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The Hardy Space of a Slit Domain

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Preface

If $\mathcal{H}$ is a Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ is a continous linear operator, a natural question to ask is: What are the closed subspaces $\mathcal{M}$ of $\mathcal{H}$ for which $T\mathcal{M} \subset \mathcal{M}$? Of course the famous invariant subspace problem asks whether or not T has *any* non-trivial invariant subspaces. This monograph is part of a long line of study of the invariant subspaces of the operator $T = M_z$ (multiplication by the independent variable z, i.e., $M_z f = zf$) on a Hilbert space of analytic functions on a bounded domain G in $\mathbb{C}$. The characterization of these M_z-invariant subspaces is particularly interesting since it entails both the properties of the functions inside the domain G, their zero sets for example, as well as the behavior of the functions near the boundary of G. The operator M_z is not only interesting in its own right but often serves as a model operator for certain classes of linear operators. By this we mean that given an operator T on $\mathcal{H}$ with certain properties (certain subnormal operators or two-isometric operators with the right spectral properties, etc.), there is a Hilbert space of analytic functions on a domain G for which T is unitarity equivalent to M_z.

Probably the first to successfully study these types of problems was Beurling [13] who gave a complete characterization of the M_z-invariant subspaces of the Hardy space of the unit disk. These are the functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ which are analytic on the open unit disk $\mathbb{D} := \{|z| < 1\}$ for which $\sum_{n \geqslant 0} |a_n|^2 < \infty$. Many others followed with a discussion, often a complete characterization, of the M_z-invariant subspaces where the Hardy space is replaced by the space of analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $\mathbb{D}$ satisfying $\sum_{n \geqslant 0} w_n |a_n|^2 < \infty$, where $(w_n)_{n \geqslant 0}$ is a sequence of positive weights. For example, when $w_n = n$, we get the classical Dirichlet space where the M_z-invariant subspaces were discussed in [60, 61, 62]. When $w_n = n^{\alpha}$ and $\alpha > 1$, we get certain weighted Dirichlet spaces where the M_z-invariant subspaces were completely characterized in [69]. See [52, 53] for some related results. When $w_n = n^{-1}$ (or more generally $w_n = n^{\alpha}, \alpha < 0$), we get the Bergman (weighted Bergman) spaces where the M_z-invariant subspaces were discussed in [8, 68]. See also [30, 42].

In Beurling's seminal paper, and the ones that followed, notice how the underlying domain of analyticity is kept fixed to be the unit disk $\mathbb{D}$, but the Hilbert space of analytic functions is changed by varying the weights w_n. In a series of papers beginning with Sarason [65], the basic type of Hilbert space is fixed but the domain of analyticity is changed. To see what we mean here, the condition $f(z) = \sum_{n \geqslant 0} a_n z^n$ is analytic on $\mathbb{D}$ and $\sum_{n \geqslant 0} |a_n|^2 < \infty$, the definition of the Hardy space of $\mathbb{D}$, can be equivalently restated as

f is analytic on $\mathbb{D}$ and there is a harmonic function U on $\mathbb{D}$ for which $|f|^2 \leqslant U$ on $\mathbb{D}$. Such a function U is called a harmonic majorant for $|f|^2$. For a general bounded domain $G \subset \mathbb{C}$, one can define the Hardy space of G to be the analytic functions f on G for which $|f|^2$ has a harmonic majorant on G. Beginning with Sarason's paper, there were several authors [6, 7, 37, 44, 64, 76, 77, 78] who characterized the M_z-invariant subspaces of the Hardy space of annular-type domains, which include an annulus, a disk with several holes removed, and a crescent domain (the region between two internally tangent circles).

Conspicuously missing from this list of domains are slit domains, for example $G = \mathbb{D} \setminus [0,1)$. In this monograph, we obtain a complete characterization of the M_z-invariant subspaces of the Hardy space of slit domains. Along the way, we give a thorough exposition of the Hardy space, and even the Hardy-Smirnov space, of a slit domain as well as several applications of our results to de Branges-type spaces and the classical backward shift operator of the Hardy space of $\mathbb{D}$. We also discuss several aspects of the operator $M_z|\mathcal{M}$, where $\mathcal{M}$ is an M_z-invariant subspace of the Hardy space of G. In particular, we explore questions about cyclicity, the spectrum, and the essential spectrum for $M_z|\mathcal{M}$.