{p(z)}{\prod_{j=1}^n (1 - \overline{\lambda_j} z)},$$

where $p(z)$ is an analytic polynomial of degree at most $n-1$.

Putting together the material we have developed so far, we obtain the following corollary to Theorem 1, which, in terms of computing the norms of truncated Toeplitz operators, is interesting in its own right.

Corollary 1. *Let Θ be a nonconstant inner function.*

(i) *If* $\varphi \in L^\infty$*, then*

$$\|A_\varphi\|_{K_\Theta \to K_\Theta} = \sup_{\substack{f \in K_\Theta \\ \|f\|=1}} \left| \frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \frac{\varphi f^2}{\Theta} dz \right|. \tag{27}$$

(ii) *If* $\varphi \in H^\infty$*, then*

$$\|A_\varphi\|_{K_\Theta \to K_\Theta} = \sup_{\substack{f \in H^2 \\ \|f\|=1}} \left| \frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \frac{\varphi f^2}{\Theta} dz \right|. \tag{28}$$

Proof. Lemma 1 asserts that the operator $A_\varphi : K_\Theta \to K_\Theta$ is C-symmetric with respect to the conjugation $Cf = \overline{fz}\Theta$ on K_Θ. Next we note that

$$\langle A_\varphi f, Cf \rangle = \langle P_\Theta(\varphi f), Cf \rangle = \langle \varphi f, Cf \rangle = \left\langle \varphi f, \overline{fz}\Theta \right\rangle = \left\langle \varphi f^2 z, \Theta \right\rangle$$

holds for any f in K_Θ. Using the identity $dz = iz|dz|$ to write the expression $\left\langle \varphi f^2 z, \Theta \right\rangle$ as a contour integral on $\partial\mathbb{D}$ and noting that $\overline{\Theta} = 1/\Theta$ a.e. on $\partial\mathbb{D}$, yields (upon an application of Theorem 1) the desired formula (27). This establishes (i).

Let us now prove (ii). It follows from assertion (i) and the reduction discussed in (26) that

$$\frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \frac{\varphi f^2}{\Theta} dz = \frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \frac{\varphi (P_\Theta f)^2}{\Theta} dz, \quad f \in H^2.$$

In other words, if the function φ belongs to H^∞, then the supremum in (27) can be taken over ball(H^2). This yields (ii). ∎

The following corollary to Theorem 1 and Corollary 1 provides the fundamental connection between truncated Toeplitz operators and the nonlinear extremal problem $\Gamma(\psi)$.

Corollary 2. *Suppose that* ψ *is a rational function having no poles on* $\partial\mathbb{D}$ *and poles* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *lying in* $\mathbb{D}$*, counted according to multiplicity. Let* Θ *denote the associated Blaschke product* (6) *whose zeros are precisely* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *and note that* $\varphi = \psi\Theta$ *belongs to* H^∞*. We then have the following:*

(i) *The nonlinear extremal problems* $\Gamma(\psi)$ *from* (1) *and* (25) *are equal. Moreover, both are equal to* $\|A_\varphi\|_{K_\Theta \to K_\Theta}$.

(ii) *There is a unit vector* $f \in K_\Theta$ *satisfying*

$$A_\varphi f = \|A_\varphi\| Cf \tag{29}$$

and any such f *is an extremal function for* $\Gamma(\psi)$*. In other words,*

$$\frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \psi f^2 \, dz = \|A_\varphi\|_{K_\Theta \to K_\Theta}.$$

(iii) *Every extremal function* f *for* $\Gamma(\psi)$ *belongs to* K_Θ *and satisfies*

$$A_\varphi f = \|A_\varphi\| Cf.$$

(iv) *An extremal function for* $\Gamma(\psi)$ *is unique, up to a sign, if and only if the kernel of the operator* $|A_\varphi| - \|A_\varphi\| I$ *is one-dimensional.*

Proof. Use Theorem 1, Corollary 1, the reduction in (26), along with the fact that K_Θ is finite dimensional so that the C-symmetric operator $T = A_{\psi\Theta}$ is compact. ∎

4. Equality of the linear and nonlinear extrema

For a rational function ψ having no poles on $\partial\mathbb{D}$, it turns out that the classical linear extremal problem $\Lambda(\psi)$ given by (2) and our nonlinear extremal problem $\Gamma(\psi)$ given by (1) yield the same answer. We begin with the observation that for any $f \in H^2$ with $\|f\| = 1$, we have $f^2 \in H^1$ and $\|f^2\|_1 = 1$. Therefore

$$\{f^2 : f \in H^2, \|f\| \leq 1\} \ \subseteq \ \{F : F \in H^1, \|F\|_1 \leq 1\},$$

which implies that

$$\Gamma(\psi) \leq \Lambda(\psi). \tag{30}$$

Recall from the introduction that Egerváry, in the special case where

$$\psi(z) = \frac{1}{z} + \frac{1}{z^2} + \cdots + \frac{1}{z^{n+1}},$$

showed (see (5)) that an extremal function for the linear extremal problem $\Lambda(\psi)$ is the square of an H^2 function whence $\Gamma(\psi) = \Lambda(\psi)$. As a somewhat easier example, we remind the reader of the family of functions (3) which all serve as extremal functions for the linear extremal problem $\Lambda(1/z^2)$. In particular, the extremal function

$$\left(\frac{z+1}{\sqrt{2}}\right)^2$$

is the square of an H^2 function and we again have $\Gamma(\psi) = \Lambda(\psi)$. As it turns out, the preceding examples are not peculiar in this regard. Indeed, S. Ya. Khavinson [23] showed, in greater generality beyond rational ψ, that one of the extremal functions for $\Lambda(\psi)$ can be taken to be a function F with no zeros in $\mathbb{D}$. Thus $f := \sqrt{F}$ will be (up to a unimodular constant) an extremal function for $\Gamma(\psi)$. Here is a new proof of the identity $\Gamma(\psi) = \Lambda(\psi)$ in the language of truncated Toeplitz operators.

Proposition 1. *If ψ is a rational function with no poles on $\partial\mathbb{D}$, then $\Lambda(\psi) = \Gamma(\psi)$. In other words,*

$$\sup_{\substack{F\in H^1\\ \|F\|_1=1}} \left|\frac{1}{2\pi i}\oint_{\partial\mathbb{D}} \psi F\, dz\right| = \sup_{\substack{f\in H^2\\ \|f\|=1}} \left|\frac{1}{2\pi i}\oint_{\partial\mathbb{D}} \psi f^2\, dz\right|.$$

Proof. Since every $F \in H^1$ can be written as $F = fg$, where $f, g \in H^2$ and $\|F\|_1 = \|f\|\,\|g\|$ (see [19, Ex. 1, Ch. 2] or [11, Thm. 3.15]), it follows that

$$\sup_{\substack{f,g\in H^2\\ \|f\|=\|g\|=1}} \left|\frac{1}{2\pi i}\oint_{\partial\mathbb{D}} \psi fg\, dz\right| = \sup_{\substack{F\in H^1\\ \|F\|_1=1}} \left|\frac{1}{2\pi i}\oint_{\partial\mathbb{D}} \psi F\, dz\right|. \tag{31}$$

As usual, let Θ be the corresponding Blaschke product from (6). Writing f, g in the preceding as $f = f_1 + \Theta h_1$ and $g = g_1 + \Theta h_2$, where $f_1, g_1 \in K_\Theta$ and $h_1, h_2 \in H^2$, we obtain

$$\frac{1}{2\pi i}\oint_{\partial\mathbb{D}} \psi fg\, dz = \frac{1}{2\pi}\oint_{\partial\mathbb{D}} \psi(f_1 + \Theta h_1)(g_1 + \Theta h_2) dz = \frac{1}{2\pi}\oint_{\partial\mathbb{D}} \psi f_1 g_1 dz. \tag{32}$$

In particular, observe that all but one of the terms in the product integrate to zero since the function $\varphi = \psi\Theta$ belongs to H^∞.

Putting this all together we find that

$$\sup_{\substack{F\in H^1\\ \|F\|_1=1}} \left|\frac{1}{2\pi i}\oint_{\partial\mathbb{D}} \psi F\, dz\right| = \sup_{\substack{f,g\in H^2\\ \|f\|=\|g\|=1}} \left|\frac{1}{2\pi i}\oint_{\partial\mathbb{D}} \psi fg\, dz\right| \qquad \text{(by (31))}$$

$$
\begin{aligned}
&= \sup_{\substack{f,g\in K_\Theta\\ \|f\|=\|g\|=1}} \left| \frac{1}{2\pi i}\oint_{\partial\mathbb{D}} \psi f g\,dz \right| && \text{(by (32))}\\
&= \sup_{\substack{f,g\in K_\Theta\\ \|f\|=\|g\|=1}} |\langle \Theta\psi f, \overline{zg}\Theta\rangle| \\
&= \sup_{\substack{f,g\in K_\Theta\\ \|f\|=\|g\|=1}} |\langle \varphi f, Cg\rangle| && (\varphi=\psi\Theta)\\
&= \sup_{\substack{f,g\in K_\Theta\\ \|f\|=\|g\|=1}} |\langle \varphi f, g\rangle| && (C^2=I)\\
&= \sup_{\substack{f,g\in K_\Theta\\ \|f\|=\|g\|=1}} |\langle P_\Theta(\varphi f), g\rangle| \\
&= \sup_{\substack{f,g\in K_\Theta\\ \|f\|=\|g\|=1}} |\langle A_\varphi f, g\rangle| \\
&= \|A_\varphi\|_{K_\Theta\to K_\Theta}\\
&= \sup_{\substack{f\in H^2\\ \|f\|=1}} \left| \frac{1}{2\pi i}\oint_{\partial\mathbb{D}} \psi f^2\,dz \right| && \text{(Corollary 2).}
\end{aligned}
$$

Thus $\Lambda(\psi)=\Gamma(\psi)$ as claimed. ■

5. Computing the supremum

Recall that Corollary 2 ensures that

$$\Gamma(\psi)=\|A_\varphi\|_{K_\Theta\to K_\Theta}$$

where φ denotes the H^∞ function $\psi\Theta$. We propose two simple and practical procedures for computing $\|A_\varphi\|$. Both methods involve computing matrix representations for the operator A_φ. In practice, these matrices are explicitly computable and hence their norms can be determined algebraically (for $n=\dim K_\Theta$ small) or numerically via `Mathematica` or any other comparable piece of software.

5.1. Using the Takenaka-Malmquist-Walsh basis. Our first approach involves representing the truncated Toeplitz operator $A_\varphi : K_\Theta \to K_\Theta$ as an $n\times n$ matrix with respect to the Takenaka-Malmquist-Walsh basis (17) for K_Θ.

(i) Let $\varphi := \psi\Theta$ and note that $\varphi\in H^\infty$ since the poles of ψ lying in $\mathbb{D}$ cancel with the zeros of Θ.

(ii) The jk-th entry $[M_{A_\varphi}]_{jk}$ of the matrix representation M_{A_φ} of A_φ with respect to the Takenaka-Malmquist-Walsh basis $\{v_1,v_2,\ldots,v_n\}$ is $\langle A_\varphi v_k, v_j\rangle$. The resulting matrix M_{A_φ} is lower triangular (see [29, Lect. V] and below).

(iii) The largest singular value of the matrix M_{A_φ} is the desired quantity $\|A_\varphi\|=\Gamma(\psi)$.

A straightforward computation confirms that the matrix representation M_{A_φ} of A_φ with respect to the Takenaka-Malmquist-Walsh basis is lower triangular. Indeed, for $j<k$ we have

$$
\begin{aligned}
[M_{A_\varphi}]_{jk} &= \langle A_\varphi v_k, v_j\rangle = \langle P_\Theta(\varphi v_k), v_j\rangle\\
&= \left\langle \varphi\left(\prod_{i=1}^{k-1} B_{\lambda_i}\right)\frac{\sqrt{1-|\lambda_k|^2}}{1-\overline{\lambda_k}z}, \left(\prod_{i=1}^{j-1} B_{\lambda_i}\right)\frac{\sqrt{1-|\lambda_j|^2}}{1-\overline{\lambda_j}z}\right\rangle
\end{aligned}
$$

$$= \sqrt{1-|\lambda_j|^2}\left\langle \underbrace{\varphi\left(\prod_{i=j}^{k-1} B_{\lambda_i}\right)\frac{\sqrt{1-|\lambda_k|^2}}{1-\overline{\lambda_k}z}}_{\in H^\infty}, \frac{1}{1-\overline{\lambda_j}z}\right\rangle = 0 \tag{33}$$

since $B_{\lambda_j}(\lambda_j) = 0$ and thus the matrix is lower triangular.

The diagonal entries of our matrix are also relatively easy to compute:

$$[M_{A_\varphi}]_{kk} = \varphi(\lambda_k), \qquad 1 \le k \le n.$$

Since M_{A_φ} is a lower triangular matrix this is to be expected since the eigenvalues $\varphi(\lambda_1), \varphi(\lambda_2), \ldots, \varphi(\lambda_n)$ of the operator A_φ must appear along the main diagonal.

For $j > k$ the computations are more involved:

$$\begin{aligned}
[M_{A_\varphi}]_{jk} &= \left\langle \varphi\left(\prod_{i=1}^{k-1} B_{\lambda_i}\right)\frac{\sqrt{1-|\lambda_k|^2}}{1-\overline{\lambda_k}z}, \left(\prod_{i=1}^{j-1} B_{\lambda_i}\right)\frac{\sqrt{1-|\lambda_j|^2}}{1-\overline{\lambda_j}z}\right\rangle \\
&= \left\langle \varphi\frac{\sqrt{1-|\lambda_k|^2}}{1-\overline{\lambda_k}z}, \left(\prod_{i=k}^{j-1} B_{\lambda_i}\right)\frac{\sqrt{1-|\lambda_j|^2}}{1-\overline{\lambda_j}z}\right\rangle \\
&= (1-|\lambda_j|^2)^{\frac12}(1-|\lambda_k|^2)^{\frac12}\frac{1}{2\pi i}\oint_{\partial\mathbb{D}} \frac{\varphi(z)\,dz}{\left(\prod_{i=k}^{j-1} B_{\lambda_i}\right)(1-\overline{\lambda_k}z)(z-\lambda_j)}.
\end{aligned} \tag{34}$$

Although the preceding can be evaluated explicitly using the residue calculus, the resulting expression is somewhat unwieldy and we choose not to write it here. We will see the matrix representation of a truncated Toeplitz operator with respect to the Takenaka-Malmquist-Walsh basis again in Section 8.

5.2. Using modified Aleksandrov-Clark bases. A similar approach using the Aleksandrov-Clark bases can also be formulated.

(i) Let $\varphi := \psi\Theta$ and note that $\varphi \in H^\infty$ since the poles of ψ lying in $\mathbb{D}$ cancel with the zeros of Θ.

(ii) Fix some β on $\partial\mathbb{D}$ and let $v_1, v_2, \ldots, v_n$ be the corresponding modified Aleksandrov-Clark basis from (20). In particular, compute the unimodular constants $\zeta_1, \zeta_2, \ldots, \zeta_n$ and $\omega_1, \omega_2, \ldots, \omega_n$ from (19).

(iii) The jk-th entry $[M_{A_\varphi}]_{jk}$ of the matrix representation M_{A_φ} of A_φ with respect to the Aleksandrov-Clark basis $\{v_1, v_2, \ldots, v_n\}$ is $\langle A_\varphi v_k, v_j\rangle$. The resulting matrix M_{A_φ} is complex symmetric (i.e., self-transpose).

(iv) The largest singular value of the matrix M_{A_φ} is the desired quantity $\|A_\varphi\| = \Gamma(\psi)$.

Let us derive a general formula for M_{A_φ} in the simple case where the zeros $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct (One can make adjustments for the general case where the zeros $\lambda_1, \lambda_2, \ldots, \lambda_n$ are not necessarily distinct). Using the fact that the functions φ, v_j, and v_k are analytic in a neighborhood of the closed unit disk $\mathbb{D}^-$, it follows from the residue calculus that

$$\begin{aligned}
\langle A_\varphi v_k, v_j\rangle &= \langle \varphi v_k, v_j\rangle \\
&= \frac{\omega_k}{\sqrt{|\Theta'(\zeta_k)|}}\frac{\overline{\omega_j}}{\sqrt{|\Theta'(\zeta_j)|}}\oint_{|z|=1}\varphi(z)\frac{1-\overline{\beta}\Theta(z)}{1-\overline{\zeta_k}z}\,\frac{1-\beta\overline{\Theta(z)}}{1-\zeta_j\overline{z}}\,\frac{dz}{2\pi i z} \\
&= \frac{\omega_k}{\sqrt{|\Theta'(\zeta_k)|}}\frac{\overline{\omega_j}}{\sqrt{|\Theta'(\zeta_j)|}}\oint_{|z|=1}\varphi(z)\frac{1-\overline{\beta}\Theta(z)}{1-\overline{\zeta_k}z}\,\frac{\Theta(z)-\beta}{\Theta(z)(z-\zeta_j)}\,\frac{dz}{2\pi i}
\end{aligned}$$

$$= \frac{\omega_k}{\sqrt{|\Theta'(\zeta_k)|}} \frac{\beta\overline{\omega_j}}{\sqrt{|\Theta'(\zeta_j)|}} \sum_{i=1}^{n} \frac{\varphi(\lambda_i)}{\Theta'(\lambda_i)(1-\overline{\zeta_k}\lambda_i)(\zeta_j-\lambda_i)}$$

$$= \frac{\omega_k}{\sqrt{|\Theta'(\zeta_k)|}} \frac{\beta\overline{\zeta_j}\overline{\omega_j}}{\sqrt{|\Theta'(\zeta_j)|}} \sum_{i=1}^{n} \frac{\varphi(\lambda_i)}{\Theta'(\lambda_i)(1-\overline{\zeta_k}\lambda_i)(1-\overline{\zeta_j}\lambda_i)} \tag{35}$$

$$= \frac{\omega_k}{\sqrt{|\Theta'(\zeta_k)|}} \frac{\omega_j}{\sqrt{|\Theta'(\zeta_j)|}} \sum_{i=1}^{n} \frac{\varphi(\lambda_i)}{\Theta'(\lambda_i)(1-\overline{\zeta_k}\lambda_i)(1-\overline{\zeta_j}\lambda_i)}. \tag{36}$$

Observe that we employed the identity $\beta\overline{\zeta_j}\overline{\omega_j} = \omega_j$ when passing from (35) to (36). Notice also from (36) that the $n \times n$ matrix $M_{A_\varphi} = (\langle A_\varphi v_k, v_j\rangle)_{j,k=1}^n$ is indeed complex symmetric as previously claimed. It follows from the preceding calculations and the fact that $v_1, v_2, \ldots, v_n$ is an orthonormal basis for K_Θ that $\|A_\varphi\|_{K_\Theta \to K_\Theta}$ is equal to

$$\left\| \left(\frac{\omega_k}{\sqrt{|\Theta'(\zeta_k)|}} \frac{\omega_j}{\sqrt{|\Theta'(\zeta_j)|}} \sum_{i=1}^{n} \frac{\varphi(\lambda_i)}{\Theta'(\lambda_i)(1-\overline{\zeta_k}\lambda_i)(1-\overline{\zeta_j}\lambda_i)} \right)_{j,k=1}^{n} \right\|. \tag{37}$$

Example 1. Let $c_1, c_2, \ldots, c_n$ be n arbitrary complex numbers and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be n *distinct* points in $\mathbb{D}$. Now form the rational function

$$\psi(z) = \sum_{i=1}^{n} \frac{c_i}{z-\lambda_i}.$$

Using the residue calculus and Corollary 2, we obtain

$$\Gamma(\psi) = \|A_\varphi\|_{K_\Theta \to K_\Theta}$$

where $\varphi = \psi\Theta$ and Θ denotes the usual finite Blaschke product (6). Observe that

$$\varphi(z) = \psi(z)\Theta(z) = \sum_{i=1}^{n} c_i \frac{\Theta(z)}{z-\lambda_i}.$$

Employing the identity

$$\varphi(\lambda_i) = c_i\Theta'(\lambda_i), \quad 1 \le i \le n$$

in (36) yields

$$\Gamma(\psi) = \left\| \left(\frac{\omega_k}{\sqrt{|\Theta'(\zeta_k)|}} \frac{\omega_j}{\sqrt{|\Theta'(\zeta_j)|}} \sum_{s=1}^{n} \frac{c_s}{(1-\overline{\zeta_k}\lambda_s)(1-\overline{\zeta_j}\lambda_s)} \right)_{j,k=1}^{n} \right\|.$$

Example 2. To better demonstrate the procedure outlined above, let us work through a specific numerical example in detail. Consider the nonlinear extremal problem

$$\sup_{\substack{f \in H^2 \\ \|f\|=1}} \left| f^2(0) + f^2(\tfrac{1}{2}) \right| = \Gamma(\psi) = \|A_\varphi\|_{K_\Theta \to K_\Theta}$$

where

$$\Theta(z) = z\frac{z-\frac{1}{2}}{1-\frac{1}{2}z}, \quad \psi = \frac{1}{z} + \frac{1}{z-\frac{1}{2}},$$

and $\varphi = \psi\Theta$ as usual. Letting $\beta = 1$ we find that the equation $\Theta(\zeta) = 1$ has the solutions

$$\zeta_1 = 1, \quad \zeta_2 = -1.$$

From the preceding, we find that the desired unimodular constants ω_1 and ω_2 are given by

$$\omega_1 = 1, \quad \omega_2 = -i.$$

For a 2×2 matrix A we have the identity

$$\|A\| = \frac{1}{2}\left(\sqrt{\operatorname{tr}(A^*A) + 2|\det(A)|} + \sqrt{\operatorname{tr}(A^*A) - 2|\det(A)|}\right). \tag{38}$$

This formula follows from the polar decomposition $A = U|A|$ where U is unitary and $|A| = \sqrt{A^*A}$ is nonnegative. If $\lambda_1, \lambda_2 \geq 0$ denote the eigenvalues of $|A|$ (i.e., the singular values of A), note that $\operatorname{tr}(A^*A) = \lambda_1^2 + \lambda_2^2$ and $|\det A| = \lambda_1\lambda_2$ since U is unitary.

Plugging this data into (36) and using (38) we find that

$$\|A_\varphi\|_{K_\Theta \to K_\Theta} = \left\| \begin{pmatrix} \frac{5}{4} & -\frac{7i}{4\sqrt{3}} \\ -\frac{7i}{4\sqrt{3}} & -\frac{13}{12} \end{pmatrix} \right\| = \frac{1}{6}(7+\sqrt{37}) \approx 2.1805.$$

In light of Proposition 1, we also have

$$\sup_{\substack{F\in H^1 \\ \|F\|_1 = 1}} \left|F(0) + F(\tfrac{1}{2})\right| = \frac{1}{6}(7+\sqrt{37}).$$

Example 3. One could generalize the above example to

$$\sup_{\substack{f\in H^2 \\ \|f\| = 1}} \left|c_1 f^2(a_1) + c_2 f^2(a_2)\right| = \Gamma(\psi) = \|A_\varphi\|_{K_\Theta \to K_\Theta},$$

where $a_1, a_2 \in \mathbb{D}$ $(a_1 \neq a_2)$, $c_1, c_2 \in \mathbb{C}\setminus\{0\}$. In this case we have

$$\Theta(z) = \frac{(z-a_1)(z-a_2)}{(1-\overline{a_1}z)(1-\overline{a_2}z)}, \quad \psi(z) = \frac{c_1}{z-a_1} + \frac{c_2}{z-a_2}, \quad \varphi = \psi\Theta.$$

We now represent our truncated Toeplitz operator with respect to the Takenaka-Malmquist-Walsh basis

$$v_1(z) = \frac{\sqrt{1-|a_1|^2}}{1-\overline{a_1}z}, \quad v_2(z) = \frac{z-a_1}{1-\overline{a_1}z}\frac{\sqrt{1-|a_2|^2}}{1-\overline{a_2}z}.$$

We then use (33) and (34) to get

$$\Gamma(\psi) = \left\| \begin{pmatrix} \frac{c_1(a_1-a_2)}{(1-|a_1|^2)(1-a_1\overline{a_2})} & 0 \\ \frac{c_1\sqrt{1-|a_2|^2}}{\sqrt{1-|a_1|^2}(1-a_1\overline{a_2})} + \frac{c_2\sqrt{1-|a_1|^2}}{\sqrt{1-|a_2|^2}(1-a_2\overline{a_1})} & \frac{c_2(a_2-a_1)}{(1-|a_2|^2)(1-a_2\overline{a_1})} \end{pmatrix} \right\|. \tag{39}$$

Example 4. Plugging $a_1 = \frac{\sqrt{2}}{4} + i\frac{\sqrt{2}}{4}$, $a_2 = \frac{1}{3}$, $c_1 = 1$, $c_2 = -\frac{1}{2}$, into (39) and using `Mathematica` reveals that

$$\begin{aligned}
&\sup_{\substack{F\in H^1 \\ \|F\|_1 = 1}} \left|F(\tfrac{\sqrt{2}}{4} + i\tfrac{\sqrt{2}}{4}) - \tfrac{1}{2}F(\tfrac{1}{3})\right| \\
&\quad = \tfrac{1}{48}\sqrt{\tfrac{3966841 - 1644636\sqrt{2} + \sqrt{15343011060049 - 9085021064952\sqrt{2}}}{2882 - 888\sqrt{2}}} \\
&\quad \approx 0.927119.
\end{aligned}$$

Example 5. For another class of examples, suppose $c_0, c_1, \ldots, c_n$ are complex numbers and consider the rational function

$$\psi(z) = \frac{c_0}{z} + \frac{c_1}{z^2} + \cdots + \frac{c_n}{z^{n+1}}.$$

The corresponding Blaschke product (6) is therefore $\Theta(z) = z^{n+1}$ and hence by Corollary 2 we see that

$$\begin{aligned}
\Gamma(\psi) &= \|A_{\psi z^{n+1}}\|_{K_{z^{n+1}} \to K_{z^{n+1}}} \\
&= \|A_{c_n + c_{n-1}z + \cdots + c_0 z^n}\|_{K_{z^{n+1}} \to K_{z^{n+1}}}.
\end{aligned}$$

The matrix representation of $A_{\psi z^{n+1}}$ with respect to the monomial basis $\{1, z, \ldots, z^n\}$ for $K_{z^{n+1}}$ is the lower triangular $(n+1) \times (n+1)$ Toeplitz matrix

$$T = \begin{pmatrix} c_n & & & & \\ \vdots & \ddots & & & \\ c_2 & \cdots & c_n & & \\ c_1 & c_2 & \cdots & c_n & \\ c_0 & c_1 & c_2 & \cdots & c_n \end{pmatrix}. \tag{40}$$

By Corollary 2 we therefore obtain

$$\sup_{\substack{f \in H^2 \\ \|f\|=1}} \left| \frac{1}{2\pi i} \oint_{\partial \mathbb{D}} \left(\frac{c_0}{z} + \cdots + \frac{c_n}{z^{n+1}} \right) f^2 dz \right| = \|T\|. \tag{41}$$

Notice how this reproves the Fejér-Egerváry results mentioned in the introduction.

Example 6. By the above calculation,

$$\begin{aligned} \Gamma\left(\frac{1}{z} + \frac{1}{z^2}\right) &= \|A_{1+z}\|_{K_{z^2} \to K_{z^2}} \\ &= \left\| \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\| \\ &= \frac{1+\sqrt{5}}{2} \\ &\approx 1.618. \end{aligned}$$

In particular, notice that this answer agrees with Egerváry's example (4) discussed in the introduction.

Example 7. This example generalizes the results of Example 6 and demonstrates the use of *antilinear eigenvalue problems* to calculate the norm of a complex symmetric operator. Consider the nonlinear extremal problem

$$\Gamma\left(\frac{a_0}{z^2} + \frac{a_1}{z}\right) = \sup_{\substack{f \in K_\Theta \\ \|f\|=1}} |2a_0 f(0) f'(0) + a_1 f(0)| \tag{42}$$

where a_0 and a_1 are fixed. To ensure that the problem (42) is nontrivial, we make the additional assumption that $a_0 \neq 0$. By the discussion in Example 5, it follows that

$$\Gamma\left(\frac{a_0}{z^2} + \frac{a_1}{z}\right) = \|A_{a_0 + a_1 z}\|_{K_{z^2} \to K_{z^2}} = \left\| \begin{pmatrix} a_0 & 0 \\ a_1 & a_0 \end{pmatrix} \right\|.$$

Although the norm of the above matrix can be computed explicitly using standard methods or the explicit formula (38), let us now illustrate our anti-linear eigenvalue technique.

The conjugation C on K_{z^2} is given by $Cf = \overline{fz} z^2 = \overline{f} z$. In other words,

$$C(c_0 + c_1 z) = \overline{c_1} + \overline{c_0} z, \qquad c_0, c_1 \in \mathbb{C}.$$

With respect to the monomial basis $\{1, z\}$ for K_{z^2} we see that the anti-linear eigenvalue problem $A_{a_0 + a_1 z} \mathbf{u} = \lambda C \mathbf{u}$ (which has a unit vector solution) is equivalent to the following $\mathbb{R}$-*linear* problem:

$$\begin{pmatrix} a_0 & 0 \\ a_1 & a_0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \lambda \overline{u_2} \\ \lambda \overline{u_1} \end{pmatrix}. \tag{43}$$

The norm of the associated Toeplitz matrix coincides with the largest $\lambda \geq 0$ for which the preceding system is consistent.

In light of the assumption that $a_0 \neq 0$, it follows that $u_2 \neq 0$ else $u_1 = u_2 = 0$. Solving the first equation in (43) for u_1 and substituting the result in the second yields

$$a_1 u_2 = \left(\lambda - \frac{|a_0|^2}{\lambda}\right)\overline{u_2}. \tag{44}$$

Since the function $\lambda \mapsto \lambda - \frac{|a_0|^2}{\lambda}$ is increasing for $\lambda > 0$, it follows from (44) that the largest positive number λ satisfying (43) must satisfy

$$|a_1| = \lambda - \frac{|a_0|^2}{\lambda}.$$

This in turn means that λ satisfies the quadratic equation

$$\lambda^2 - |a_1|\lambda - |a_0|^2 = 0.$$

Solving this yields the explicit answer

$$\Gamma\left(\frac{a_0}{z^2} + \frac{a_1}{z}\right) = \frac{|a_1| + \sqrt{|a_1|^2 + 4|a_0|^2}}{2}, \tag{45}$$

which agrees with (38). On the other hand, a direct attack on the problem would involve computing the eigenvalues of the matrix

$$\begin{pmatrix} \overline{a_0} & \overline{a_1} \\ 0 & \overline{a_0} \end{pmatrix}\begin{pmatrix} a_0 & 0 \\ a_1 & a_0 \end{pmatrix} = \begin{pmatrix} |a_0|^2 & a_0\overline{a_1} \\ \overline{a_0}a_1 & |a_0|^2 + |a_1|^2 \end{pmatrix}$$

and then explicitly computing their square roots. This involves considerably more symbolic computation. For instance, `Mathematica` yields the seemingly more complicated answer

$$\sqrt{|a_0|^2 + \frac{1}{2}\left(|a_1|^2 - |a_1|\sqrt{|a_1|^2 + 4|a_0|^2}\right)},$$

which is, of course, equal to (45).

6. Computing an extremal function

By Corollary 2 we have

$$\Gamma(\psi) = \|A_{\psi\Theta}\|_{K_\Theta \to K_\Theta} = \langle A_{\psi\Theta}u, Cu\rangle,$$

where $u \in K_\Theta$ is a unit vector solution to the anti-linear eigenvalue problem

$$A_{\psi\Theta}u = \|A_{\psi\Theta}\|Cu$$

(such a vector exists). Here is a procedure which, at least in principle, can be used to compute an extremal function u for $\Gamma(\psi)$.

As before, let $M_{A_{\psi\Theta}}$ denote the matrix representation of the truncated Toeplitz operator $A_{\psi\Theta}$ with respect to the modified Aleksandrov-Clark basis $\{v_1, v_2, \ldots, v_n\}$ in (20). In particular, recall that $Cv_j = v_j$ for all $1 \leq j \leq n$ by (21). If $u \in K_\Theta$ is a unit vector, then

$$u = c_1v_1 + c_2v_2 + \cdots + c_nv_n, \qquad |c_1|^2 + |c_2|^2 + \cdots + |c_n|^2 = 1.$$

If, in addition, u is a solution to

$$A_{\psi\Theta}u = \|A_{\psi\Theta}\|Cu,$$

then the coefficients $c_1, c_2, \ldots, c_n$ satisfy the equation

$$M_{A_{\psi\Theta}}\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \|M_{A_{\psi\Theta}}\|\begin{pmatrix} \overline{c_1} \\ \vdots \\ \overline{c_n} \end{pmatrix} \tag{46}$$

where $M_{A_{\psi\Theta}}$ is a complex symmetric matrix. If we write $c_j = x_j + iy_j$ where $x_j, y_j \in \mathbb{R}$, the above equation can be written as a linear system in $2n$ real variables and then solved using linear algebra (then normalizing so that $|c_1|^2 + \cdots + |c_n|^2 = 1$).

Remark 1. If one only wishes to find the extremal function up to a unimodular constant, then one can avoid the conjugation C as follows: If $\{v_1, \ldots, v_n\}$ is *any* orthonormal basis for K_Θ and M is the matrix representation of $A_{\psi\Theta} : K_\Theta \to K_\Theta$ with respect to $\{v_1, \ldots, v_n\}$, it follows from the proof of part (iii) of Theorem 1 that if $(c_1, \ldots, c_n) \in \mathbb{C}^n$ is a unit vector and

$$M^*M \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \|M\|^2 \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix},$$

then, for some unimodular constant ζ,

$$u = \zeta(c_1 v_1 + \cdots + c_n v_n)$$

is an extremal function for $\Gamma(\psi)$.

Example 8. Let us return to the data of Example 2. In particular, recall that we have $\Gamma(\psi) = \|A_{\psi\Theta}\|_{K_\Theta \to K_\Theta} \approx 2.1805$, where

$$\Theta = z\frac{z - 1/2}{1 - z/2}, \quad \psi = \frac{1}{z} + \frac{1}{z - 1/2}.$$

Moreover, we found that the desired supremum is equal to

$$\left\| \begin{pmatrix} \frac{5}{4} & -\frac{7i}{4\sqrt{3}} \\ -\frac{7i}{4\sqrt{3}} & -\frac{13}{12} \end{pmatrix} \right\| = \frac{1}{6}(7 + \sqrt{37}) \approx 2.1805.$$

With $\beta = 1$ and $\zeta_1 = 1, \zeta_2 = -1, \omega_1 = 1, \omega_2 = -i$, we can use (20) to compute the modified Aleksandrov-Clark basis

$$v_1(z) = -1 - \frac{3}{z - 2}, \quad v_2(z) = -i\sqrt{3} - \frac{i\sqrt{3}}{z - 2}$$

for the model space K_Θ. An extremal function u is any unit vector solution to

$$A_{\psi\Theta} u = \|A_{\psi\Theta}\| C u.$$

To compute $u = c_1 v_1 + c_2 v_2$, we need to find a unit vector solution to the anti-linear eigenvalue problem

$$\begin{pmatrix} \frac{5}{4} & -\frac{7i}{4\sqrt{3}} \\ -\frac{7i}{4\sqrt{3}} & -\frac{13}{12} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{6}(7 + \sqrt{37}) \begin{pmatrix} \overline{c_1} \\ \overline{c_2} \end{pmatrix}.$$

A computation with `Mathematica` yields the coefficients

$$c_1 = \frac{1}{2}\sqrt{2 + \frac{1}{\sqrt{37}}}, \quad c_2 = \frac{1}{2} i \sqrt{2 - \frac{1}{\sqrt{37}}}.$$

Therefore, the desired unit vector $u = c_1 v_1 + c_2 v_2$ equals

$$\frac{z\sqrt{2 - \frac{11}{\sqrt{37}}}}{z - 2} - \frac{\sqrt{2 + \frac{10}{\sqrt{37}}}}{z - 2}$$

Another computation with `Mathematica` will show that $A_{\psi\Theta}$ has two distinct singular values and hence the kernel of $|A_{\psi\Theta}| - \|A_{\psi\Theta}\| I$ is one-dimensional. By Corollary 2 it follows that this extremal solution u for $\Gamma(\psi)$ is unique up to a sign.

When examining the extremal problem

$$\Gamma\left(\frac{c_0}{z}+\frac{c_1}{z^2}+\cdots+\frac{c_n}{z^{n+1}}\right),$$

we saw from our previous discussion that this supremum is equal to the norm of the lower triangular Toeplitz matrix T from (40), which is simply the matrix representation of $A_{\psi z^{n+1}}: K_{z^{n+1}} \to K_{z^{n+1}}$, where

$$\psi(z)=\frac{c_0}{z}+\cdots+\frac{c_n}{z^{n+1}},$$

with respect to the orthonormal basis $\{1, z, \ldots, z^n\}$ for $K_{z^{n+1}}$. Observing that

$$C(a_0+a_1z+\cdots+a_nz^n)=\overline{a_n}+\overline{a_{n-1}}z+\cdots+\overline{a_0}z^n,$$

we see that in order to find a unit vector solution u to

$$A_{\psi z^{n+1}}u=\|A_{\psi z^{n+1}}\|Cu,$$

we need to find a unit vector solution $(a_0, a_1, \ldots, a_n)$ to

$$\begin{pmatrix} c_n & & & & \\ \vdots & \ddots & & & \\ c_2 & \cdots & c_n & & \\ c_1 & c_2 & \cdots & c_n & \\ c_0 & c_1 & c_2 & \cdots & c_n \end{pmatrix}\begin{pmatrix} a_0\\ a_1\\ \vdots\\ a_{n-1}\\ a_n\end{pmatrix}=\|A_{\psi z^{n+1}}\|\begin{pmatrix} \overline{a_n}\\ \overline{a_{n-1}}\\ \vdots\\ \overline{a_1}\\ \overline{a_0}\end{pmatrix}. \tag{47}$$

As before, writing $a_j = x_j + iy_j$, $x_j, y_j \in \mathbb{R}$, this can be solved for $a_0, a_1, \ldots, a_n$ using linear algebra (and then normalizing so that $|a_0|^2 + |a_1|^2 + \cdots + |a_n|^2 = 1$).

Example 9. We saw from Example 6 that

$$\Gamma\left(\frac{1}{z}+\frac{1}{z^2}\right)=\|A_{1+z}\|_{K_{z^2}\to K_{z^2}}=\left\|\begin{pmatrix}1&0\\1&1\end{pmatrix}\right\|=\frac{1+\sqrt{5}}{2}\approx 1.618.$$

To find an extremal function $u = a_0 + a_1 z \in K_{z^2}$, we need to find a unit vector solution to

$$\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}a_0\\a_1\end{pmatrix}=\frac{1+\sqrt{5}}{2}\begin{pmatrix}\overline{a_1}\\\overline{a_0}\end{pmatrix}$$

A short computation yields the extremal function

$$u(z)=\frac{2z+\sqrt{5}+1}{\sqrt{2\left(5+\sqrt{5}\right)}}.$$

Recalling that

$$\Gamma\left(\frac{1}{z}+\frac{1}{z^2}\right)=\Lambda\left(\frac{1}{z}+\frac{1}{z^2}\right),$$

observe that u is the square root of the H^1 extremal function from Egerváry's example (5).

Another computation with `Mathematica` shows that A_{1+z} has two distinct singular values and so the kernel of $|A_{1+z}| - \|A_{1+z}\|I$ is one dimensional. By Corollary 2, we conclude that the extremal solution u for $\Gamma(\psi)$ is unique up to a sign.

Example 10. Consider the rational kernel $\psi = 1/z^2$, which yields the nonlinear extremal problem

$$\Gamma(\psi)=\sup_{\substack{f\in K_\Theta\\ \|f\|=1}}\left|\frac{1}{2\pi i}\oint_{\partial\mathbb{D}}\frac{f^2(z)}{z^2}\,dz\right|=2\sup_{\substack{f\in K_\Theta\\ \|f\|=1}}|f(0)f'(0)|.$$

According to our recipe, the corresponding Blaschke product is $\Theta = z^2$. We therefore have $\varphi = \psi\Theta = 1$ whence $A_\varphi = I$ and

$$\Gamma\left(\frac{1}{z^2}\right) = \left\|\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\| = 1.$$

In this case,

$$f_1 = \frac{1}{\sqrt{2}}(1+z), \quad f_2 = \frac{1}{\sqrt{2}}(-1+z) \tag{48}$$

are linearly independent extremal functions for $\Gamma(\psi)$. Despite the fact that the corresponding model space $K_{z^2} = \bigvee\{1, z\}$ is two-dimensional, this does not imply that *every* function $f(z) = a_0 + a_1 z$ satisfying $|a_0|^2 + |a_1|^2 = 1$ is extremal. Indeed, simply consider the functions 1 or z. This behavior is not unexpected, since the underlying extremal problem is nonlinear.

Example 11. Recall the nonlinear extremal problem

$$\Gamma\left(\frac{a_0}{z^2} + \frac{a_1}{z}\right) = \frac{|a_1| + \sqrt{|a_1|^2 + 4|a_0|^2}}{2}$$

from Example 7. With λ denoting the above extremum, a computation with `Mathematica` shows that a solution to the corresponding antilinear eigenvalue problem (43) is given by

$$u_1 = i\frac{|a_0|}{a_0}\frac{-\lambda^2 - \lambda a_1 + a_0^2}{\sqrt{|\lambda^2 - a_1\lambda + a_0^2|^2 + |\lambda^2 + \lambda a_1 - a_0^2|^2}},$$

$$u_2 = \frac{|a_0|}{a_0}\frac{-\lambda^2 + a_1\lambda - a_0^2}{\sqrt{|\lambda^2 - a_1\lambda + a_0^2|^2 + |\lambda^2 + \lambda a_1 - a_0^2|^2}}.$$

This means that an extremal function is $f = u_1 + u_2 z$. One can check that the singular values of

$$\begin{pmatrix} a_0 & 0 \\ a_1 & a_0 \end{pmatrix}$$

are distinct if and only if $a_1 = 0$. Therefore f is the unique extremal function, up to sign, for (42) if and only if $a_1 = 0$. When $a_1 = 0$, the supremum is equal to $|a_0|$ and one can check by direct computation that (48) are two linearly independent solutions to (42).

7. Norm attaining symbols

Since the truncated Toeplitz operator $A_\varphi : K_\Theta \to K_\Theta$ is a multiplication operator followed by an orthogonal projection, it follows immediately that

$$\|A_\varphi\|_{K_\Theta \to K_\Theta} \leq \|\varphi\|_\infty, \qquad \varphi \in H^\infty. \tag{49}$$

For a rational function ψ having no poles on $\partial\mathbb{D}$, we have shown that the nonlinear extremal problem $\Gamma(\psi)$ given by (1) yields the supremum $\Gamma(\psi) = \|A_\varphi\|_{K_\Theta \to K_\Theta}$ where Θ is the corresponding Blaschke product (6) and $\varphi = \psi\Theta$. In particular, it follows that

$$\Gamma(\psi) = \sup_{\substack{f \in H^2 \\ \|f\|=1}} \left|\frac{1}{2\pi i}\oint_{\partial\mathbb{D}} \psi f^2 dz\right| = \|A_\varphi\|_{K_\Theta \to K_\Theta} \leq \|\varphi\|_\infty. \tag{50}$$

We now investigate conditions under which equality holds in (49) and (50).

We say that a unit vector x is a *maximal vector* for a bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ if $\|Tx\| = \|T\|$. It is clear that maximal vectors exist for any compact operator and thus for any operator on a finite-dimensional space. From our perspective, the importance of maximal vectors stems from the following important lemma [37, Prop. 5.1]:

Lemma 4 (Sarason)**.** *Let Θ be any inner function and $T : K_\Theta \to K_\Theta$ be a linear operator of unit norm that commutes with $A_z : K_\Theta \to K_\Theta$. If T has a maximal vector, then there is a unique function $\varphi \in H^\infty$ such that $\|\varphi\|_\infty = 1$ and $A_\varphi = T$. Moreover, φ is both an inner function and the quotient of two functions from K_Θ.*

We say that a symbol $\varphi \in L^\infty$ for a truncated Toeplitz operator A_φ on K_Θ is *norm-attaining* if $\|A_\varphi\| = \|\varphi\|_\infty$. It is important to note that this definition depends upon the particular inner function Θ corresponding to the model space K_Θ upon which A_φ acts. For Θ a finite Blaschke product, the next result says exactly when we have equality in (49):

Theorem 2. *Let Θ be a finite Blaschke product. A symbol φ in H^∞ is norm-attaining with respect to K_Θ if and only if φ is a scalar multiple of the inner factor of a function from K_Θ.*

Proof. ($\Rightarrow$) Without loss of generality, we assume that $\|A_\varphi\| = \|\varphi\|_\infty = 1$. Since Θ is a finite Blaschke product, it follows that K_Θ is finite-dimensional whence the truncated Toeplitz operator $A_\varphi : K_\Theta \to K_\Theta$ has a maximal vector. Now recall the well-known fact [37, Thm. 1] that the commutant of $A_z : K_\Theta \to K_\Theta$ is

$$\{A_\varphi : K_\Theta \to K_\Theta : \varphi \in H^\infty\}$$

whence A_φ commutes with A_z. By Lemma 4, it follows immediately that φ is an inner function and $\varphi = f/g$, for some $f, g \in K_\Theta$. Thus $f = \varphi g = \varphi I_g F_g$, where I_g is the inner factor of g and F_g is the outer factor. A short argument now reveals that

$$A_{\overline{I_g}} f = P_\Theta(\overline{I_g} \varphi I_g F_g) = P_\Theta(\varphi F_g) = \varphi F_g \tag{51}$$

since the function φF_g belongs to H^2 and is orthogonal to ΘH^2. Indeed, since f belongs to K_Θ we have, for any $h \in H^2$,

$$\langle \varphi F_g, \Theta h\rangle = \langle \varphi I_g F_g, \Theta(I_g h)\rangle = \langle f, \Theta(I_g h)\rangle = 0.$$

We conclude from (51) that φ is indeed the inner factor of a function from K_Θ.

($\Leftarrow$) Conversely, let φ be the inner factor of a function from K_Θ. In other words, suppose that there exists an outer function F of unit norm such that φF belongs to K_Θ. This implies that

$$\|A_\varphi F\| = \|P_\Theta(\varphi F)\| = \|\varphi F\| = \|F\| = 1$$

whence $\|A_\varphi\| = 1$. Thus φ is a norm-attaining symbol with respect to Θ. ■

Let us make a few important remarks concerning possible generalizations of the preceding theorem. First observe that the ($\Leftarrow$) implication of Theorem 2 remains true for any inner function Θ. Furthermore, the ($\Rightarrow$) implication of the theorem remains true for general inner Θ and $\varphi \in H^\infty$ as long as the operator $A_\varphi : K_\Theta \to K_\Theta$ has a maximal vector. This occurs, for instance, if A_φ is compact.

Corollary 3. *Let Θ be a finite Blaschke product. If $\varphi \in H^\infty$ is not a scalar multiple of a Blaschke product, then $\|A_\varphi\| < \|\varphi\|_\infty$.*

Corollary 4. *If Θ is an inner function and φ is a finite Blaschke product of degree $< \dim K_\Theta$, then φ is a norm attaining symbol for K_Θ.*

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote the zeros of φ and note that $n < \dim K_\Theta$. Consequently there exists a nonzero function $f \in K_\Theta$ such that f is orthogonal to the n kernel functions $k_{\lambda_1}, k_{\lambda_2}, \ldots, k_{\lambda_n}$. In other words, there exists a nonzero function $f \in K_\Theta$ which vanishes at each $\lambda_1, \lambda_2, \ldots, \lambda_n$. Since inner factors of functions in K_Θ can be removed without leaving K_Θ, we may assume that the inner factor of f is precisely φ. By the remarks preceding Corollary 3, it follows that φ is a norm-attaining symbol for K_Θ. ■

Example 12. If Θ is a singular inner function, then there exists a Blaschke product φ having simple zeros such that φ is a norm-attaining symbol for K_Θ. Indeed, an argument using Frostman's Theorem [30, Thm. 3.10.2] (see also [15, Sec. 3.3]) produces many such Blaschke products which occur as inner factors of functions in K_Θ. Now employ the observations made prior to Corollary 3.

The following corollary of Theorem 2 (along with Corollary 2) tells us when we have equality in (50).

Corollary 5. *For a rational function ψ having no poles in $\partial\mathbb{D}$, the following are equivalent:*

(i) $\Lambda(\psi) = \Gamma(\psi) = \max_{\zeta\in\partial\mathbb{D}} |\psi(\zeta)|$,

(ii) *There exist a $c \in \mathbb{C}$ and finite Blaschke products B_1 and B_2 having no common zeros and satisfying* $\deg B_1 < \deg B_2$ *so that*

$$\psi = c\frac{B_1}{B_2}.$$

Proof. If we assume (i), then, adopting the notation from Corollary 2, we see that the symbol $\varphi = \psi\Theta$ is norm-attaining with respect to K_Θ. Thus by Theorem 2 it follows that $\psi\Theta = cB$ where $c \in \mathbb{C}$ and B is the inner factor of some function $f \in K_\Theta$. Using the representation (10) we find that

$$f(z) = \frac{p(z)}{\prod_{j=1}^{n}(1-\overline{\lambda_j}z)}, \tag{52}$$

where $p(z)$ is a polynomial with $\deg p < n = \deg\Theta$. It follows that B is a finite Blaschke product with $\deg B < n$. The fact that Θ and B have no common zeros follows from the definition of Θ from (6).

Conversely, if we assume (ii), then $\psi = c\frac{B}{\Theta}$, where B and Θ are finite Blaschke products with $\deg B < \deg\Theta$. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the zeros of Θ (repeated according to multiplicity), then every $f \in K_\Theta$ takes the form (52). If we take $p(z)$ to be a polynomial whose zeros are precisely those of B, which is possible since $\deg B < n$, then B will be the inner factor of a function from K_Θ. By Theorem 2 it then follows that the symbol $\psi\Theta$ is norm attaining with respect to K_Θ. To conclude the proof, simply appeal to Corollary 2. ■

Example 13. In Example 6 we showed that

$$\Gamma\left(\frac{1}{z}+\frac{1}{z^2}\right) = \frac{1+\sqrt{5}}{2} \approx 1.618.$$

Observe that with $\psi(z) = 1/z + 1/z^2$, we have

$$\max_{\zeta\in\partial\mathbb{D}} |\psi(\zeta)| = 2 > \Gamma(\psi).$$

This strict inequality is expected since the rational function

$$\psi(z) = \frac{z+1}{z^2}$$

cannot be written as B_1/B_2 where $\deg B_1 < \deg B_2$. Indeed, ψ is the quotient of the nonconstant outer function $z+1$ and the finite Blaschke product z^2.

Example 14. In a very similar way, we know from Example 2 that

$$\Gamma\left(\frac{1}{z}+\frac{1}{z-1/2}\right) = \frac{1}{6}(7+\sqrt{37}) \approx 2.1805.$$

With $\psi(z) = 1/z + 1/(z - \frac{1}{2})$, a brief calculation shows that

$$\max_{\zeta \in \partial\mathbb{D}} |\psi(\zeta)| = 3 > \Gamma(\psi).$$

As in the preceding example, the strict inequality is to be expected since

$$\psi(z) = 2\frac{z - \frac{1}{4}}{z(z - \frac{1}{2})}$$

cannot be written in the form B_1/B_2 where $\deg B_1 < \deg B_2$.

Example 15. Let $\varphi(z) = z$ and

$$\Theta(z) = z\left(\frac{z - \frac{1}{2}}{1 - \frac{1}{2}z}\right). \tag{53}$$

Since $\|A_z\| \leq \|z\|_\infty = 1$ and

$$\|A_z(1)\| = \|P_\Theta(z1)\| = \|z\| = 1,$$

it follows immediately that $\|A_z\| = 1$. That z is a norm-attaining symbol for K_Θ is to be expected, since z is indeed the inner factor of a function from K_Θ.

This can also be verified by a direct computation. Using $\beta = 1$ in our recipe, we find that $\zeta_1 = 1$ and $\zeta_2 = -1$ whence $\omega_1 = 1$ and $\omega_2 = -i$. Putting this all together yields the modified Aleksandrov-Clark basis

$$v_1(z) = \frac{1}{2}\frac{1 - \Theta(z)}{1 - z}, \quad v_2(z) = -i\frac{\sqrt{3}}{2}\frac{1 - \Theta(z)}{1 + z}$$

with respect to which A_z has the matrix representation

$$M_{A_z} = \begin{pmatrix} \frac{3}{4} & -\frac{i\sqrt{3}}{4} \\ -\frac{i\sqrt{3}}{4} & -\frac{1}{4} \end{pmatrix}$$

which, via (38), has norm 1, as expected.

Example 16. Let $\varphi(z) = \frac{1-z}{2}$ and let Θ as in (53). Using the data from the previous example, we see that

$$M_{A_\varphi} = \begin{pmatrix} \frac{1}{8} & \frac{i\sqrt{3}}{8} \\ \frac{i\sqrt{3}}{8} & \frac{5}{8} \end{pmatrix},$$

which has norm equal to

$$\frac{1}{4}(1 + \sqrt{3}) \approx 0.683013 < 1 = \|\varphi\|_\infty .$$

This is to be expected since φ is outer.

8. Best constants

As an application of the identity

$$\Lambda(\psi) = \Gamma(\psi) = \|A_{\psi\Theta}\|_{K_\Theta \to K_\Theta},$$

we can obtain the best constant in various pointwise estimates of the derivatives of H^1 functions. Such estimates have been studied before [12, 20, 26, 27] (The papers [2, 4, 25, 36] contain related results). using different methods and function theoretic language. Our discussion uses operator theoretic language. To begin, first observe that the Cauchy integral formula [11, p. 40] says that

$$F^{(n)}(\lambda) = \frac{n!}{2\pi i}\oint_{\partial\mathbb{D}} \frac{F(z)}{(z - \lambda)^{n+1}}dz, \qquad F \in H^1,\ \lambda \in \mathbb{D}. \tag{54}$$

This yields, via the inequality $|z-\lambda| \geq 1-|\lambda|$ for $|z|=1$, the pointwise estimate

$$|F^{(n)}(\lambda)| \leq \frac{n!}{(1-|\lambda|)^{n+1}} \|F\|_1 . \tag{55}$$

This is not the best that we can do, however. Going back to (54) we see that the best (sharpest) constant $c_{n,\lambda}$ in the pointwise inequality

$$|F^{(n)}(\lambda)| \leq c_{n,\lambda} \|F\|_1 , \quad F \in H^1, \tag{56}$$

is

$$c_{n,\lambda} = \Lambda\left(\frac{n!}{(z-\lambda)^{n+1}}\right) = \Gamma\left(\frac{n!}{(z-\lambda)^{n+1}}\right).$$

By Corollary 1, this constant is the operator norm of the truncated Toeplitz operator

$$A_{\frac{n!}{(1-\overline{\lambda}z)^{n+1}}} : K_\Theta \to K_\Theta$$

where

$$\Theta(z) = \left(\frac{z-\lambda}{1-\overline{\lambda}z}\right)^{n+1}. \tag{57}$$

Putting this all together, we obtain the following theorem:

Theorem 3. *For $n = 0, 1, 2, \ldots$ and $\lambda \in \mathbb{D}$, we have*

$$c_{n,\lambda} = \frac{n!}{(1-|\lambda|^2)^{n+1}} \|T\|,$$

where T is the $(n+1) \times (n+1)$ lower triangular Toeplitz matrix whose entries are

$$[T]_{ij} = \begin{cases} |\lambda|^{i-j}\binom{n+1}{i-j} & \text{if } i \geq j, \\ 0 & \text{if } i < j. \end{cases} \tag{58}$$

Furthermore, an extremal function for $\Lambda(\frac{n!}{(z-\lambda)^{n+1}})$, when $\lambda = r \in (0,1)$, is of the form

$$F(z) = \zeta \frac{1-r^2}{(1-rz)^2}\left[\alpha_0 + \alpha_1 \frac{z-r}{1-rz} + \cdots + \alpha_n \left(\frac{z-r}{1-rz}\right)^n\right]^2, \tag{59}$$

where ζ is a certain unimodular constant and $(\alpha_0, \ldots, \alpha_n) \in \mathbb{C}^{n+1}$ is a unit vector solution to

$$T^*T \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \\ \alpha_n \end{pmatrix} = \|T\|^2 \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \\ \alpha_n \end{pmatrix}. \tag{60}$$

Before proceeding with the proof of Theorem 3, let us make a few remarks.

(i) Once the unit vector solution $(\alpha_0, \alpha_1, \ldots, \alpha_n)$ to (60) is fixed, the unimodular constant ζ in (59) is determined by the condition

$$\frac{1}{2\pi i}\oint_{\partial\mathbb{D}} \frac{n!}{(z-r)^{n+1}} F(z)dz > 0.$$

(ii) Since the matrix (58) in the formula from Theorem 3 is a *real* Toeplitz matrix, its norm can be computed by considering the corresponding *selfadjoint* Hankel matrix obtained by reversing the order of its rows.

(iii) The theorem above was first proved by Golusin [20] using a function-theoretic approach.

Proof. To prove (58) we may, by radial symmetry, assume that $\lambda = r \in (0,1)$. Let Θ denote the finite Blaschke product (57) and note that $\dim K_\Theta = n+1$. Next, define $\psi = \frac{n!}{(z-r)^{n+1}}$ so that

$$\varphi = \psi\Theta = \frac{n!}{(1-rz)^{n+1}}.$$

Recall now that the corresponding Takenaka-Malmquist-Walsh basis

$$\{v_1, v_2, \ldots, v_{n+1}\}$$

for K_Θ from (18) is given by

$$v_k(z) = \frac{\sqrt{1-r^2}}{(1-rz)^k}(z-r)^{k-1}, \quad 1 \leq k \leq n+1,$$

whence the ijth entry $[M]_{ij}$ of the matrix representation $M = M_{A_\varphi}$ of the truncated Toeplitz operator $A_\varphi : K_\Theta \to K_\Theta$ is

$$[M]_{ij} = \langle A_\varphi v_j, v_i \rangle .$$

As we have seen, the matrix M is lower-triangular (see Subsection 5.1) so that $[M]_{ij} = 0$ whenever $i < j$. Let us compute $[M]_{ij}$ for $i \geq j$:

$$\begin{aligned}
[M]_{ij} &= \langle A_\varphi v_j, v_i \rangle = \langle \varphi v_j, v_i \rangle \\
&= (1-r^2)\frac{n!}{2\pi i} \oint_{\partial\mathbb{D}} \frac{1}{(1-rz)^{n+1}} \frac{(z-r)^{j-1}}{(1-rz)^j} \frac{\overline{(z-r)}^{i-1}}{\overline{(1-rz)}^i} \frac{dz}{z} \\
&= (1-r^2)\frac{n!}{2\pi i} \oint_{\partial\mathbb{D}} \frac{1}{(1-rz)^{n+1}} \frac{(z-r)^{j-1}}{(1-rz)^j} \frac{(\overline{z}-r)^{i-1}}{(1-r\overline{z})^i} \frac{z^{i-1}}{z^i} dz \\
&= (1-r^2)\frac{n!}{2\pi i} \oint_{\partial\mathbb{D}} \frac{(z-r)^{j-1}}{(1-rz)^{n+j+1}} \frac{(1-rz)^{i-1}}{(z-r)^i} dz \\
&= (1-r^2)\frac{n!}{2\pi i} \oint_{\partial\mathbb{D}} \frac{1}{(1-rz)^{n+j-i+2}} \frac{1}{(z-r)^{i-j+1}} dz. \qquad (61)
\end{aligned}$$

For $p, q = 0, 1, 2, \ldots$, an application of the Cauchy integral formula shows that

$$\frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \frac{1}{(1-rz)^p} \frac{1}{(z-r)^{q+1}} dz = \frac{(p+q-1)!}{q!(p-1)!} \frac{r^q}{(1-r^2)^{p+q}}.$$

Applying this identity with $p = n+j-i+2$ and $q = i-j$ and continuing from (61) we find that

$$\begin{aligned}
[M]_{ij} &= n!(1-r^2)\frac{(n+1)!}{(i-j)!(n+j-i+1)!} \frac{r^{i-j}}{(1-r^2)^{n+2}} \\
&= \frac{n!}{(1-r^2)^{n+1}} \binom{n+1}{i-j} r^{i-j}
\end{aligned}$$

for $i \geq j$. This yields the desired formula (58).

The second part of the theorem follows from Remark 1 and the fact that if f is an extremal function for $\Gamma(\frac{n!}{(z-r)^{n+1}})$, then $F = f^2$ is an extremal function for $\Lambda(\frac{n!}{(z-r)^{n+1}})$. ■

Remark 2. The coefficients $\alpha_0, \ldots, \alpha_n$ from (60) can also be computed by solving the $\mathbb{R}$-linear system

$$T \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \\ \alpha_n \end{pmatrix} = \|T\| \begin{pmatrix} \overline{\alpha_n} \\ \overline{\alpha_{n-1}} \\ \vdots \\ \overline{\alpha_1} \\ \overline{\alpha_0} \end{pmatrix}.$$

To see this, notice that T is the matrix representation, with respect to the monomial basis $\{1, z, z^2, \ldots, z^{n-1}\}$, for the truncated Toeplitz operator $A_\varphi : K_{z^n} \to K_{z^n}$, where

$$\varphi(z) = \sum_{k=0}^{n-1} \binom{n}{k} r^k z^k.$$

The isomorphism $U : K_{z^n} \to K_\Theta$, $Uz^j = v_{j+1}$, where v_{j+1} is one of the Takenaka-Malmquist-Walsh basis elements, shows that $A_\varphi : K_{z^n} \to K_{z^n}$ is unitarily equivalent to $A_{\frac{n!}{(1-rz)^{n+1}}} : K_\Theta \to K_\Theta$. Moreover, the extremal vectors for each of these truncated Toeplitz operators get mapped to each other via U. The extremal vector for A_φ on K_{z^n} can be solved by (47).

Example 17. Consider the simple case $n = 0$. Let

$$\varphi(z) = \frac{1}{1 - rz}, \quad \Theta(z) = \frac{z - r}{1 - rz}$$

so that $\psi(z) = 1/(z - r)$. Using the fact that the model space K_Θ is precisely the span of the single function $1/(1 - rz)$, it follows from the Cauchy integral formula that

$$\begin{aligned} c_{0,r} &= \Lambda\left(\frac{1}{z-r}\right) = \Gamma\left(\frac{1}{z-r}\right) \\ &= \frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \frac{1/(1-rz)^2}{z-r}\, dz \\ &= \frac{1}{1-r^2}. \end{aligned}$$

Replacing r with $|\lambda|$, where $\lambda \in \mathbb{D}$, we obtain the *sharp* estimate

$$|F(\lambda)| \leq \frac{1}{1 - |\lambda|^2} \|F\|_1, \qquad F \in H^1.$$

In particular, this is a substantial improvement over the naive estimate (55). Moreover, the preceding reproduces (using entirely different techniques) a result of Egerváry [12] (see also [26, 27]). An extremal function for the linear extremal problem $\Lambda(\frac{1}{z-\lambda})$ is simply

$$F(z) = \frac{1 - |\lambda|^2}{(1 - \overline{\lambda} z)^2}.$$

Notice how this is the square of the H^2 function

$$f(z) = \frac{\sqrt{1 - |\lambda|^2}}{1 - \overline{\lambda} z}$$

which is a unimodular scalar multiple of an extremal function for the corresponding nonlinear problem $\Gamma(\frac{1}{z-\lambda})$.

Example 18. We now consider the slightly more complicated case $n = 1$. Using Theorem 3 we get

$$c_{1,r} = \frac{1}{(1 - r^2)^2} \left\| \begin{pmatrix} 1 & 0 \\ 2r & 1 \end{pmatrix} \right\|.$$

Using (38), we find that the norm of the 2×2 Toeplitz matrix

$$T = \begin{pmatrix} 1 & 0 \\ 2r & 1 \end{pmatrix}$$

is given by the formula

$$\left\| \begin{pmatrix} 1 & 0 \\ 2r & 1 \end{pmatrix} \right\| = r + \sqrt{1 + r^2}$$

whence

$$c_{1,r} = \frac{r + \sqrt{1+r^2}}{(1-r^2)^2}.$$

Replacing r by $|\lambda|$ and going back to (56) we establish the following *sharp* estimate:

$$|F'(\lambda)| \leq \frac{|\lambda| + \sqrt{1+|\lambda|^2}}{(1-|\lambda|^2)^2} \|F\|_1, \quad F \in H^1,$$

This estimate was originally proved by Macintyre and Rogosinski in [26] using entirely different methods.

An extremal function f for $\Gamma(\frac{1}{(z-r)^2})$ must be of the form

$$f = \zeta(a_1 v_1 + a_2 v_2),$$

where $|\zeta| = 1$, $|a_1|^2 + |a_2|^2 = 1$, $\{v_1, v_2\}$ is the Takenaka-Malmquist-Walsh basis

$$v_1(z) = \frac{\sqrt{1-r^2}}{1-rz}, \quad v_2(z) = \frac{\sqrt{1-r^2}}{(1-rz)^2}(z-r),$$

for

$$H^2 \ominus \left(\frac{z-r}{1-rz}\right)^2 H^2,$$

and (a_1, a_2) satisfy

$$\begin{pmatrix} 1 & 2r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2r & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \left\| \begin{pmatrix} 1 & 0 \\ 2r & 1 \end{pmatrix} \right\|^2 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

A `Mathematica` computation shows that

$$a_1 = \frac{r + \sqrt{r^2+1}}{\sqrt{\left(r + \sqrt{r^2+1}\right)^2 + 1}}, \quad a_2 = \frac{1}{\sqrt{\left(r + \sqrt{r^2+1}\right)^2 + 1}}.$$

Thus an extremal function for $\Lambda(\frac{1}{(z-r)^2})$ is $F = f^2$.

Example 19. Macintyre and Rogosinski [27, Sec. 11] note several qualitative properties of $c_{2,r}$, although they do not compute it explicitly. Using our techniques, we can compute $c_{2,r}$ explicitly. When $n = 2$, we apply Theorem 3 to get

$$c_{2,r} = \frac{2!}{(1-r^2)^3} \left\| \begin{pmatrix} 1 & 0 & 0 \\ 3r & 1 & 0 \\ 3r^2 & 3r & 1 \end{pmatrix} \right\| = \frac{2!}{(1-r^2)^3} \left\| \begin{pmatrix} 3r^2 & 3r & 1 \\ 3r & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\|.$$

Using `Mathematica` to symbolically compute the norm of the preceding 3×3 selfadjoint Hankel matrix, we find that $c_{2,r}$ is $\frac{2!}{(1-r^2)^3}$ times the quantity

$$r^2 + \frac{2}{3}\sqrt{9r^4 + 24r^2 + 4}\cos\left(\frac{1}{3}\arg\left(9r\left(6r^5 + 24r^3 + 11r + i\sqrt{3\left(36\left(r^2+4\right)r^2 + 109\right)r^2 + 96}\right) - 16\right)\right) + \frac{1}{3}.$$

As in the previous example, an extremal function f for $\Gamma(\frac{2!}{(z-r)^3})$ is

$$f(z) = \zeta(a_1 v_1(z) + a_2 v_2(z) + a_3 v_3(z)),$$

where ζ is a certain unimodular constant,

$$v_1(z) = \frac{\sqrt{1-r^2}}{1-rz}, \quad v_2(z) = \frac{\sqrt{1-r^2}}{(1-rz)^2}(z-r), \quad v_3(z) = \frac{\sqrt{1-r^2}}{(1-rz)^3}(z-r)^2,$$

and (a_1, a_2, a_3) is a unit vector solution to

$$\begin{pmatrix} 1 & 0 & 0 \\ 3r & 1 & 0 \\ 3r^2 & 3r & 1 \end{pmatrix} \begin{pmatrix} 1 & 3r & 3r^2 \\ 0 & 1 & 3r \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \left\| \begin{pmatrix} 1 & 0 & 0 \\ 3r & 1 & 0 \\ 3r^2 & 3r & 1 \end{pmatrix} \right\|^2 \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

or equivalently

$$\begin{pmatrix} 1 & 0 & 0 \\ 3r & 1 & 0 \\ 3r^2 & 3r & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \left\| \begin{pmatrix} 1 & 0 & 0 \\ 3r & 1 & 0 \\ 3r^2 & 3r & 1 \end{pmatrix} \right\| \begin{pmatrix} \overline{a_3} \\ \overline{a_2} \\ \overline{a_1} \end{pmatrix}.$$

Once the unit vector solution (a_1, a_2, a_3) to the above system is fixed, the unimodular constant ζ is determined by the condition that

$$\frac{1}{2\pi i} \oint_{\partial \mathbb{D}} \frac{2!}{(z-r)^3} f(z)^2 dz > 0.$$

Example 20. In a similar fashion we obtain

$$c_{3,r} = \frac{3!}{(1-r^2)^4} \left\| \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4r & 1 & 0 & 0 \\ 6r^2 & 4r & 1 & 0 \\ 4r^3 & 6r^2 & 4r & 1 \end{pmatrix} \right\|,$$

$$c_{4,r} = \frac{4!}{(1-r^2)^5} \left\| \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 5r & 1 & 0 & 0 & 0 \\ 10r^2 & 5r & 1 & 0 & 0 \\ 10r^3 & 10r^2 & 5r & 1 & 0 \\ 5r^4 & 10r^3 & 10r^2 & 5r & 1 \end{pmatrix} \right\|,$$

$$c_{5,r} = \frac{5!}{(1-r^2)^6} \left\| \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 6r & 1 & 0 & 0 & 0 & 0 \\ 15r^2 & 6r & 1 & 0 & 0 & 0 \\ 20r^3 & 15r^2 & 6r & 1 & 0 & 0 \\ 15r^4 & 20r^3 & 15r^2 & 6r & 1 & 0 \\ 6r^5 & 15r^4 & 20r^3 & 15r^2 & 6r & 1 \end{pmatrix} \right\|.$$

Unfortunately, evaluating these expressions explicitly in terms of r is prohibitive (if not impossible). However, it is clear that these quantities can be computed numerically, for a specified value of r, to arbitrarily high precision since this involves nothing more than computing the eigenvalues of a selfadjoint Hankel matrix.

We also remark that if V is the $(n+1) \times (n+1)$ matrix

$$V = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix},$$

then

$$T = (I + rV)^{n+1}. \tag{62}$$

If $\tau_{n,r}$ is the norm of the Toeplitz matrix from (58) with $|\lambda| = r$ (so that $c_{n,r} = \frac{n!}{(1-r^2)^{n+1}} \tau_{n,r}$), Golusin [20] (see also [23]) proved that for fixed n the function

$$r \mapsto \tau_{n,r}$$

is an increasing function on $(0,1)$ and

$$\lim_{r\to 1^-} \tau_{n,r} \leq 2^{n+1}.$$

For an $N \times N$ matrix A, let

$$\|A\|_F := \left(\sum_{i,j=1}^{N} |A_{i,j}|^2 \right)^{1/2}$$

denote the *Frobenius norm* (also known as the *Hilbert-Schmidt norm*) of A and note that

$$\frac{1}{\sqrt{N}} \|A\|_F \leq \|A\| \leq \|A\|_F$$

(see [22, p. 314] for further details). Using this inequality and the definition of $\tau_{n,r}$, we get a small improvement of Golusin's estimate, namely

$$\left(\frac{1}{n+1} \sum_{k=0}^{n} (n+1-k) \binom{n+1}{k}^2 r^{2k} \right)^{1/2} \leq \tau_{n,r} \leq \left(\sum_{k=0}^{n} (n+1-k) \binom{n+1}{k}^2 r^{2k} \right)^{1/2}.$$

Going back to the simplistic estimate in (55), Corollary 5 implies that

$$c_{n,r} < \frac{n!}{(1-r)^{n+1}} = \sup_{\zeta\in\partial\mathbb{D}} \left| \frac{n!}{(1-r\zeta)^{n+1}} \right|. \tag{63}$$

9. Final comments

If ψ belongs to $L^\infty(\partial\mathbb{D})$ but is not necessarily rational, one can still consider the extremal problems $\Lambda(\psi)$ and $\Gamma(\psi)$. At this level of generality, several technical problems arise. To begin with, there are functions $\psi \in L^\infty$ for which the linear extremal problem $\Lambda(\psi)$ given by (2) does not have an extremal function [11, p. 134]. In addition, we do not know if $\Lambda(\psi)$ is equal to $\Gamma(\psi)$ nor do we know if it is possible for an extremal function to exist for $\Lambda(\psi)$ but not for $\Gamma(\psi)$. On the other hand, we still obtain the inequality $\Gamma(\psi) \leq \Lambda(\psi)$ from (30).

When ψ is continuous on $\partial\mathbb{D}$, it is known [23, p. 33] that an extremal function F for $\Lambda(\psi)$ exists and that F can be chosen so that $|F| > 0$ on $\mathbb{D}$. It follows, by noting that $f := \sqrt{F}$ is an extremal function for $\Gamma(\psi)$ (see Section 4 above), that $\Lambda(\psi) = \Gamma(\psi)$.

If $\psi \in L^\infty(\partial\mathbb{D})$ such that there is an inner Θ (not necessarily a finite Blaschke product) for which $\psi\Theta \in H^\infty$, then one can show that the proofs of Theorem 1 (i), Corollary 2 (i), and Proposition 1 are still valid. In particular, this implies that

$$\Lambda(\psi) = \Gamma(\psi) = \|A_{\psi\Theta}\|_{K_\Theta \to K_\Theta}.$$

Using the remarks preceding Corollary 3, we can also show that if $\psi = \lambda \frac{I_f}{\Theta}$, where $\lambda \in \mathbb{C}$ and I_f is the inner factor of a function f from K_Θ, then

$$\Lambda(\psi) = \Gamma(\psi) = \|\psi\|_\infty.$$

Unfortunately, any potential statements from our results concerning the existence and possible uniqueness of extremal functions for $\Gamma(\psi)$ must require different proofs since it is not clear whether or not $A_{\psi\Theta} : K_\Theta \to K_\Theta$ has a maximal vector. Additionally, one does not necessarily have a practical matrix representation for $A_{\psi\Theta}$ with which to compute the norm.

We end with some results which begin to extend our work beyond rational ψ.

Lemma 5. *If $\varphi \in H^\infty$ and Θ is an interpolating Blaschke product with zeros $\lambda_1, \lambda_2, \ldots$ (necessarily distinct) then $A_\varphi : K_\Theta \to K_\Theta$ is compact if and only if $\lim_{n\to\infty} \varphi(\lambda_n) = 0$.*

Proof. By Carleson's Interpolation Theorem, it follows that the kernels k_{λ_n} form a Riesz basis for K_Θ [29, p. 132-135] whence there exists a bounded invertible operator $Q : l^2(\mathbb{N}) \to K_\Theta$ such that $Qe_n = k_{\lambda_n}$ for $n \in \mathbb{N}$. Here e_n denotes the n-th standard basis vector for $l^2(\mathbb{N})$. Since $A_\varphi^* k_{\lambda_n} = \overline{\varphi(\lambda_n)} k_{\lambda_n}$ (see (23)) for all n, it follows that $Q^{-1}A_\varphi^* Q e_n = \overline{\varphi(\lambda_n)} e_n$ for all n. Thus A_φ^* is similar to the diagonal operator $\operatorname{diag}(\overline{\varphi(\lambda_1)}, \overline{\varphi(\lambda_2)}, \ldots)$ on $l^2(\mathbb{N})$ which is compact if and only if $\lim_{n\to\infty} \varphi(\lambda_n) = 0$. ■

Observe that if the associated truncated Toeplitz operator $A_\varphi : K_\Theta \to K_\Theta$ is compact, then A_φ has a maximal vector and thus most of our proofs can proceed as before. In particular, we get the following:

Corollary 6. *If $\psi = \frac{\varphi}{\Theta}$, where $\varphi \in H^\infty$, Θ is an interpolating Blaschke product with zeros $\lambda_1, \lambda_2, \ldots$, and $\lim_{n\to\infty} \varphi(\lambda_n) = 0$, then $\Lambda(\psi) = \Gamma(\psi)$ and there exists an extremal function for $\Gamma(\psi)$ and hence for $\Lambda(\psi)$.*

Corollary 7. *If $\psi = \frac{\varphi}{\Theta}$, where $\varphi \in H^\infty$, Θ is an interpolating Blaschke product with zeros $\lambda_1, \lambda_2, \ldots$, and $\lim_{n\to\infty} \varphi(\lambda_n) = 0$, then $\Lambda(\psi) = \Gamma(\psi) = \|\psi\|_\infty$ if and only if $\psi = \lambda \frac{I}{\Theta}$, where $\lambda \in \mathbb{C}$ and I is the inner factor of a function from K_Θ.*

References

1. D. Aharonov, C. Bénéteau, D. Khavinson, and H. Shapiro, *Extremal problems for nonvanishing functions in Bergman spaces*, Selected topics in complex analysis, Oper. Theory Adv. Appl., vol. 158, Birkhäuser, Basel, 2005, pp. 59–86.
2. J. M. Anderson and J. Rovnyak, *On generalized Schwarz-Pick estimates*, Mathematika **53** (2006), no. 1, 161–168 (2007).
3. V. Andreev and A. L. Matheson, *Quadratic extremal problems on the Dirichlet space*, Complex Var. Theory Appl. **48** (2003), no. 9, 767–776.
4. C. Bénéteau, A. Dahlner, and D. Khavinson, *Remarks on the Bohr phenomenon*, Comput. Methods Funct. Theory **4** (2004), no. 1, 1–19.
5. C. Bénéteau and D. Khavinson, *A survey of certain extremal problems for non-vanishing analytic functions*, Complex and harmonic analysis, DEStech Publ., Inc., Lancaster, PA, 2007, pp. 45–61.
6. J. A. Cima, A. L. Matheson, and W. T. Ross, *The Cauchy transform*, Mathematical Surveys and Monographs, vol. 125, American Mathematical Society, Providence, RI, 2006.
7. J. A. Cima and W. T. Ross, *The backward shift on the Hardy space*, Mathematical Surveys and Monographs, vol. 79, American Mathematical Society, Providence, RI, 2000.
8. D. N. Clark, *One dimensional perturbations of restricted shifts*, J. Analyse Math. **25** (1972), 169–191.
9. J. B. Conway, *A course in functional analysis*, Graduate Texts in Mathematics, vol. 96, Springer-Verlag, New York, 1985.
10. R. G. Douglas, H. S. Shapiro, and A. L. Shields, *Cyclic vectors and invariant subspaces for the backward shift operator.*, Ann. Inst. Fourier (Grenoble) **20** (1970), no. fasc. 1, 37–76.
11. P. L. Duren, *Theory of H^p spaces*, Academic Press, New York, 1970.
12. E. Egerváry, *Über gewisse Extremumprobleme der Funktionentheorie*, Math. Ann. **99** (1928), no. 1, 542–561.
13. L. Fejér, *Über gewisse Minimumprobleme der Funktionentheorie*, Math. Ann. **97** (1927), no. 1, 104–123.
14. S. D. Fisher, *A quadratic extremal problem on the Dirichlet space*, Complex Variables Theory Appl. **26** (1995), no. 4, 367–380.
15. S. R. Garcia, *Conjugation and Clark operators*, Recent advances in operator-related function theory, Contemp. Math., vol. 393, Amer. Math. Soc., Providence, RI, 2006, pp. 67–111.
16. ______, *Approximate antilinear eigenvalue problems and related inequalities*, Proc. Amer. Math. Soc. **136** (2008), no. 1, 171–179 (electronic).
17. S. R. Garcia and M. Putinar, *Complex symmetric operators and applications*, Trans. Amer. Math. Soc. **358** (2006), no. 3, 1285–1315 (electronic).
18. ______, *Complex symmetric operators and applications. II*, Trans. Amer. Math. Soc. **359** (2007), no. 8, 3913–3931 (electronic).
19. J. Garnett, *Bounded analytic functions*, first ed., Graduate Texts in Mathematics, vol. 236, Springer, New York, 2007.

20. G. Golusin, *Estimates for analytic functions with bounded mean of the modulus*, Trav. Inst. Math. Stekloff **18** (1946), 87 pp.
21. ______, *On the problem of Carathéodory-Fejér and similar problems*, Rec. Math. [Mat. Sbornik] N. S. **18(60)** (1946), 213–226.
22. R. Horn and C. Johnson, *Matrix analysis*, Cambridge University Press, Cambridge, 1985.
23. S. Ja Khavinson, *Two papers on extremal problems in complex analysis*, Amer. Math. Soc. Transl. **129** (1986), no. 2.
24. P. Koosis, *Introduction to H_p spaces*, second ed., Cambridge Tracts in Mathematics, vol. 115, Cambridge University Press, Cambridge, 1998.
25. B. D. MacCluer, K. Stroethoff, and R. Zhao, *Schwarz-Pick type estimates*, Complex Var. Theory Appl. **48** (2003), no. 8, 711–730.
26. A. J. Macintyre and W. W. Rogosinski, *Some elementary inequalities in function theory*, Edinburgh Math. Notes **1945** (1945), no. 35, 1–3.
27. ______, *Extremum problems in the theory of analytic functions*, Acta Math. **82** (1950), 275–325.
28. J. M. McDougall, *Orthogonal polynomials and quadratic extremal problems*, Trans. Amer. Math. Soc. **354** (2002), no. 6, 2341–2357 (electronic).
29. N. Nikolski, *Treatise on the shift operator*, Springer-Verlag, Berlin, 1986.
30. ______, *Operators, functions, and systems: an easy reading. Vol. 1*, Mathematical Surveys and Monographs, vol. 92, American Mathematical Society, Providence, RI, 2002.
31. ______, *Operators, functions, and systems: an easy reading. Vol. 2*, Mathematical Surveys and Monographs, vol. 93, American Mathematical Society, Providence, RI, 2002.
32. A. Poltoratski and D. Sarason, *Aleksandrov-Clark measures*, Recent advances in operator-related function theory, Contemp. Math., vol. 393, Amer. Math. Soc., Providence, RI, 2006, pp. 1–14.
33. M. Reed and B. Simon, *Methods of modern mathematical physics. I*, Second ed., Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1980, Functional analysis.
34. W. W. Rogosinski and H. S. Shapiro, *On certain extremum problems for analytic functions*, Acta Math. **90** (1953), 287–318.
35. W. T. Ross and H. S. Shapiro, *Generalized analytic continuation*, University Lecture Series, vol. 25, American Mathematical Society, Providence, RI, 2002.
36. St. Ruscheweyh, *Two remarks on bounded analytic functions*, Serdica **11** (1985), no. 2, 200–202.
37. D. Sarason, *Generalized interpolation in H^∞*, Trans. Amer. Math. Soc. **127** (1967), 179–203.
38. ______, *Sub-Hardy Hilbert spaces in the unit disk*, University of Arkansas Lecture Notes in the Mathematical Sciences, 10, John Wiley & Sons Inc., New York, 1994, , A Wiley-Interscience Publication.
39. ______, *Algebraic properties of truncated Toeplitz operators*, Operators and Matrices **1** (2007), no. 4, 491–526.
40. S. Takenaka, *On the orthonormal functions and a new formula of interpolation*, Jap. J. Math. **2** (1925), 129 – 145.
41. V. M. Terpigoreva, *Extremal problems in certain subclasses of analytic functions of bounded form*, Litovsk. Mat. Sb. **10** (1970), 171–187.

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