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Global convergence of *a posteriori* error estimates for a discontinuous Galerkin method for one-dimensional linear hyperbolic problems

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SUPERCONVERGENCE AND *A POSTERIORI* ERROR ESTIMATES OF A LOCAL DISCONTINUOUS GALERKIN METHOD FOR THE FOURTH-ORDER INITIAL-BOUNDARY VALUE PROBLEMS ARISING IN BEAM THEORY

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Abstract. In this paper, we investigate the superconvergence properties and *a posteriori* error estimates of a local discontinuous Galerkin (LDG) method for solving the one-dimensional linear fourth-order initial-boundary value problems arising in study of transverse vibrations of beams. We present a local error analysis to show that the leading terms of the local spatial discretization errors for the k-degree LDG solution and its spatial derivatives are proportional to $(k+1)$-degree Radau polynomials. Thus, the k-degree LDG solution and its derivatives are $\mathcal{O}(h^{k+2})$ superconvergent at the roots of $(k+1)$-degree Radau polynomials. Computational results indicate that global superconvergence holds for LDG solutions. We discuss how to apply our superconvergence results to construct efficient and asymptotically exact *a posteriori* error estimates in regions where solutions are smooth. Finally, we present several numerical examples to validate the superconvergence results and the asymptotic exactness of our *a posteriori* error estimates under mesh refinement. Our results are valid for arbitrary regular meshes and for P^k polynomials with $k \geq 1$, and for various types of boundary conditions.

Key words. Local discontinuous Galerkin method; fourth-order initial-boundary value problems; Euler-Bernoulli beam equation; superconvergence; *a posteriori* error estimates.

1. Introduction

The goal of this paper is to investigate the superconvergence properties and develop a simple procedure to compute *a posteriori* error estimates of the spatial errors for the local discontinuous Galerkin (LDG) method applied to the following linear fourth-order initial-boundary value problem in one space dimension:

$$u_{tt} + u_{xxxx} = f(x,t), \quad x \in [0,L], \ t \in [0,T], \tag{1.1a}$$

subject to the initial conditions

$$u(x,0) = g(x), \quad u_t(x,0) = h(x), \quad x \in [0,L], \tag{1.1b}$$

and to one of the following four kinds of boundary conditions which are commonly encountered in practice ($t \in [0,T]$):

$$u(0,t) = u_1(t), \ u_{xx}(0,t) = u_2(t), \ u_x(L,t) = u_3(t), \ u_{xxx}(L,t) = u_4(t), \tag{1.1c}$$

$$u(0,t) = u_1(t), \ u_{xx}(0,t) = u_2(t), \ u(L,t) = u_3(t), \ u_{xx}(L,t) = u_4(t), \tag{1.1d}$$

$$u(0,t) = u_1(t), \ u_x(0,t) = u_2(t), \ u(L,t) = u_3(t), \ u_x(L,t) = u_4(t), \tag{1.1e}$$

$$u(0,t) = u(L,t), \ u_x(0,t) = u_x(L,t), \ u_{xx}(0,t) = u_{xx}(L,t), \ u_{xxx}(0,t) = u_{xxx}(L,t). \tag{1.1f}$$

In our analysis we assume that the interval $[0,T]$ is a finite time interval, and select the side conditions and the source, $f(x,t)$, such that the exact solution, $u(x,t)$, is

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a smooth function on $[0, L] \times [0, T]$. Even though the analysis in this paper is restricted to (1.1a), the same results can be directly generalized to the well-known Euler-Bernoulli beam equation with constant and variable geometrical and physical properties

$$(E(x)I(x)u_{xx})_{xx} + \rho(x)A(x)u_{tt} = f(x,t),$$

where $u(x,t)$ is the deflection of the neutral axis of the beam, $E(x)$ is the Young's modulus of elasticity, $I(x)$ is the area moment of inertia of the cross section with respect to its neutral midplane, $A(x)$ is the cross section in the yz-plane, $\rho(x)$ is the mass density per unit length, and $f(x,t)$ is the transverse load.

The fourth-order Euler-Bernoulli beam equation considered in this paper plays a very important role in both theory and applications. This is due to its use to describe a large number of physical and engineering phenomenons such as the flexural vibrations of a slender isotropic beam within the framework of Euler-Bernoulli assumptions. Several numerical schemes are proposed in the literature for solving (1.1a). Consult [11, 12, 14, 35, 36, 37, 41, 42, 47] and the references cited therein for more details. In this paper, we develop, analyze and test a superconvergent LDG method for solving (1.1). The proposed scheme is based on the fourth-order Runge-Kutta method approximation in time and on the LDG approximation in the spatial discretization. Our proposed scheme for solving the beam equation extends our previous work [16, 23] in which we investigated the convergence properties and the error estimates of the LDG method applied to the second-order wave and convection-diffusion equations in one space dimension.

The main motivation for the LDG method proposed in this paper originates from the LDG techniques which have been developed for convection-diffusion equations. The LDG finite element method considered here is an extension of the discontinuous Galerkin (DG) method aimed at solving ordinary and partial differential equations (PDEs) containing higher than first-order spatial derivatives. The DG method is a class of finite element methods using completely discontinuous piecewise polynomials for the numerical solution and the test functions. With discontinuous finite element bases, they capture discontinuities in, *e.g.*, hyperbolic systems with high accuracy and efficiency; simplify adaptive $h-$, $p-$, $r-$, refinements and produce efficient parallel solution procedures. The DG method was initially introduced by Reed and Hill in 1973 as a technique to solve neutron transport problems [44]. Lesaint and Raviart [40] presented the first numerical analysis of the method for a linear advection equation. Since then, DG methods have been used to solve ordinary and partial differential equations. Consult [32, 17] and the references cited therein for a detailed discussion of the history of DG method and a list of important citations on the DG method and its applications.

The LDG method for solving convection-diffusion problems was first introduced by Cockburn and Shu in [33]. They further studied the stability and error estimates for the LDG method. Castillo *et al.* [26] presented the first *a priori* error analysis for the LDG method for a model elliptic problem. They considered arbitrary meshes with hanging nodes and elements of various shapes and studied general numerical fluxes. They showed that, for smooth solutions, the L^2 errors in ∇u and in u are of order k and $k+1/2$, respectively, when polynomials of total degree not exceeding k are used. Cockburn *et al.* [31] presented a superconvergence result for the LDG method for a model elliptic problem on Cartesian grids. They identified a special numerical flux for which the L^2-norms of the gradient and the potential are of orders $k+1/2$ and $k+1$, respectively, when tensor product polynomials of degree at most k are used. Several LDG schemes have been developed for various high order PDEs including the convection-diffusion equations [33], second-order wave equations [16, 20], nonlinear KdV type equations [48, 50], and beam equation [18, 19]. More details about the LDG methods for high order time dependent equations can be found in the review paper [49] and the recent proceeding of Shu [46]. Furthermore, some LDG methods for other high order wave equations were developed by Yan and Shu [51], which were high order accurate and stable schemes.

The study of superconvergence and *a posteriori error* estimates of DG methods has been an area of active research in both mathematics and engineering, see *e.g.* [15]. A knowledge of superconvergence properties can be used to (i) construct simple and asymptotically exact *a posteriori* estimates of discretization errors and (ii) help detect discontinuities to find elements needing limiting, stabilization and/or refinement. Typically, *a posteriori* error estimators employ the known numerical solution to derive estimates of the actual solution errors. They are also used to steer adaptive schemes where either the mesh is locally refined (h-refinement) or the polynomial degree is raised (p-refinement). For an introduction to the subject of *a posteriori* error estimation see the monograph of Ainsworth and Oden [13]. Superconvergence properties for DG methods have been studied in [34, 40] for ordinary differential equations, [4, 16, 8, 9] for hyperbolic problems and [2, 3, 5, 9, 10, 24, 25, 27, 28] for diffusion and convection-diffusion problems. Several *a posteriori* DG error estimates are known for hyperbolic [29, 30, 38, 22, 7] and diffusive [39, 45] problems.

Adjerid and Baccouch [4] investigated the global convergence of the implicit residual-based *a posteriori* error estimates of Adjerid *et al.* [8]. They proved that, for smooth solutions, these *a posteriori* error estimates at a fixed time t converge to the true spatial error in the L^2-norm under mesh refinement. Recently, Adjerid and Baccouch [6, 5] showed that LDG solutions are superconvergent at Radau points for two-dimensional convection-diffusion problems. They used these results to construct asymptotically correct *a posteriori* error estimates. In [16], we presented new superconvergence results for the semi-discrete LDG method applied to the second-order scalar wave equation in one space dimension. We performed an error analysis on one element and showed that the k-degree LDG solution and its spatial derivative are $\mathcal{O}(h^{k+2})$ superconvergent at the roots of $(k+1)$-degree right and left Radau polynomials, respectively. Computational results showed that global superconvergence holds for LDG solutions. We used these results to construct asymptotically correct *a posteriori* error estimates by solving local steady problem with no boundary conditions on each element. However, we only presented several numerical results suggesting that the global spatial error estimates converge to the true errors under mesh refinement where temporal errors are assumed to be negligible. More recently, Baccouch [21, 20] analyzed the superconvergence properties of the LDG formulation applied to transient convection-diffusion and wave equations in one space dimension. The author proved that the leading error term on each element for the solution is proportional to a $(k+1)$-degree right Radau polynomial while the leading error term for the solution's derivative is proportional to a $(k+1)$-degree left Radau polynomial, when polynomials of degree at most k are used. He further analyzed the convergence of *a posteriori* error estimates and proved that these error estimates are globally asymptotically exact under mesh refinement.

The goals of this paper are to (i) design a superconvergent LDG method for solving the fourth-order initial-boundary value problems, (ii) investigate the superconvergence properties of LDG solutions, and (iii) develop computationally simple *a posteriori* error estimates. We show that the local discretization errors for the k-degree LDG solution and its derivatives up to third order converge as $\mathcal{O}(h^{k+2})$ at the roots of Radau polynomials of degree $k+1$ on each element. More precisely, a local error analysis reveals that the leading terms of the spatial discretization errors for the LDG solution and its derivatives, using k-degree polynomial approximations, are proportional to $(k+1)$-degree (either right or left) Radau polynomials. We use these results to construct asymptotically exact *a posteriori* error estimates in regions where solutions are smooth. The leading terms of the discretization errors for the solution and its spatial derivatives are estimated by solving a local steady problem with no boundary conditions on each element. The four coefficients of the leading terms of the spatial discretization errors are functions of the time variable and obtained from a 4-by-4 linear algebraic system on each element. Several numerical simulations are performed to validate the theory.

This paper is organized as follows: In section 2 we define the LDG scheme and we introduce some notations and definitions which will be used in our error analysis. In section 3, we present the LDG error analysis and prove our main superconvergence results. In section 4, we discuss our error estimation procedure. In section 5, we present numerical results to confirm the global superconvergence results and the asymptotic exactness of our *a posteriori* error estimates under mesh refinement. We conclude and discuss our results in section 6.

2. The LDG scheme

In order to construct the LDG scheme, we first introduce three auxiliary variables $q = u_x$, $p = q_x$, $r = p_x$ and rewrite our model problem (1.1a) as a first-order system in space

$$u_{tt} + r_x = f, \quad r - p_x = 0, \quad p - q_x = 0, \quad q - u_x = 0. \tag{2.1}$$

In order to obtain a weak LDG formulation we partition the interval $I = [0, L]$ into a quasi-uniform mesh, $\Delta_N = \{0 = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_N = L\}$, having N subintervals $I_i = [x_{i-1}, x_i],\ i = 1, \cdots, N$ with length $h_i = x_i - x_{i-1}$. The length of the largest subinterval is denoted by $h = \max_{1\le i\le N} h_i$. Throughout this paper, $v\big|_i$ denotes the value of the function $v = v(x,t)$ at $x = x_i$. We also define $v^-\big|_i$ and $v^+\big|_i$ to be the left limit and the right limit of the function v at the discontinuity point x_i, *i.e.*,

$$v^-\big|_i = v^-(x_i,t) = \lim_{s\to 0^-} v(x_i+s,t), \qquad v^+\big|_i = v^+(x_i,t) = \lim_{s\to 0^+} v(x_i+s,t).$$

We define a finite element space consisting of piecewise k^{th}-degree polynomial functions $V_h^k = \{v : v|_{I_i} \in P^k(I_i)\}$, where $P^k(I_i)$ is the space of polynomials of degree not exceeding k on I_i. Note that polynomials in the space V_h^k are allowed to have discontinuities across element boundaries.

Let us multiply the four equations in (2.1) by test functions $v,\ w,\ s$, and z, respectively, integrate over an arbitrary subinterval I_i, and use integration by parts to write

$$\int_{I_i} u_{tt} v dx - \int_{I_i} r v_x dx + rv\big|_i - rv\big|_{i-1} = \int_{I_i} f v dx, \tag{2.2a}$$

$$\int_{I_i} r w dx + \int_{I_i} p w_x dx - pw\big|_i + pw\big|_{i-1} = 0, \tag{2.2b}$$

$$\int_{I_i} p s dx + \int_{I_i} q s_x dx - qs\big|_i + qs\big|_{i-1} = 0, \tag{2.2c}$$

$$\int_{I_i} q z dx + \int_{I_i} u z_x dx - uz\big|_i + uz\big|_{i-1} = 0. \tag{2.2d}$$

Next, we approximate the exact solutions $u(.,t)$, $q(.,t)$, $p(.,t)$, and $r(.,t)$ by piecewise polynomials $u_h(.,t) \in V_h^k$, $q_h(.,t) \in V_h^k$, $p_h(.,t) \in V_h^k$, and $r_h(.,t) \in V_h^k$, respectively, whose restriction to I_i are in $P^k(I_i)$. Here u_h, q_h, p_h, and r_h are not necessarily continuous at the endpoints of I_i. The semi-discrete LDG method consists of finding u_h, q_h, p_h, $r_h \in V_h^k$ such that $\forall\ i = 1, \ldots, N$,

$$\int_{I_i} (u_h)_{tt} v dx - \int_{I_i} r_h v_x dx + \hat{r}_h v^-\big|_i - \hat{r}_h v^+\big|_{i-1} = \int_{I_i} f v\ dx, \quad \forall\ v \in V_h^k, \tag{2.3a}$$

$$\int_{I_i} r_h w\ dx + \int_{I_i} p_h w_x\ dx - \hat{p}_h w^-\big|_i + \hat{p}_h w^+\big|_{i-1} = 0, \quad \forall\ w \in V_h^k, \tag{2.3b}$$

$$\int_{I_i} p_h s\ dx + \int_{I_i} q_h s_x\ dx - \hat{q}_h s^-\big|_i + \hat{q}_h s^+\big|_{i-1} = 0, \quad \forall\ s \in V_h^k, \tag{2.3c}$$

$$\int_{I_i} q_h z\ dx + \int_{I_i} u_h z_x\ dx - \hat{u}_h z^-\big|_i + \hat{u}_h z^+\big|_{i-1} = 0, \quad \forall\ z \in V_h^k, \tag{2.3d}$$

where the hatted terms, $\hat{u}_h$, $\hat{q}_h$, $\hat{p}_h$, and $\hat{r}_h$ are the so-called numerical fluxes. These numerical fluxes are single-valued functions defined on the boundaries of I_i and should be designed to ensure numerical stability.
For the boundary conditions (1.1c), we choose the following alternating fluxes

$$\hat{u}_h\big|_i = \begin{cases} u_1(t), & i = 0, \\ u_h^-\big|_i, & i = 1, \ldots, N, \end{cases} \qquad \hat{q}_h\big|_i = \begin{cases} q_h^+\big|_i, & i = 0, \ldots, N-1, \\ u_3(t), & i = N, \end{cases}$$

$$\hat{p}_h\big|_i = \begin{cases} u_2(t), & i = 0, \\ p_h^-\big|_i, & i = 1, \ldots, N, \end{cases} \qquad \hat{r}_h\big|_i = \begin{cases} r_h^+\big|_i, & i = 0, \ldots, N-1, \\ u_4(t), & i = N. \end{cases} \tag{2.3e}$$

If other boundary conditions are chosen, the numerical fluxes can be easily designed. For instance the numerical fluxes associated with the boundary conditions (1.1d) can be taken as

$$\hat{u}_h\big|_i = \begin{cases} u_1(t), & i = 0, \\ u_h^-\big|_i, & i = 1, \ldots, N-1, \\ u_3(t), & i = N, \end{cases} \quad \hat{q}_h\big|_i = \begin{cases} q_h^+\big|_i, & i = 0, \ldots, N-1, \\ \left(q_h^- - \delta_2(u_h^- - u_3)\right)\big|_i, & i = N, \end{cases}$$

$$\hat{p}_h\big|_i = \begin{cases} u_2(t), & i = 0, \\ p_h^-\big|_i, & i = 1, \ldots, N-1, \\ u_4(t), & i = N, \end{cases} \quad \hat{r}_h\big|_i = \begin{cases} r_h^+\big|_i, & i = 0, \ldots, N-1, \\ \left(r_h^- - \delta_2(p_h^- - u_4)\right)\big|_i, & i = N, \end{cases} \tag{2.3f}$$

where the stabilization parameters δ_1 and δ_2 for the LDG method are given by $\delta_1 = \frac{k}{h_i}$ and $\delta_2 = \frac{k}{h_i}$.

Similarly, the numerical fluxes associated with the boundary conditions (1.1e) can be taken as

$$\hat{u}_h\big|_i = \begin{cases} u_1(t), & i=0,\\ u_h^-\big|_i, & i=1,\ldots,N-1,\\ u_3(t), & i=N,\end{cases} \quad \hat{q}_h\big|_i = \begin{cases} u_2(t), & i=0,\\ q_h^+\big|_i, & i=1,\ldots,N-1,\\ u_4(t), & i=N,\end{cases}$$

$$\hat{p}_h\big|_i = \begin{cases} \left(p_h^+ + \delta_1(q_h^+ - u_2)\right)\big|_i, & i=0,\\ p_h^-\big|_i, & i=1,\ldots,N,\end{cases} \quad \hat{r}_h\big|_i = \begin{cases} r_h^+\big|_i, & i=0,\ldots,N-1,\\ \left(r_h^- - \delta_2(u_h^- - u_3)\right)\big|_i, & i=N.\end{cases} \tag{2.3g}$$

where the stabilization parameters δ_1 and δ_2 for the LDG method are given by $\delta_1 = \frac{k}{h_i}$ and $\delta_2 = \frac{k}{h_i^3}$.

For the periodic boundary conditions (1.1f), we choose the following alternating fluxes (*e.g.*, see [43])

$$\hat{u}_h\big|_i = \begin{cases} u_h^-\big|_N, & i=0,\\ u_h^-\big|_i, & i=1,\ldots,N,\end{cases} \quad \hat{q}_h\big|_i = \begin{cases} q_h^+\big|_i, & i=0,\ldots,N-1,\\ q_h^+\big|_0, & i=N,\end{cases}$$

$$\hat{p}_h\big|_i = \begin{cases} p_h^-\big|_N, & i=0,\\ p_h^-\big|_i, & i=1,\ldots,N,\end{cases} \quad \hat{r}_h\big|_i = \begin{cases} r_h^+\big|_i, & i=0,\ldots,N-1,\\ r_h^+\big|_0, & i=N.\end{cases} \tag{2.3h}$$

We note that this choice is not unique. For instance the following choice is also fine

$$\hat{u}_h\big|_i = \begin{cases} u_h^+\big|_i, & i=0,\ldots,N-1,\\ u_h^+\big|_0, & i=N,\end{cases} \quad \hat{q}_h\big|_i = \begin{cases} q_h^-\big|_N, & i=0,\\ q_h^-\big|_i, & i=1,\ldots,N,\end{cases}$$

$$\hat{p}_h\big|_i = \begin{cases} p_h^+\big|_i, & i=0,\ldots,N-1,\\ p_h^+\big|_0, & i=N,\end{cases} \quad \hat{r}_h\big|_i = \begin{cases} r_h^-\big|_N, & i=0,\\ r_h^-\big|_i, & i=1,\ldots,N.\end{cases} \tag{2.3i}$$

In order to complete the definition of the semi-discrete LDG method we need to design the initial conditions of our numerical scheme. In this paper, the initial conditions $u_h(x,0) \in V_h^k$ and $(u_h)_t(x,0) \in V_h^k$ are obtained by interpolating the exact initial conditions $u(x,0) = g(x)$ and $u_t(x,0) = h(x)$ as

$$u_h(x,0) = \pi^+ g(x), \quad (u_h)_t(x,0) = \pi^+ h(x), \quad x \in I_i, \quad i = 1,\cdots,N, \tag{2.4}$$

where $\pi^+ v$ is the k-degree polynomial that interpolates v at the roots of $(k+1)$-degree right Radau polynomial which will be defined later.

Remark: In our numerical experiments we approximated the initial conditions of the numerical scheme by the polynomials that interpolate the exact initial conditions at the roots of the right Radau polynomial of degree $k+1$. However, numerical experiments suggest that if we use the standard L^2 projection of the initial conditions as our numerical initial conditions instead, the convergence and superconvergence rates do not converge to the desired $k+1$ and $k+2$ accuracy, respectively. We observed that the order of accuracy for the solution and the auxiliary variables is oscillating. Furthermore, we did not observe any pointwise superconvergence. We would like to emphasize that our special choice of initial conditions (2.4) is essential to obtain the desired superconvergence rate of the proposed LDG method.

In order to discretize in time, we first solve for the auxiliary variables q_h, p_h, and r_h in terms of u_h in an element-by-element fashion using (2.3b)-(2.3d). Substituting the resulting expressions for q_h, p_h, and r_h into (2.3a), then expressing $u_h(x,t) = \sum_{j=0}^{k} c_{j,i}(t) L_{j,i}(x)$, $x \in I_i$, as a linear combination of orthogonal basis $L_{j,i}(x)$, $j = 0,\ldots,k$, where $L_{j,i}$ denotes the j^{th}-degree Legendre polynomial on I_i, and choosing the test functions $v = L_{j,i}$, $j = 0,\ldots,k$, we obtain the following linear second-order ordinary differential system:

$$M_i \frac{d^2\mathbf{C}_i''(t)}{dt^2} = A_i\mathbf{C}_i(t) + \mathbf{b}_i(t), \quad i = 1,\cdots,N,$$

where $\mathbf{C}_i(t) = [c_{0,i}(t), c_{1,i}(t), \cdots, c_{k,i}(t)]$ denotes the solution vector at time t, M_i denotes the mass matrix, A_i is a matrix, and $\mathbf{b}_i(t)$ is a vector which depends on the source term and the boundary conditions but independent of solution. We introduced the superscript i to emphasize that these systems can be solved on each element I_i using *e.g.*, the classical fourth-order Runge-Kutta method. As our interest is in the effect of the spatial discretization, we determine the time-step Δt so that temporal errors are small relative to spatial errors. We do not discuss the influence of the time discretization error in this paper.

In our analysis we need the k^{th}-degree Legendre polynomial defined by Rodrigues formula [1]

$$L_k(\xi) = \frac{1}{2^k k!}\frac{d^k}{d\xi^k}[(\xi^2-1)^k], \quad -1 \le \xi \le 1,$$

which satisfies the following properties: $L_k(1) = 1$, $L_k(-1) = (-1)^k$ and

$$\int_{-1}^{1} L_k(\xi)L_p(\xi)d\xi = \frac{2}{2k+1}\delta_{kp}, \quad \text{where } \delta_{kp} \text{ is the Kronecker symbol.} \tag{2.5}$$

Next, we define the $(k+1)-$degree right Radau polynomial as $\tilde{R}^+_{k+1}(\xi) = L_{k+1}(\xi) - L_k(\xi)$, $-1 \le \xi \le 1$, which has $k+1$ real distinct roots, $-1 < \xi_0^+ < \cdots < \xi_k^+ = 1$. We also define the $(k+1)$-degree left Radau polynomial $\tilde{R}^-_{k+1}(\xi) = L_{k+1}(\xi) + L_k(\xi)$, $-1 \le \xi \le 1$, which has $k+1$ real distinct roots, $-1 = \xi_0^- < \cdots < \xi_k^- < 1$. Mapping the element $I_i = [x_{i-1}, x_i]$ into a reference element $[-1,1]$ by the standard affine mapping

$$x(\xi, h_i) = \frac{x_i + x_{i-1}}{2} + \frac{h_i}{2}\xi, \tag{2.6}$$

we obtain the shifted Radau polynomials $R^{\pm}_{k+1,i}(x) = \tilde{R}^{\pm}_{k+1}\left(\frac{2x - x_i - x_{i-1}}{h_i}\right)$ on I_i.

In this paper, we define the L^2 inner product of two integrable functions, $u = u(x,t)$ and $v = v(x,t)$, depending on x and t on the intervals $I_i = [x_{i-1}, x_i]$ and $I = [0, L]$ as

$$(u(.,t), v(.,t))_i = \int_{I_i} u(x,t)v(x,t)dx, \qquad (u(.,t), v(.,t)) = \int_{I} u(x,t)v(x,t)dx,$$

and the subsequent induced norms are $\|u(.,t)\|_i^2 = (u(.,t), u(.,t))_i$ and $\|u(.,t)\|^2 = (u(.,t), u(.,t))$. In the remainder of this paper we will omit the notation $(.,t)$ used in the subsequent induced norms unless needed for clarity. Thus we use $\|u\|$ instead of $\|u(.,t)\|$ etc.

3. Superconvergence error analysis

In this section we investigate the superconvergence properties of the LDG method. We show that u_h and p_h are $\mathcal{O}(h^{k+2})$ superconvergent at the $(k+1)$-degree right-Radau polynomial and q_h and r_h are $\mathcal{O}(h^{k+2})$ superconvergent at the $(k+1)$-degree left-Radau polynomial. The local superconvergence results are proved and the global superconvergence results are confirmed numerically.

Throughout this paper, e_u, e_q, e_p, and e_r, respectively, denote the errors between the exact solutions of (2.1) and the numerical solutions defined in (2.3) *i.e.*,

$$e_u = u - u_h, \quad e_q = q - q_h, \quad e_p = p - p_h, \quad e_r = r - r_h.$$

We subtract (2.3) from (2.2) with $v,\ w,\ s,\ z \in V_h^k$ to obtain the LDG orthogonality conditions for the errors $e_u,\ e_q,\ e_p$, and e_r on I_i

$$\int_{I_i}(e_u)_{tt}v dx - \int_{I_i} e_r v_x dx + \hat{e}_r v^-\big|_i - \hat{e}_r v^+\big|_{i-1} = 0, \tag{3.1a}$$

$$\int_{I_i} e_r w\ dx + \int_{I_i} e_p w_x\ dx - \hat{e}_p w^-\big|_i + \hat{e}_p w^+\big|_{i-1} = 0, \tag{3.1b}$$

$$\int_{I_i} e_p s\ dx + \int_{I_i} e_q s_x\ dx - \hat{e}_q s^-\big|_i + \hat{e}_q s^+\big|_{i-1} = 0, \tag{3.1c}$$

$$\int_{I_i} e_q z\ dx + \int_{I_i} e_u z_x\ dx - \hat{e}_u z^-\big|_i + \hat{e}_u z^+\big|_{i-1} = 0. \tag{3.1d}$$

Using the mapping of $I_i = [x_{i-1}, x_i]$ onto the canonical element $[-1,1]$ defined by (2.6) and denoting $\tilde{e}_u(\xi,t,h_i) = e_u(x(\xi,h_i),t)$, $\tilde{e}_q(\xi,t,h_i) = e_q(x(\xi,h_i),t)$, $\tilde{e}_p(\xi,t,h_i) = e_p(x(\xi,h_i),t)$, $\tilde{e}_r(\xi,t,h_i) = e_r(x(\xi,h_i),t)$, we obtain the LDG orthogonality condition (3.1) on the reference element $[-1,1]$

$$\frac{h_i}{2}\int_{-1}^{1}(\tilde{e}_u)_{tt}\tilde{v}d\xi - \int_{-1}^{1}\tilde{e}_r\tilde{v}_\xi d\xi + \tilde{\hat{e}}_r\tilde{v}^-\big|_i - \tilde{\hat{e}}_r\tilde{v}^+\big|_{i-1} = 0, \tag{3.2a}$$

$$\frac{h_i}{2}\int_{-1}^{1}\tilde{e}_r\tilde{w}d\xi + \int_{-1}^{1}\tilde{e}_p\tilde{w}_\xi d\xi - \tilde{\hat{e}}_p\tilde{w}^-\big|_i + \tilde{\hat{e}}_p\tilde{w}^+\big|_{i-1} = 0, \tag{3.2b}$$

$$\frac{h_i}{2}\int_{-1}^{1}\tilde{e}_p\tilde{s}d\xi + \int_{-1}^{1}\tilde{e}_q\tilde{s}_\xi d\xi - \tilde{\hat{e}}_q\tilde{s}^-\big|_i + \tilde{\hat{e}}_q\tilde{s}^+\big|_{i-1} = 0, \tag{3.2c}$$

$$\frac{h_i}{2}\int_{-1}^{1}\tilde{e}_q\tilde{z}d\xi + \int_{-1}^{1}\tilde{e}_u\tilde{z}_\xi d\xi - \tilde{\hat{e}}_u\tilde{z}^-\big|_i + \tilde{\hat{e}}_u\tilde{z}^+\big|_{i-1} = 0. \tag{3.2d}$$

If the exact solution u is analytic, the LDG solutions (u_h, q_h, p_h, r_h) on I_i are also analytic with respect to x since they are polynomials in x. We further note that $\tilde{u}_h(\xi,t,h_i) = u_h(x(\xi,h_i),t)$, $\tilde{q}_h(\xi,t,h_i) = q_h(x(\xi,h_i),t)$, $\tilde{p}_h(\xi,t,h_i) = p_h(x(\xi,h_i),t)$, and $\tilde{r}_h(\xi,t,h_i) = r_h(x(\xi,h_i),t)$ are analytic with respect to h_i by transforming the local LDG weak problem to the reference element and solving for the finite element coefficients which are analytic functions of h_i. Thus, at fixed time t, we can expand the local errors in Maclaurin series with respect to h_i as

$$\tilde{e}_u(\xi,t,h_i) = \sum_{j=0}^{\infty}\tilde{U}_j(\xi,t)h_i^j, \quad \tilde{e}_q(\xi,t,h_i) = \sum_{j=0}^{\infty}\tilde{Q}_j(\xi,t)h_i^j, \tag{3.3a}$$

$$\tilde{e}_p(\xi,t,h_i) = \sum_{j=0}^{\infty}\tilde{P}_j(\xi,t)h_i^j, \quad \tilde{e}_r(\xi,t,h_i) = \sum_{j=0}^{\infty}\tilde{R}_j(\xi,t)h_i^j, \tag{3.3b}$$

where $\tilde{U}_j(.,t),\ \tilde{Q}_j(.,t),\ \tilde{P}_j(.,t)$, and $\tilde{R}_j(.,t) \in P^j([-1,1])$ are polynomials of degree j in the variable ξ and are obtained by applying the chain rule as

$$\tilde{U}_j(\xi,t) = \frac{1}{j!}\frac{d^j\tilde{e}_u}{dh_i^j}(\xi,t,0) = \frac{1}{j!}\sum_{l=0}^{j}\frac{\xi^l}{2^l}\binom{j}{l}\partial_x^l\partial_h^{j-l}\tilde{e}_u(0,t,0),$$

$$\tilde{Q}_j(\xi,t) = \frac{1}{j!}\frac{d^j\tilde{e}_q}{dh_i^j}(\xi,t,0) = \frac{1}{j!}\sum_{l=0}^{j}\frac{\xi^l}{2^l}\binom{j}{l}\partial_x^l\partial_h^{j-l}\tilde{e}_q(0,t,0),$$

$$\tilde{P}_j(\xi,t) = \frac{1}{j!}\frac{d^j\tilde{e}_p}{dh_i^j}(\xi,t,0) = \frac{1}{j!}\sum_{l=0}^{j}\frac{\xi^l}{2^l}\binom{j}{l}\partial_x^l\partial_h^{j-l}\tilde{e}_p(0,t,0),$$

$$\tilde{R}_j(\xi,t)=\frac{1}{j!}\frac{d^j\tilde{e}_r}{dh_i^j}(\xi,t,0)=\frac{1}{j!}\sum_{l=0}^{j}\frac{\xi^l}{2^l}\binom{j}{l}\partial_x^l\partial_h^{j-l}\tilde{e}_r(0,t,0),$$

where the binomial coefficient $\binom{j}{l}$ is defined by $\binom{j}{l}=\frac{j!}{l!\,(j-l)!}$ for $0\le l\le j$.
For simplicity, we present a local error analysis on the element $[0,h]$. For this, we consider the problem (1.1a) on $[0,h]$ subject to the initial conditions (1.1b), where $x\in[0,h]$ and to either the boundary conditions (1.1c) or (1.1d). For each case, the proof is presented separately. Similar results hold when using the boundary conditions (1.1f) and (1.1e). The proofs are very similar to proofs provided for the first two cases, and are therefore omitted to save space. Several numerical examples are included to validate these results globally.

3.1. Case 1. In this subsection, we consider the problem (1.1a) in $[0,h]$ subject to the initial conditions (1.1b) and the boundary conditions (1.1c). In the next theorem, we state and prove the following pointwise superconvergence results.

Theorem 3.1. *Let (u,q,p,r) and (u_h,q_h,p_h,r_h), respectively, be the solutions of (2.1) and (2.3) in $[0,h]$ with the numerical fluxes (2.3e) subject to (1.1b) and (1.1c). If we apply the mapping of $[0,h]$ onto the canonical element $[-1,1]$ defined by (2.6), then, the local finite element errors can be written as*

$$\text{(3.4a)}\qquad \tilde{e}_u(\xi,t,h)=\sum_{j=k+1}^{\infty}\tilde{U}_j(\xi,t)h^j,\quad \tilde{e}_q(\xi,t,h)=\sum_{j=k+1}^{\infty}\tilde{Q}_j(\xi,t)h^j,$$

$$\text{(3.4b)}\qquad \tilde{e}_p(\xi,t,h)=\sum_{j=k+1}^{\infty}\tilde{P}_j(\xi,t)h^j,\quad \tilde{e}_r(\xi,t,h)=\sum_{j=k+1}^{\infty}\tilde{R}_j(\xi,t)h^j,$$

where the leading terms of the discretization errors are given by

$$\text{(3.4c)}\qquad \tilde{U}_{k+1}(\xi,t)=a_{k+1}(t)\tilde{R}^+_{k+1}(\xi),\quad \tilde{Q}_{k+1}(\xi,t)=b_{k+1}(t)\tilde{R}^-_{k+1}(\xi),$$

$$\text{(3.4d)}\qquad \tilde{P}_{k+1}(\xi,t)=c_{k+1}(t)\tilde{R}^+_{k+1}(\xi),\quad \tilde{R}_{k+1}(\xi,t)=d_{k+1}(t)\tilde{R}^-_{k+1}(\xi).$$

In the remainder of this paper we will omit the $\tilde{}$ unless we feel it is needed for clarity. Since we consider one element, we will omit the $^\pm$, for instance, $v^+(-1)=v(-1)$ and $v^-(1)=v(1)$, etc.

Proof. Since we consider one element $[0,h]$, the numerical fluxes (2.3e) using the boundary conditions (1.1c) become

$$\begin{gathered}\hat{u}_h(-1,t,h)=u_1(t),\quad \hat{u}_h(1,t,h)=u_h(1,t,h),\\ \hat{q}_h(-1,t,h)=q_h(1,t,h),\quad \hat{q}_h(1,t,h)=u_3(t),\\ \hat{p}_h(-1,t,h)=u_2(t),\quad \hat{p}_h(1,t,h)=p_h(1,t,h),\\ \hat{r}_h(-1,t,h)=r_h(1,t,h),\quad \hat{r}_h(1,t,h)=u_4(t).\end{gathered}$$

Thus, the LDG orthogonality conditions (3.2) for the local errors can be simplified to

$$\text{(3.5a)}\qquad \frac{h}{2}\int_{-1}^{1}(e_u)_{tt}v\,d\xi-\int_{-1}^{1}e_r v_\xi d\xi-e_r(-1,t,h)v(-1)=0,$$

$$\text{(3.5b)}\qquad \frac{h}{2}\int_{-1}^{1}e_r w\,d\xi+\int_{-1}^{1}e_p w_\xi d\xi-e_p(1,t,h)w(1)=0,$$

$$\text{(3.5c)}\qquad \frac{h}{2}\int_{-1}^{1}e_p s\,d\xi+\int_{-1}^{1}e_q s_\xi d\xi+e_q(-1,t,h)s(-1)=0,$$

$$\frac{h}{2}\int_{-1}^{1} e_q z d\xi + \int_{-1}^{1} e_u z_\xi d\xi - e_u(1,t,h)z(1) \quad = \quad 0. \tag{3.5d}$$

Substituting (3.3) into (3.5) and collecting terms having the same powers of h lead to

$$-\int_{-1}^{1} R_0 v_\xi d\xi - R_0(-1,t)v(-1) + \sum_{j=1}^{k} h^j \left(\frac{1}{2}\int_{-1}^{1}(U_{j-1})_{tt} v d\xi - \int_{-1}^{1} R_j v_\xi d\xi - R_j(-1,t)v(-1)\right)$$

$$+ \sum_{j=k+1}^{\infty} h^j \left(\frac{1}{2}\int_{-1}^{1}(U_{j-1})_{tt} v d\xi - \int_{-1}^{1} R_j v_\xi d\xi - R_j(-1,t)v(-1)\right) = 0, \tag{3.6a}$$

$$\int_{-1}^{1} P_0 w_\xi d\xi - P_0(1,t)w(1) + \sum_{j=1}^{k} h^j \left(\frac{1}{2}\int_{-1}^{1} R_{j-1} w d\xi + \int_{-1}^{1} P_j w_\xi d\xi - P_j(1,t)w(1)\right) +$$

$$\sum_{j=k+1}^{\infty} h^j \left(\frac{1}{2}\int_{-1}^{1} R_{j-1} w d\xi + \int_{-1}^{1} P_j w_\xi d\xi - P_j(1,t)w(1)\right) = 0, \tag{3.6b}$$

$$\int_{-1}^{1} Q_0 s_\xi d\xi + Q_0(-1,t)s(-1) + \sum_{j=1}^{k} h^j \left(\frac{1}{2}\int_{-1}^{1} P_{j-1} s d\xi + \int_{-1}^{1} Q_j s_\xi d\xi + Q_j(-1,t)s(-1)\right) +$$

$$\sum_{j=k+1}^{\infty} h^j \left(\frac{1}{2}\int_{-1}^{1} P_{j-1} s d\xi + \int_{-1}^{1} Q_j s_\xi d\xi + Q_j(-1,t)s(-1)\right) = 0, \tag{3.6c}$$

$$\int_{-1}^{1} U_0 z_\xi d\xi - U_0(1,t)z(1) + \sum_{j=1}^{k} h^j \left(\frac{1}{2}\int_{-1}^{1} Q_{j-1} z d\xi + \int_{-1}^{1} U_j z_\xi d\xi - U_j(1,t)z(1)\right) +$$

$$\sum_{j=k+1}^{\infty} h^j \left(\frac{1}{2}\int_{-1}^{1} Q_{j-1} z d\xi + \int_{-1}^{1} U_j z_\xi d\xi - U_j(1,t)z(1)\right) = 0. \tag{3.6d}$$

Setting each term of the power series zero, the polynomials $U_j \in P^j([-1,1])$, $Q_j \in P^j([-1,1])$, $P_j \in P^j([-1,1])$ and $R_j \in P^j([-1,1])$, $j = 0, \cdots, k$, satisfy the following conditions: $\forall\ v, w, s, z \in P^k([-1,1])$,

$$-\int_{-1}^{1} R_0 v_\xi d\xi - R_0(-1,t)v(-1) = 0, \tag{3.7a}$$

$$\frac{1}{2}\int_{-1}^{1}(U_{j-1})_{tt} v d\xi - \int_{-1}^{1} R_j v_\xi d\xi - R_j(-1,t)v(-1) = 0, \quad j = 1, \cdots, k, \tag{3.7b}$$

$$\int_{-1}^{1} P_0 w_\xi d\xi - P_0(1,t)w(1) = 0, \tag{3.7c}$$

$$\frac{1}{2}\int_{-1}^{1} R_{j-1} w d\xi + \int_{-1}^{1} P_j w_\xi d\xi - P_j(1,t)w(1) = 0, \quad j = 1, \cdots, k, \tag{3.7d}$$

$$\int_{-1}^{1} Q_0 s_\xi d\xi + Q_0(-1,t)s(-1) = 0, \tag{3.7e}$$

$$\frac{1}{2}\int_{-1}^{1} P_{j-1} s d\xi + \int_{-1}^{1} Q_j s_\xi d\xi + Q_j(-1,t)s(-1) = 0, \quad j = 1, \cdots, k, \tag{3.7f}$$

$$\int_{-1}^{1} U_0 z_\xi d\xi - U_0(1,t)z(1) = 0, \tag{3.7g}$$

$$\frac{1}{2}\int_{-1}^{1} Q_{j-1} z d\xi + \int_{-1}^{1} U_j z_\xi d\xi - U_j(1,t)z(1) = 0, \quad j = 1, \cdots, k. \tag{3.7h}$$

Next, we will use induction to prove

$$U_j(\xi,t)=Q_j(\xi,t)=P_j(\xi,t)=R_j(\xi,t)=0,\quad 0\le\ j\le k. \tag{3.8}$$

Taking $v=w=s=z=1$ in (3.7a), (3.7c), (3.7e), and (3.7g), respectively, gives

$$U_0(1,t)=Q_0(-1,t)=P_0(1,t)=R_0(-1,t)=0.$$

Since $U_0,Q_0,P_0,R_0\in P^0([-1,1])$ are constant polynomials of degree 0, we have $U_0(\xi,t)=Q_0(\xi,t)=P_0(\xi,t)=R_0(\xi,t)=0$. Thus, (3.8) is true for $j=0$. Now we assume that $U_j(\xi,t)=Q_j(\xi,t)=P_j(\xi,t)=R_j(\xi,t)=0,\quad 0\le\ j\le k-1$ and prove that $U_k(\xi,t)=Q_k(\xi,t)=P_k(\xi,t)=R_k(\xi,t)=0$.
Next, we note that (3.7b), (3.7d), (3.7f), and (3.7h) for $j=k$ become

$$-\int_{-1}^{1}R_k v_\xi d\xi-R_k(-1,t)v(-1)=0,\quad \forall\ v\in P^k([-1,1]), \tag{3.9a}$$

$$\int_{-1}^{1}P_k w_\xi d\xi-P_k(1,t)w(1)=0,\quad \forall\ w\in P^k([-1,1]), \tag{3.9b}$$

$$\int_{-1}^{1}Q_k s_\xi d\xi+Q_k(-1,t)s(-1)=0,\quad \forall\ s\in P^k([-1,1]), \tag{3.9c}$$

$$\int_{-1}^{1}U_k z_\xi d\xi-U_k(1,t)z(1)=0,\quad \forall\ z\in P^k([-1,1]). \tag{3.9d}$$

Setting $v=w=s=z=1$ in (3.9a), (3.9b), (3.9c), and (3.9d), respectively, yields

$$U_k(1,t)=Q_k(-1,t)=P_k(1,t)=R_k(-1,t)=0. \tag{3.10}$$

Combining (3.9) and (3.10) we obtain: $\forall\ v,w,s,z\in P^k([-1,1])$,

$$\int_{-1}^{1}R_k v_\xi d\xi=0,\quad \int_{-1}^{1}P_k w_\xi d\xi=0,\quad \int_{-1}^{1}Q_k s_\xi d\xi=0,\quad \int_{-1}^{1}U_k z_\xi d\xi=0. \tag{3.11}$$

Writing $U_k(\xi,t)$, $Q_k(\xi,t)$, $P_k(\xi,t)$, and $R_k(\xi,t)$ as a linear combination of Legendre polynomials,

$$U_k(\xi,t)=\sum_{j=0}^{k}a_j(t)L_j(\xi),\ \ Q_k(\xi,t)=\sum_{j=0}^{k}b_j(t)L_j(\xi),$$

$$P_k(\xi,t)=\sum_{j=0}^{k}c_j(t)L_j(\xi),\ \ R_k(\xi,t)=\sum_{j=0}^{k}d_j(t)L_j(\xi),$$

and using (3.11) and the orthogonality relation (2.5), we arrive at

$$U_k(\xi,t)=a_k(t)L_k(\xi),\quad Q_k(\xi,t)=b_k(t)L_k(\xi),$$

$$P_k(\xi,t)=c_k(t)L_k(\xi),\quad R_k(\xi,t)=d_k(t)L_k(\xi).$$

Using (3.10) and the properties of Legendre polynomial $L_k(1)=1$, $L_k(-1)=(-1)^k$, we get

$$0=U_k(1,t)=a_k(t)L_k(1)=a_k(t),\quad 0=Q_k(-1,t)=b_k(t)L_k(-1)=(-1)^k b_k(t),$$
$$0=P_k(1,t)=c_k(t)L_k(1)=c_k(t),\quad 0=R_k(-1,t)=d_k(t)L_k(-1)=(-1)^k d_k(t).$$

Thus,

$$U_k(\xi,t)=Q_k(\xi,t)=P_k(\xi,t)=R_k(\xi,t)=0. \tag{3.12}$$

Next, after using (3.12), the $\mathcal{O}(h^{k+1})$ terms in (3.6) yield

$$-\int_{-1}^{1} R_{k+1} v_\xi d\xi - R_{k+1}(-1,t)v(-1) = 0, \quad \forall\, v \in P^k([-1,1]), \tag{3.13a}$$

$$\int_{-1}^{1} P_{k+1} w_\xi d\xi - P_{k+1}(1,t)w(1) = 0, \quad \forall\, w \in P^k([-1,1]), \tag{3.13b}$$

$$\int_{-1}^{1} Q_{k+1} s_\xi d\xi + Q_{k+1}(-1,t)s(-1) = 0, \quad \forall\, s \in P^k([-1,1]), \tag{3.13c}$$

$$\int_{-1}^{1} U_{k+1} z_\xi d\xi - U_{k+1}(1,t)z(1) = 0, \quad \forall\, z \in P^k([-1,1]). \tag{3.13d}$$

Taking $v = w = s = z = 1$ in (3.13a), (3.13b), (3.13c), and (3.13d), respectively, we get

$$U_{k+1}(1,t) = Q_{k+1}(-1,t) = P_{k+1}(1,t) = R_{k+1}(-1,t) = 0. \tag{3.14}$$

Therefore, (3.13) becomes: $\forall\, v, w, s, z \in P^k([-1,1])$,

$$\begin{array}{ll} \int_{-1}^{1} R_{k+1} v_\xi d\xi = 0, & \int_{-1}^{1} P_{k+1} w_\xi d\xi = 0, \\ \int_{-1}^{1} Q_{k+1} s_\xi d\xi = 0, & \int_{-1}^{1} U_{k+1} z_\xi d\xi = 0. \end{array} \tag{3.15}$$

Expanding $U_{k+1},\ Q_{k+1},\ P_{k+1},\ R_{k+1} \in P^{k+1}([-1,1])$ in series of Legendre polynomials *i.e.*,

$$U_{k+1}(\xi,t) = \sum_{j=0}^{k+1} a_j(t) L_j(\xi), \quad Q_{k+1}(\xi,t) = \sum_{j=0}^{k+1} b_j(t) L_j(\xi), \tag{3.16a}$$

$$P_{k+1}(\xi,t) = \sum_{j=0}^{k+1} c_j(t) L_j(\xi), \quad R_{k+1}(\xi,t) = \sum_{j=0}^{k+1} d_j(t) L_j(\xi), \tag{3.16b}$$

and using the orthogonality relation (2.5), we obtain

$$U_{k+1}(\xi,t) = a_k(t)L_k(\xi) + a_{k+1}(t)L_{k+1}(\xi), \quad Q_{k+1}(\xi,t) = b_k(t)L_k(\xi) + b_{k+1}(t)L_{k+1}(\xi),$$
$$P_{k+1}(\xi,t) = c_k(t)L_k(\xi) + c_{k+1}(t)L_{k+1}(\xi), \quad R_{k+1}(\xi,t) = d_k(t)L_k(\xi) + d_{k+1}(t)L_{k+1}(\xi).$$

Using (3.14) and the properties $L_k(1) = 1$, $L_k(-1) = (-1)^k$, we have

$$\begin{array}{lll} 0 & = & U_{k+1}(1,t) = a_k(t)L_k(1) + a_{k+1}(t)L_{k+1}(1) = a_k(t) + a_{k+1}(t), \\ 0 & = & Q_{k+1}(-1,t) = b_k(t)L_k(-1) + b_{k+1}(t)L_{k+1}(1) = (-1)^k b_k(t) + (-1)^{k+1} b_{k+1}(t), \\ 0 & = & P_{k+1}(1,t) = c_k(t)L_k(1) + c_{k+1}(t)L_{k+1}(1) = c_k(t) + c_{k+1}(t), \\ 0 & = & R_{k+1}(-1,t) = d_k(t)L_k(-1) + d_{k+1}(t)L_{k+1}(1) = (-1)^k d_k(t) + (-1)^{k+1} d_{k+1}(t), \end{array}$$

which give $a_{k+1}(t) = -a_k(t)$, $\quad b_{k+1}(t) = b_k(t)$, $\quad c_{k+1}(t) = -c_k(t)$, $\quad d_{k+1}(t) = d_k(t)$. Thus, the leading terms of the discretization errors can be written as

$$\begin{aligned} U_{k+1}(\xi,t) &= a_{k+1}(t)\left(L_{k+1}(\xi) - L_k(\xi)\right) = a_{k+1}(t) R^+_{k+1}(\xi), \\ Q_{k+1}(\xi,t) &= b_{k+1}(t)\left(L_{k+1}(\xi) + L_k(\xi)\right) = b_{k+1}(t) R^-_{k+1}(\xi), \\ P_{k+1}(\xi,t) &= c_{k+1}(t)\left(L_{k+1}(\xi) - L_k(\xi)\right) = c_{k+1}(t) R^+_{k+1}(\xi), \\ R_{k+1}(\xi,t) &= d_{k+1}(t)\left(L_{k+1}(\xi) + L_k(\xi)\right) = d_{k+1}(t) R^-_{k+1}(\xi), \end{aligned}$$

which complete the proof of the Theorem. $\square$

3.2. Case 2. Here, we consider the problem (1.1a) in $[0,h]$ subject to the initial conditions (1.1b) and to the boundary conditions (1.1d). In the following theorem, we show that the results of Theorem 3.1 still hold.

Theorem 3.2. *Let (u,q,p,r) and (u_h,q_h,p_h,r_h), respectively, be the solutions of (2.1) and (2.3) in $[0,h]$ subject to the initial conditions (1.1b) and the boundary conditions (1.1d). Let $x(\xi,h)=\frac{h}{2}(\xi+1)$ be the mapping of $[0,h]$ onto $[-1,1]$. Then (3.4) holds.*

Proof. Since we consider one element $[0,h]$, the numerical fluxes (2.3f) using the boundary conditions (1.1d) with $\delta_1=\frac{k}{h}$ and $\delta_2=\frac{k}{h}$ become

$$\begin{aligned}
&\hat{u}_h(0,t)=u_1(t), \quad \hat{u}_h(h,t)=u_3(t),\\
&\hat{q}_h(0,t)=q_h(0,t), \quad \hat{q}_h(h,t)=q_h(h,t)-\frac{k}{h}(u_h(h,t)-u_3(t)),\\
&\hat{p}_h(0,t)=u_2(0,t), \quad \hat{p}_h(h,t)=u_4(t),\\
&\hat{r}_h(0,t)=r_h(0,t), \quad \hat{r}_h(h,t)=r_h(h,t)-\frac{k}{h}(p_h(h,t)-u_4(t)),
\end{aligned}$$

where u_h,q_h,p_h and r_h are the LDG solutions on $[0,h]$. Thus,

$$\begin{aligned}
\hat{e}_u(0,t)&=u(0,t)-\hat{u}_h(0,t)=u_1(t)-u_1(t)=0,\\
\hat{e}_u(h,t)&=u(h,t)-\hat{u}_h(h,t)=u_3(t)-u_3(t)=0,\\
\hat{e}_q(0,t)&=q(0,t)-\hat{q}_h(0,t)=q(0,t)-q_h(0,t)=e_q(0,t),\\
\hat{e}_q(h,t)&=q(h,t)-\hat{q}_h(h,t)=q(h,t)-q_h(h,t)+\frac{k}{h}(u_h(h,t)-u_3(t))\\
&=e_q(h,t)-\frac{k}{h}e_u(h,t),\\
\hat{e}_p(0,t)&=p(0,t)-\hat{p}_h(0,t)=u_2(t)-u_2(t)=0,\\
\hat{e}_p(h,t)&=p(h,t)-\hat{p}_h(h,t)=u_4(t)-u_4(t)=0,\\
\hat{r}_h(0,t)&=r(0,t)-\hat{r}_h(0,t)=r(0,t)-r_h(0,t)=e_r(0,t),\\
\hat{e}_r(h,t)&=r(h,t)-r_h(h,t)=r(h,t)-r_h(h,t)+\frac{k}{h}(p_h(h,t)-u_4(t))\\
&=e_r(h,t)-\frac{k}{h}e_p(h,t).
\end{aligned}$$

Using the mapping of $[0,h]$ onto $[-1,1]$ given by (2.6), we have

$$\begin{aligned}
&\hat{e}_u(-1,t,h)=0, \quad \hat{e}_u(1,t,h)=0,\\
&\hat{e}_q(-1,t,h)=e_q(-1,t,h), \quad \hat{e}_q(1,t,h)=e_q(1,t,h)-\frac{k}{h}e_u(1,t,h),\\
&\hat{e}_p(-1,t,h)=0, \quad \hat{e}_p(1,t,h)=0,\\
&\hat{e}_r(-1,t,h)=e_r(-1,t,h), \quad \hat{e}_r(1,t,h)=e_r(1,t,h)-\frac{k}{h}e_p(1,t,h).
\end{aligned}$$

The LDG orthogonality conditions (3.2) with the boundary conditions (1.1d) and numerical fluxes (2.3f) become

$$\frac{h}{2}\int_{-1}^{1}(e_u)_{tt}v\,d\xi-\int_{-1}^{1}e_rv_\xi d\xi+\left(e_r(1,t,h)-\frac{k}{h}e_p(1,t,h)\right)v(1)$$

$$-e_r(-1,t,h)v(-1)=0, \tag{3.17a}$$

$$\frac{h}{2}\int_{-1}^{1}e_rw\,d\xi+\int_{-1}^{1}e_pw_\xi d\xi=0, \tag{3.17b}$$

$$\frac{h}{2}\int_{-1}^{1} e_p s d\xi + \int_{-1}^{1} e_q s_\xi d\xi - \left(e_q(1,t,h) - \frac{k}{h}e_u(1,t,h)\right)s(1)$$

$$+e_q(-1,t,h)s(-1) = 0, \tag{3.17c}$$

$$\frac{h}{2}\int_{-1}^{1} e_q z d\xi + \int_{-1}^{1} e_u z_\xi d\xi = 0. \tag{3.17d}$$

Substituting the series (3.3) in the LDG orthogonality condition (3.17) and collecting terms having the same powers of h we get

$$-kP_0(1,t)v(1) + (-\int_{-1}^{1} R_0 v_\xi d\xi + (R_0(1,t) - kP_1(1,t))\,v(1) - R_0(-1,t)v(-1))h$$

$$+\sum_{j=2}^{\infty} h^j(\frac{1}{2}\int_{-1}^{1}(U_{j-2})_{tt} v d\xi - \int_{-1}^{1} R_{j-1}v_\xi d\xi + (R_{j-1}(1,t) - kP_j(1,t))\,v(1)$$

$$-R_{j-1}(-1,t)v(-1)) = 0,$$

$$\int_{-1}^{1} P_0 w_\xi d\xi + \sum_{j=1}^{\infty} h^j\left(\frac{1}{2}\int_{-1}^{1} R_{j-1} w d\xi + \int_{-1}^{1} P_j w_\xi d\xi\right) = 0,$$

$$kU_0(1,t)s(1) + (\int_{-1}^{1} Q_0 s_\xi d\xi - (Q_0(1,t) - kU_1(1,t))\,s(1) + Q_0(-1,t)s(-1))h+$$

$$\sum_{j=2}^{\infty} h^j(\frac{1}{2}\int_{-1}^{1} P_{j-2} s d\xi + \int_{-1}^{1} Q_{j-1} s_\xi d\xi - (Q_{j-1}(1,t) - kU_j(1,t))\,s(1)$$

$$+Q_{j-1}(-1,t)s(-1)) = 0,$$

$$\int_{-1}^{1} U_0 z_\xi d\xi + \sum_{j=1}^{\infty} h^j\left(\frac{1}{2}\int_{-1}^{1} Q_{j-1} z d\xi + \int_{-1}^{1} U_j z_\xi d\xi\right) = 0.$$

Setting each term of the power series zero yields the orthogonality conditions: $\forall\ v, w, s, z \in P^k([-1,1])$,

$$-kP_0(1,t)v(1) = \int_{-1}^{1} P_0 w_\xi d\xi = kU_0(1,t)s(1) = \int_{-1}^{1} U_0 z_\xi d\xi = 0, \tag{3.18a}$$

$$-\int_{-1}^{1} R_0 v_\xi d\xi + (R_0(1,t) - kP_1(1,t))\,v(1) - R_0(-1,t)v(-1) = 0, \tag{3.18b}$$

$$\int_{-1}^{1} Q_0 s_\xi d\xi - (Q_0(1,t) - kU_1(1,t))\,s(1) + Q_0(-1,t)s(-1) = 0, \tag{3.18c}$$

$$\frac{1}{2}\int_{-1}^{1}(U_{j-2})_{tt} v d\xi - \int_{-1}^{1} R_{j-1} v_\xi d\xi + (R_{j-1}(1,t) - kP_j(1,t))\,v(1)$$

$$-R_{j-1}(-1,t)v(-1) = 0, \quad j \ge 2, \tag{3.18d}$$

$$\frac{1}{2}\int_{-1}^{1} R_{j-1} w d\xi + \int_{-1}^{1} P_j w_\xi d\xi = 0, \quad j \ge 1, \tag{3.18e}$$

$$\frac{1}{2}\int_{-1}^{1} P_{j-2} s d\xi + \int_{-1}^{1} Q_{j-1} s_\xi d\xi - (Q_{j-1}(1,t) - kU_j(1,t))\,s(1)$$

$$+Q_{j-1}(-1,t)s(-1) = 0, \quad j \ge 2, \tag{3.18f}$$

$$\frac{1}{2}\int_{-1}^{1} Q_{j-1} z d\xi + \int_{-1}^{1} U_j z_\xi d\xi = 0, \quad j \ge 1. \tag{3.18g}$$

Again, by induction we will prove that

$$U_j(\xi,t) = Q_j(\xi,t) = P_j(\xi,t) = R_j(\xi,t) = 0, \quad 0 \le\ j \le k. \tag{3.19}$$

Taking $v = 1$ and $s = 1$ in (3.18a) and using the fact that $U_0,\ P_0 \in P^0([-1,1])$ yields $U_0(\xi,t) = P_0(\xi,t) = 0$. Similarly, choosing $w = z = 1$ in (3.18e) and (3.18g) for $j = 1$ leads to $Q_0(\xi,t) = R_0(\xi,t) = 0$.
Taking $v = 1$ and $s = 1$ in (3.18b) and (3.18c), respectively, and using the fact that $Q_0(\xi,t) = R_0(\xi,t) = 0$, we obtain

$$U_1(1,t) = P_1(1,t) = 0. \tag{3.20}$$

Letting $w = \xi$ in (3.18e) for $j = 1$ and $z = \xi$ in (3.18g) for $j = 1$ and using $Q_0(\xi,t) = R_0(\xi,t) = 0$, we obtain

$$\int_{-1}^{1} U_1 d\xi = \int_{-1}^{1} P_1 d\xi = 0. \tag{3.21}$$

Combining (3.20) with (3.21) we get $U_1(\xi,t) = P_1(\xi,t) = 0$. Now, we assume that

$$U_j(\xi,t) = Q_j(\xi,t) = P_j(\xi,t) = R_j(\xi,t) = 0, \quad 0 \le\ j \le k-1, \tag{3.22}$$

and use induction to show that $U_k(\xi,t) = Q_k(\xi,t) = P_k(\xi,t) = R_k(\xi,t) = 0$.
Next, we note that (3.22), (3.18d), (3.18e), (3.18f), and (3.18g) for $j = k$ gives: $\forall\ v, w, s, z \in P^k([-1,1])$,

$$P_k(1,t)v(1) = \int_{-1}^{1} P_k w_\xi d\xi = U_k(1,t)s(1) = \int_{-1}^{1} U_k z_\xi d\xi = 0. \tag{3.23}$$

Letting $v = s = 1$ in (3.23), we get

$$U_k(1,t) = P_k(1,t) = 0. \tag{3.24}$$

Writing $U_k(\xi,t)$ and $P_k(\xi,t)$ as a linear combination of Legendre polynomials, $U_k(\xi,t) = \sum_{j=0}^{k} a_j(t)L_j(\xi)$, $P_k(\xi,t) = \sum_{j=0}^{k} c_j(t)L_j(\xi)$, using (3.23), and the orthogonality (2.5), we obtain $U_k(\xi,t) = a_k(t)L_k(\xi)$ and $P_k(\xi,t) = c_k(t)L_k(\xi)$, which, after applying (3.24) and the fact that $L_k(1) = 1$, give $0 = U_k(1,t) = a_k(t)L_k(1) = a_k(t)$ and $0 = P_k(1,t) = c_k(t)L_k(1) = c_k(t)$. Thus,

$$U_k(\xi,t) = P_k(\xi,t) = 0. \tag{3.25}$$

Combining (3.18d) and (3.18f) for $j = k+1$ with (3.22) and (3.25) yields ($\forall\ s \in P^k([-1,1])$)

$$-\int_{-1}^{1} R_k v_\xi d\xi + \left(R_k(1,t) - kP_{k+1}(1,t)\right) v(1) - R_k(-1,t)v(-1) = 0, \tag{3.26}$$

$$\int_{-1}^{1} Q_k s_\xi d\xi - \left(Q_k(1,t) - kU_{k+1}(1,t)\right) s(1) + Q_k(-1,t)s(-1) = 0. \tag{3.27}$$

Testing against $w = z = 1$ in (3.18e) and (3.18g), respectively, for $j = k+1$ yield

$$\int_{-1}^{1} R_k d\xi = 0, \quad \int_{-1}^{1} Q_k d\xi = 0. \tag{3.28}$$

Testing against $v = \xi - 1$ in (3.26) and $s = \xi - 1$ in (3.27), we get

$$-\int_{-1}^{1} R_k d\xi + 2R_k(-1,t) = 0, \quad \int_{-1}^{1} Q_k d\xi - 2Q_k(-1,t) = 0. \tag{3.29}$$

Combining (3.29) with (3.28), we arrive at

$$R_k(-1,t) = Q_k(-1,t) = 0. \tag{3.30}$$

Now, (3.26) and (3.27) become

$$-\int_{-1}^{1} R_k v_\xi d\xi + (R_k(1,t) - kP_{k+1}(1,t))\, v(1) = 0, \tag{3.31}$$

$$\int_{-1}^{1} Q_k s_\xi d\xi - (Q_k(1,t) - kU_{k+1}(1,t))\, s(1) = 0. \tag{3.32}$$

Choosing $v = s = (\xi-1)^i,\ 1 \le i \le k$, the second terms in (3.31) and (3.32) vanish and

$$-\int_{-1}^{1} R_k (\xi-1)^{i-1} d\xi = 0, \quad \int_{-1}^{1} Q_k (\xi-1)^{i-1} d\xi = 0, \quad \forall\ 1 \le i \le k.$$

Expanding $Q_k(\xi,t) = \sum_{j=0}^{k} b_j(t) L_j(\xi)$, $R_k(\xi,t) = \sum_{j=0}^{k} d_j(t) L_j(\xi)$, and $(\xi-1)^{i-1} = \sum_{j=0}^{i-1} e_j(t) L_j(\xi)$, $1 \le i \le k$, in series of Legendre polynomials and using the orthogonality relation (2.5), we arrive at

$$Q_k(\xi,t) = b_k(t) L_k(\xi), \quad R_k(\xi,t) = d_k(t) L_k(\xi).$$

Applying (3.30) and using the property $L_k(-1) = (-1)^k$, we obtain

$$0 = Q_k(-1,t) = b_k(t) L_k(-1) = (-1)^k b_k(t),$$
$$0 = R_k(-1,t) = d_k(t) L_k(-1) = (-1)^k d_k(t),$$

which yield $b_k = d_k = 0$. We conclude that $Q_k(\xi,t) = R_k(\xi,t) = 0$, which complete the proofs of (3.4a) and (3.4b). Next we will show (3.4c) and (3.4d). On the one hand, since $Q_k(\xi,t) = R_k(\xi,t) = 0$, (3.26) and (3.27) give

$$U_{k+1}(1,t) = P_{k+1}(1,t) = 0. \tag{3.33}$$

On the other hand, (3.18e) and (3.18g) for $j = k+1$ yield

$$\int_{-1}^{1} P_{k+1} w_\xi d\xi = 0, \quad \int_{-1}^{1} U_{k+1} z_\xi d\xi = 0, \quad \forall\ w, z \in P^k([-1,1]). \tag{3.34}$$

Expanding $U_{k+1} \in P^{k+1}([-1,1])$ and $P_{k+1} \in P^{k+1}([-1,1])$ in series of Legendre polynomials $U_{k+1}(\xi,t) = \sum_{j=0}^{k+1} a_j(t) L_j(\xi)$, $P_{k+1}(\xi,t) = \sum_{j=0}^{k+1} c_j(t) L_j(\xi)$, using (3.34) and the orthogonality relation (2.5), we arrive at

$$U_{k+1}(\xi,t) = a_k(t) L_k(\xi) + a_{k+1}(t) L_{k+1}(\xi),$$
$$P_{k+1}(\xi,t) = c_k(t) L_k(\xi) + c_{k+1}(t) L_{k+1}(\xi).$$

Applying (3.33) and the fact that $L_k(1) = L_{k+1}(1) = 1$, we obtain

$$\begin{aligned} 0 &= U_{k+1}(1,t) = a_k(t) L_k(1) + a_{k+1}(t) L_{k+1}(1) = a_k(t) + a_{k+1}(t), \\ 0 &= P_{k+1}(1,t) = c_k(t) L_k(1) + c_{k+1}(t) L_{k+1}(1) = c_k(t) + c_{k+1}(t), \end{aligned}$$

which give $a_{k+1}(t) = -a_k(t)$ and $c_{k+1}(t) = -c_k(t)$. Thus,

$$U_{k+1}(\xi,t) = a_{k+1}(t) \left(L_{k+1}(\xi) - L_k(\xi)\right), \quad P_{k+1}(\xi,t) = c_{k+1}(t) \left(L_{k+1}(\xi) - L_k(\xi)\right).$$

Next, (3.18d) and (3.18f) for $j = k+2$ become

$$-\int_{-1}^{1} R_{k+1} v_\xi d\xi + (R_{k+1}(1,t) - kP_{k+2}(1,t))\, v(1) - R_{k+1}(-1,t) v(-1) = 0,$$
$$\forall\ v \in P^k([-1,1]), \tag{3.35a}$$

$$\int_{-1}^{1} Q_{k+1} s_\xi d\xi - (Q_{k+1}(1,t) - kU_{k+2}(1,t))\, s(1) + Q_{k+1}(-1,t) s(-1) = 0,$$
$$\forall\ s \in P^k([-1,1]). \tag{3.35b}$$

Taking $v = s = \xi - 1$ gives

$$(3.36)\qquad -\int_{-1}^{1} R_{k+1} d\xi + 2R_{k+1}(-1,t) = 0, \quad \int_{-1}^{1} Q_{k+1} d\xi - 2Q_{k+1}(-1,t) = 0.$$

Testing against $w = z = 1$ in (3.18e) and (3.18g) for $j = k+1$, we obtain

$$(3.37)\qquad \int_{-1}^{1} R_{k+1} d\xi = 0, \quad \int_{-1}^{1} Q_{k+1} d\xi = 0.$$

Combining (3.36) and (3.37), we get

$$(3.38)\qquad Q_{k+1}(-1,t) = R_{k+1}(-1,t) = 0.$$

Thus, (3.35) becomes

$$(3.39a)\ -\int_{-1}^{1} R_{k+1} v_\xi d\xi + (R_{k+1}(1,t) - kP_{k+2}(1,t))\, v(1) = 0, \quad \forall\ v \in P^k([-1,1]),$$

$$(3.39b)\ \int_{-1}^{1} Q_{k+1} s_\xi d\xi - (Q_{k+1}(1,t) - kU_{k+2}(1,t))\, s(1) = 0, \quad \forall\ s \in P^k([-1,1]).$$

Testing against $v = s = (\xi-1)^i,\ 1 \le i \le k$, the second terms in (3.39) are 0 and

$$(3.40)\quad -\int_{-1}^{1} R_{k+1}(\xi-1)^{i-1} d\xi = 0, \quad \int_{-1}^{1} Q_{k+1}(\xi-1)^{i-1} d\xi = 0, \quad \forall\ 1 \le i \le k.$$

Expanding $Q_{k+1}(\xi,t) = \sum_{j=0}^{k+1} b_j(t) L_j(\xi)$, $R_{k+1}(\xi,t) = \sum_{j=0}^{k+1} d_j(t) L_j(\xi)$, and $(\xi - 1)^{i-1} = \sum_{j=0}^{i-1} e_j(t) L_j(\xi),\ 1 \le i \le k$ in series of Legendre polynomials and using the orthogonality relation (2.5), (3.40) yields

(3.41)

$$Q_{k+1}(\xi,t) = b_k(t)L_k(\xi) + b_{k+1}(t)L_{k+1}(\xi), R_{k+1}(\xi,t) = d_k(t)L_k(\xi) + d_{k+1}(t)L_{k+1}(\xi).$$

Combining (3.38) with (3.41) we conclude that $b_{k+1} = b_k$ and $d_{k+1} = d_k$, which complete the proof of the Theorem. □

In the previous section, we proved that the k-degree LDG solutions u_h and p_h are $\mathcal{O}(h^{k+2})$ superconvergent at the roots of the $(k+1)$-degree right Radau polynomial and q_h and r_h are $\mathcal{O}(h^{k+2})$ superconvergent at the roots of the $(k+1)$-degree left Radau polynomial. Now, let us note that a global superconvergence error analysis is yet to be performed and will be investigated in the future. We expect that a similar superconvergence result of Shu *et al.* [52, 28, 43] will be needed.

4. *A posteriori* error estimation

In this section, we present a technique to compute asymptotically exact *a posteriori* estimates of the LDG errors. The LDG error estimates under investigation are computed by solving a local steady problem with no boundary conditions on each element. Our numerical examples show that the LDG discretization error estimates converge to the true spatial errors in the L^2-norm as $h \to 0$.

Before we present the weak finite element formulations to compute *a posteriori* error estimates for the beam equation (1.1a), we state and prove some results which will be needed in our *a posteriori* error analysis.

Lemma 4.1. *The $(k+1)$-degree Radau polynomials on I_i, $R^{\pm}_{k+1,i}(x)$, $x \in I_i$, satisfy*

$$(4.1)\qquad \int_{I_i} \frac{dR^+_{k+1,i}}{dx} R^+_{k+1,i} dx = -2, \qquad \int_{I_i} \frac{dR^-_{k+1,i}}{dx} R^-_{k+1,i} dx = 2.$$

Proof. Since $L_k(1) = 1$ and $L_k(-1) = (-1)^k$, we have $\tilde{R}^+_{k+1}(1) = R^+_{k+1,i}(x_i) = 0$ and $\tilde{R}^+_{k+1}(-1) = R^+_{k+1,i}(x_{i-1}) = 2(-1)^{k+1}$. Thus,

$$\begin{aligned}\int_{I_i} \frac{dR^+_{k+1,i}}{dx} R^+_{k+1,i} dx &= \frac{1}{2}(R^+_{k+1,i})^2(x_i) - \frac{1}{2}(R^+_{k+1,i})^2(x_{i-1}) \\ &= -\frac{1}{2}(R^+_{k+1,i})^2(x_{i-1}) = -2. \end{aligned} \tag{4.2}$$

Similarly, since $\tilde{R}^-_{k+1}(1) = R^-_{k+1,i}(x_i) = 2$ and $\tilde{R}^-_{k+1}(-1) = R^-_{k+1,i}(x_{i-1}) = 0$, we have

$$\begin{aligned}\int_{I_i} \frac{dR^-_{k+1,i}}{dx} R^-_{k+1,i} dx &= \frac{1}{2}(R^-_{k+1,i})^2(x_i) - \frac{1}{2}(R^-_{k+1,i})^2(x_{i-1}) \\ &= \frac{1}{2}(R^-_{k+1,i})^2(x_i) = 2. \end{aligned} \tag{4.3}$$

Thus, we have completed the proof the Lemma. □

Next, we present the weak finite element formulations to compute *a posteriori* error estimates for the beam equation (1.1a). Multiplying the four equations in (2.1) by test functions v, w, s, and z, respectively, integrating over an arbitrary element I_i, and replacing u by $u_h + e_u$, q by $q_h + e_q$, p by $p_h + e_p$, and r by $r_h + e_r$, we get

$$\int_{I_i} (e_r)_x v dx = \int_{I_i} (R_{h,1} - (e_u)_{tt})\, v dx, \tag{4.4a}$$

$$-\int_{I_i} (e_p)_x w dx = \int_{I_i} (R_{h,2} - e_r)\, w dx, \tag{4.4b}$$

$$-\int_{I_i} (e_q)_x s dx = \int_{I_i} (R_{h,3} - e_p)\, s dx, \tag{4.4c}$$

$$-\int_{I_i} (e_u)_x z dx = \int_{I_i} (R_{h,4} - e_q)\, z dx, \tag{4.4d}$$

where

$$R_{h,1} = f - (u_h)_{tt} - (r_h)_x, \quad R_{h,2} = (p_h)_x - r_h, \quad R_{h,3} = (q_h)_x - p_h, \quad R_{h,4} = (u_h)_x - q_h.$$

Since the true errors can be split into significant parts and less significant parts as shown in Theorem 3.1, our error estimate procedure consists of approximating the true errors on each element I_i by the leading terms as

$$\begin{aligned} e_u &\approx E_u = a_{k+1}(t)R^+_{k+1,i}(x), \quad e_q \approx E_q = b_{k+1}(t)R^-_{k+1,i}(x), \quad x \in I_i, \quad t \in [0,T], \\ e_p &\approx E_p = c_{k+1}(t)R^+_{k+1,i}(x), \quad e_r \approx E_r = d_{k+1}(t)R^-_{k+1,i}(x), \quad x \in I_i, \quad t \in [0,T], \end{aligned}$$

where the coefficients of the leading terms of the errors, $a_{k+1}, b_{k+1}, c_{k+1}, d_{k+1}$ can be obtained from (4.4) as follows: (i) Neglecting the unknowns errors $(e_u)_{tt}, e_q, e_p$ and e_r which will be justified in Remark 1, (ii) replacing e_u by E_u, e_q by E_q, e_p by E_p, and e_r by E_r in (4.4), and (iii) choosing $v = s = R^-_{k+1,i}(x)$, $w = z = R^+_{k+1,i}(x)$, we obtain the following equations for $a_{k+1}, b_{k+1}, c_{k+1}$, and d_{k+1}

$$d_{k+1}\int_{I_i}\frac{dR^-_{k+1,i}}{dx}R^-_{k+1,i}dx = \int_{I_i}R_{h,1}R^-_{k+1,i}dx, \tag{4.5a}$$

$$-c_{k+1}\int_{I_i}\frac{dR^+_{k+1,i}}{dx}R^+_{k+1,i}dx = \int_{I_i}R_{h,2}R^+_{k+1,i}dx, \tag{4.5b}$$

$$-b_{k+1}\int_{I_i}\frac{dR^-_{k+1,i}}{dx}R^-_{k+1,i}dx = \int_{I_i}R_{h,3}R^-_{k+1,i}dx, \tag{4.5c}$$

$$-a_{k+1}\int_{I_i}\frac{dR^+_{k+1,i}}{dx}R^+_{k+1,i}dx = \int_{I_i}R_{h,4}R^+_{k+1,i}dx. \tag{4.5d}$$

Using the properties in (4.1) and solving for $a_{k+1}, b_{k+1}, c_{k+1}$, and d_{k+1}, we get

$$a_{k+1}(t) = \frac{1}{2}\int_{I_i}R_{h,4}R^+_{k+1,i}dx, \quad b_{k+1}(t) = -\frac{1}{2}\int_{I_i}R_{h,3}R^-_{k+1,i}dx, \tag{4.6a}$$

$$c_{k+1}(t) = \frac{1}{2}\int_{I_i}R_{h,2}R^+_{k+1,i}dx, \quad d_{k+1}(t) = \frac{1}{2}\int_{I_i}R_{h,1}R^-_{k+1,i}dx. \tag{4.6b}$$

Remark 1. *Numerical experiments show that neglecting the terms involving* $(e_u)_{tt}$, e_q, e_p *and* e_r *does not affect the quality of the a posteriori error estimates for* u, q, p *and* r. *In fact, the terms involving* $(e_u)_{tt}, e_q, e_p$ *and* e_r *on the right-hand side of (4.4) can be neglected since the terms on the left-hand side contain the derivative of the errors with respect to* x.

An accepted efficiency measure of *a posteriori* error estimates is the effectivity index. In this paper, we use the global effectivity indices

$$\theta_u(t) = \frac{\|E_u\|}{\|e_u\|}, \quad \theta_q(t) = \frac{\|E_q\|}{\|e_q\|}, \quad \theta_p(t) = \frac{\|E_p\|}{\|e_p\|}, \quad \theta_r(t) = \frac{\|E_r\|}{\|e_r\|}.$$

Ideally, the global effectivity indices should stay close to one and should converge to one under mesh refinement.

5. Numerical examples

In this section, we provide some numerical examples to demonstrate the global superconvergence results and the asymptotic exactness of our *a posteriori* errors estimates under mesh refinement. The initial conditions are determined using (2.4). Temporal integration is performed by the fourth-order classical explicit Runge-Kutta method. A time step Δt is chosen so that temporal errors are small relative to spatial errors. We do not discuss the influence of the time discretization error in this paper. We compute the maximum LDG errors $\|e_u\|^*$ and $\|e_p\|^*$ at shifted roots of $(k+1)$-degree right-Radau polynomial on each element I_i and then take the maximum over all elements $I_i,\ i = 1, \cdots, N$. Similarly, the maximum LDG errors $\|e_q\|^*$ and $\|e_r\|^*$ are computed at shifted roots of $(k+1)$-degree left-Radau polynomial on each element and by taking the maximum over all elements *i.e.*,

$$\|e_u\|^* = \max_{1\le i\le N}\left(\max_{1\le j\le k+1}|e_u(x^+_{j,i},t)|\right), \quad \|e_q\|^* = \max_{1\le i\le N}\left(\max_{1\le j\le k+1}|e_q(x^-_{j,i},t)|\right),$$

$$\|e_p\|^* = \max_{1\le i\le N}\left(\max_{1\le j\le k+1}|e_p(x^+_{k,i},t)|\right), \quad \|e_r\|^* = \max_{1\le i\le N}\left(\max_{1\le j\le k+1}|e_r(x^-_{j,i},t)|\right),$$

where $x^\pm_{j,i}$ are the shifted roots of $R^\pm_{k+1,i}$ on I_i.

Example 5.1. *We consider the following problem with mixed boundary conditions*

$$\begin{cases} u_{tt} + u_{xxxx} = 2e^{x+t}, \quad x \in [0,1],\ t \in [0,1], \\ u(x,0) = e^{x}, \quad u_t(x,0) = e^{x}, \quad x \in [0,1], \\ u(0,t) = e^{t}, \quad u_x(1,t) = e^{1+t}, \quad u_{xx}(0,t) = e^{t}, \quad u_{xxx}(0,t) = e^{1+t}, \quad t \in [0,1]. \end{cases}$$

This problem has the exact solution $u(x,t) = e^{x+t}$. We implement the proposed LDG method with the numerical fluxes (2.3e). We consider the case of uniform meshes having $N = 4, 8, 12, 16, 20$ elements and using the spaces P^k with $k = 1, 2$ and 3. The L^2-norm of the true errors e_u, e_q, e_p and e_r at final time $T = 1$ are presented in Table 1. This indicates that the order of convergence is $k+1$ for P^k spaces. Next, we present the zero-level curves of the true errors $e_u,\ e_q,\ e_p$, and e_r in Figures 1-3 at time $t = 1$ and for k ranging from 1 to 3. The Radau points of degree $k+1$ are shown on each element as $\times$ signs. We observe that the zero-level curves pass close to the superconvergence points marked by $\times$ and the roots of the true errors get closer to the roots of Radau polynomials with increasing k and N. The maximum errors at the superconvergence points as well as their order of convergence shown in Table 2 indicate that the LDG errors $e_u,\ e_q,\ e_p$, and e_r at time $t = 1$ are $\mathcal{O}(h^{k+2})$ superconvergent at Radau points. This example demonstrates that the local superconvergence results hold globally. On each element we apply the error estimation procedure (4.6) to compute error estimates for the LDG solution and its derivatives up to third order. The global effectivity indices at $t = 1$ shown in Table 3 indicate that, for smooth solutions, our *a posteriori* error estimates converge to the true errors under both h- and p-refinements. We repeated this experiment with all parameters kept unchanged except for the boundary conditions where we used

(1) $u(0,t) = e^{t}, \quad u_x(0,t) = e^{t}, \quad u(1,t) = e^{1+t}, \quad u_x(1,t) = e^{1+t}, \quad t \in [0,1]$,
(2) $u(0,t) = e^{t}, \quad u_{xx}(0,t) = e^{t}, \quad u(1,t) = e^{1+t}, \quad u_{xx}(1,t) = e^{1+t}, \quad t \in [0,1]$.

The exact solution for both cases is given by $u(x,t) = e^{x+t}$. Similar results have been observed. These results are not included to save space.

Example 5.2. *In this example we consider the following problem subject to the periodic boundary conditions*

$$\begin{cases} u_{tt} + u_{xxxx} = 2e^{t}\cos x, \quad x \in [0,2\pi],\ t \in [0,5], \\ u(x,0) = \cos(x), \quad u_t(x,0) = \cos(x), \quad x \in [0,2\pi], \end{cases}$$

The exact solution is given by $u(x,t) = e^{t}\cos x$. We solve this problem using the LDG method on uniform meshes having $N = 4, 8, 12, 16, 20$ elements and using the spaces P^k with $k = 1, 2$ and 3. Table 4 shows that the true errors e_u, e_q, e_p and e_r at $t = 5$ are $\mathcal{O}(h^{k+1})$ convergent in L^2 norm. We present the zero-level curves of the true errors $e_u,\ e_q,\ e_p$, and e_r in Figures 4-6 at time $t = 5$ and for k ranging from 1 to 3. The Radau points of degree $k+1$ are shown on each element as $\times$ signs. We observe that all errors vanish at points close to the Radau points for all solutions. The maximum errors at the superconvergence points as well as their order of convergence shown in Table 5 indicates that the LDG errors $e_u,\ e_q,\ e_p$, and e_r at time $t = 5$ are $\mathcal{O}(h^{k+2})$ superconvergent at Radau points. This is in full agreement with the theory. On each element we apply the error estimation procedure (4.6) to compute error estimates for the LDG solution and its derivatives up to third order. The results shown in Table 6 indicate that the global effectivity indices at $t = 5$ converge to unity under h-refinement.

TABLE 1. $||e_u||$, $||e_q||$, $||e_p||$ and $||e_r||$ errors at $t = 1$ and orders of convergence for Example 5.1 on uniform meshes having $N = 4,\ 8,\ 12,\ 16,\ 20$ elements using P^k, $k = 1$ to 3.

N	$k=1$		$k=2$		$k=3$	
	$\|\|e_u\|\|$	order	$\|\|e_u\|\|$	order	$\|\|e_u\|\|$	order
4	1.8113e-2		3.6721e-4		5.6546e-6	
8	4.5847e-3	1.9821	4.6154e-5	2.9921	3.5501e-7	3.9935
12	2.0440e-3	1.9923	1.3693e-5	2.9968	7.0196e-8	3.9975
16	1.1513e-3	1.9953	5.7801e-6	2.9980	2.2219e-8	3.9987
20	7.3736e-4	1.9968	2.9603e-6	2.9987	9.1030e-9	3.9990
	$\|\|e_q\|\|$	order	$\|\|e_q\|\|$	order	$\|\|e_q\|\|$	order
4	1.8611e-2		3.7030e-4		5.6800e-6	
8	4.6400e-3	2.0040	4.6346e-5	2.9982	3.5581e-7	3.9967
12	2.0596e-3	2.0031	1.3731e-5	3.0002	7.0300e-8	3.9994
16	1.1577e-3	2.0025	5.7922e-6	3.0003	2.2244e-8	3.9999
20	7.4063e-4	2.0018	2.9653e-6	3.0005	9.1102e-9	4.0005
	$\|\|e_p\|\|$	order	$\|\|e_p\|\|$	order	$\|\|e_p\|\|$	order
4	1.8425e-2		3.6783e-4		5.6595e-6	
8	4.6037e-3	2.0008	4.6174e-5	2.9939	3.5514e-7	3.9942
12	2.0477e-3	1.9981	1.3695e-5	2.9975	7.0199e-8	3.9983
16	1.1525e-3	1.9980	5.7807e-6	2.9981	2.2184e-8	4.0043
20	7.3785e-4	1.9985	2.9606e-6	2.9987	9.2522e-9	3.9190
	$\|\|e_r\|\|$	order	$\|\|e_r\|\|$	order	$\|\|e_r\|\|$	order
4	1.7364e-2		3.5928e-4		5.5682e-6	
8	4.5394e-3	1.9355	4.5744e-5	2.9735	3.4562e-7	4.0100
12	2.0287e-3	1.9864	1.3673e-5	2.9784	6.9158e-8	3.9681
16	1.1432e-3	1.9937	5.7653e-6	3.0018	2.1749e-8	4.0212
20	7.3413e-4	1.9848	2.9420e-6	3.0150	8.8850e-9	4.0118

6. Concluding remarks

In this paper, we investigated the superconvergence properties of the LDG method applied to the fourth-order initial-boundary value problems in one space dimension. We performed a local error analysis to show that the leading terms of the spatial discretization errors for the LDG solution and its spatial derivatives up to third order using k-degree polynomial approximations are proportional to $(k+1)$-degree Radau polynomials. As a consequence, the local discretization errors converge as $\mathcal{O}(h^{k+2})$ at the roots of Radau polynomials of degree $k+1$ on each element. These results are used to construct simple, efficient, and asymptotically exact *a posteriori* error estimates. These LDG error estimates are computationally simple and are obtained by solving a local steady problem with no boundary conditions on each element. Our *a posteriori* error estimates are tested on several problems to show their efficiency and accuracy under mesh refinement. Even though the analysis in this paper is restricted to the classical Euler-Bernoulli beam equation with constant coefficients, the same superconvergence results can be directly generalized to the fourth-order Euler-Bernoulli beam equation with variable coefficients. Superconvergence properties of the LDG method applied to two-dimensional problems on rectangular meshes are currently under investigation and will be reported in a future paper. The generalization to nonlinear equations and to two space dimensions on triangular meshes involve several technical difficulties. These will be investigated in the future.

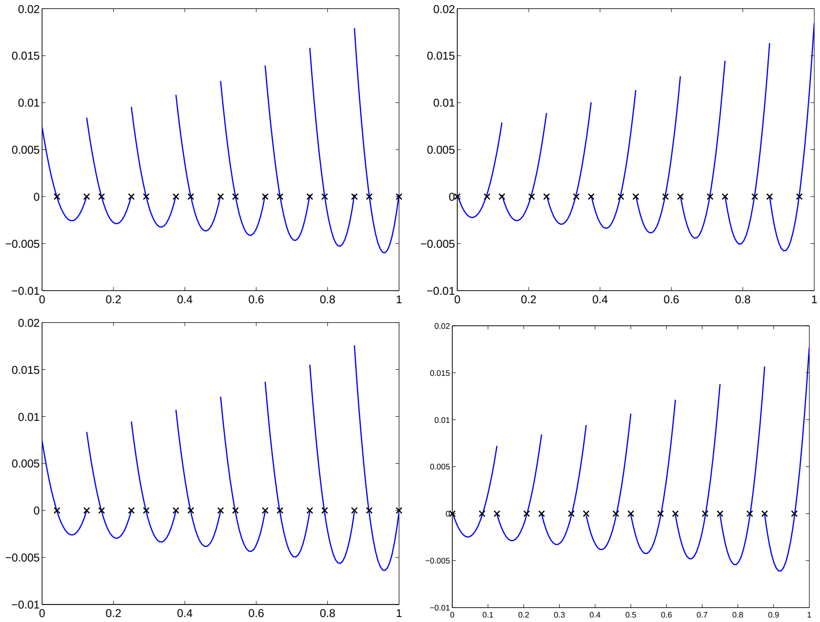

FIGURE 1. Zero-level curves of $e_u(\cdot, t = 1)$, $e_q(\cdot, t = 1)$, $e_p(\cdot, t = 1)$, $e_r(\cdot, t = 1)$ (from upper left to lower right) for Example 5.1 using P^k, $k = 1$ on uniform meshes having $N = 8$ elements.

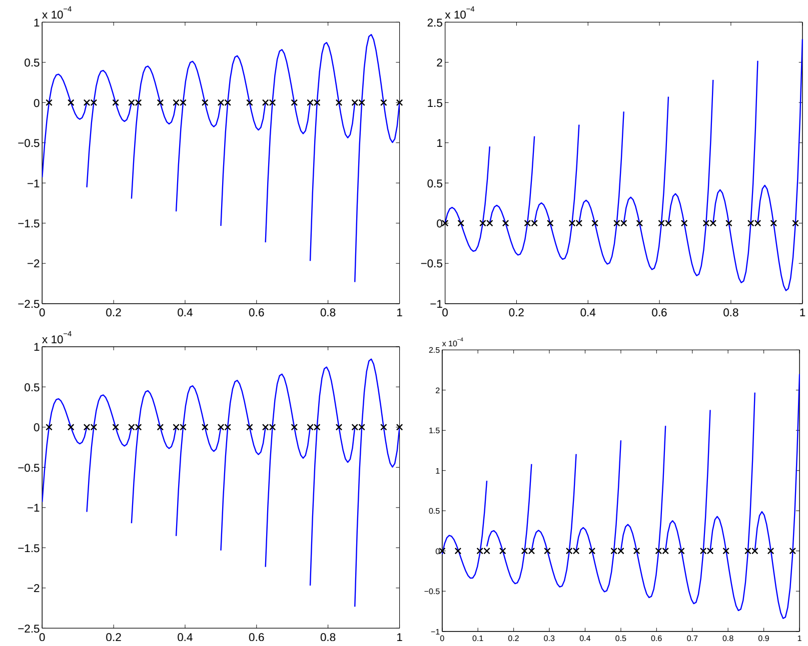

FIGURE 2. Zero-level curves of $e_u(\cdot, t = 1)$, $e_q(\cdot, t = 1)$, $e_p(\cdot, t = 1)$, $e_r(\cdot, t = 1)$ (from upper left to lower right) for Example 5.1 using P^k, $k = 2$ on uniform meshes having $N = 8$ elements.

References

[1] M. Abramowitz, I. A. Stegun, Handbook of Mathematical Functions, Dover, New York, 1965.

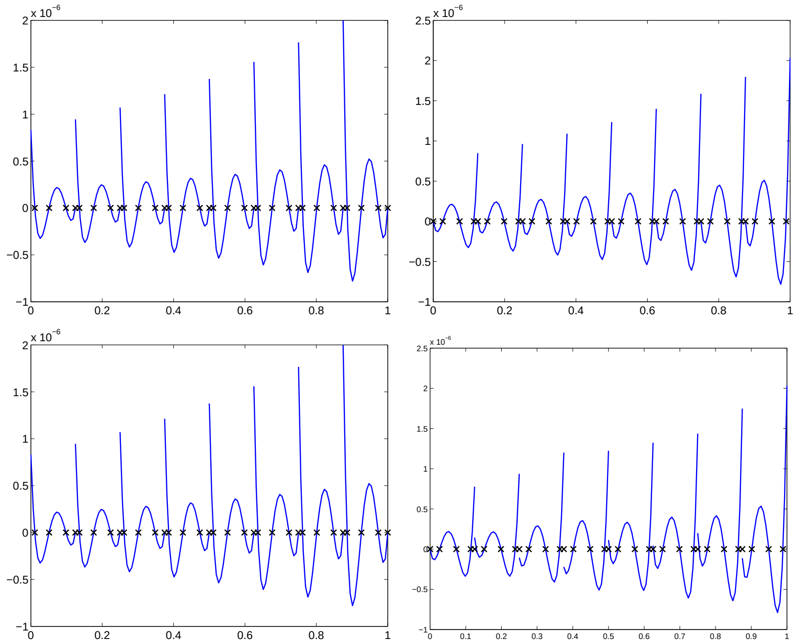

FIGURE 3. Zero-level curves of $e_u(\cdot, t = 1)$, $e_q(\cdot, t = 1)$, $e_p(\cdot, t = 1)$, $e_r(\cdot, t = 1)$ (from upper left to lower right) for Example 5.1 using P^k, $k = 3$ on uniform meshes having $N = 8$ elements.

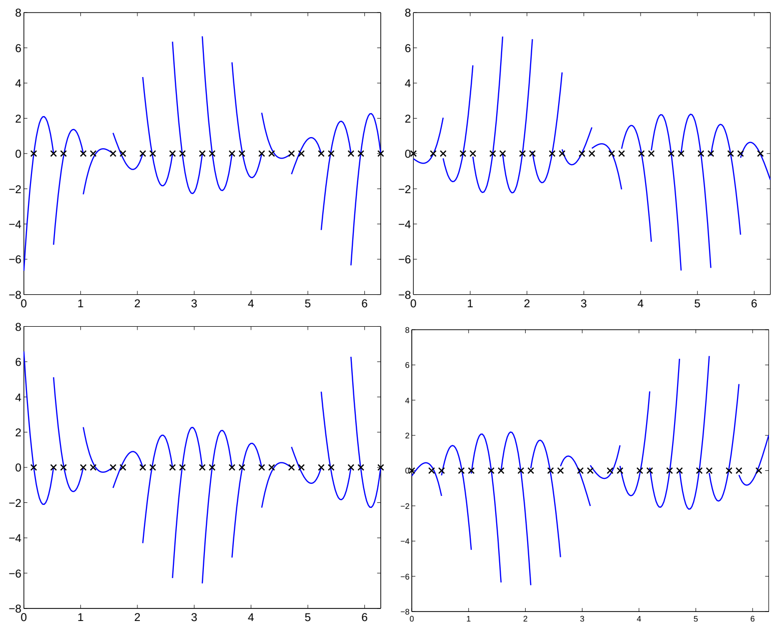

FIGURE 4. Zero-level curves of $e_u(\cdot, t = 5)$, $e_q(\cdot, t = 5)$, $e_p(\cdot, t = 5)$, $e_r(\cdot, t = 5)$ (from upper left to lower right) for Example 5.2 using P^k, $k = 1$ on uniform meshes having $N = 12$ elements.

[2] S. Adjerid, M. Baccouch, The discontinuous Galerkin method for two-dimensional hyperbolic problems. I: Superconvergence error analysis, Journal of Scientific Computing 33 (2007) 75–113.

TABLE 2. Maximum errors and orders of convergence of e_u, e_q, e_p and e_r at Radau points and $t = 1$ for Example 5.1 on uniform meshes having $N = 4,\ 8,\ 12,\ 16,\ 20$ elements using P^k, $k = 1-3$.

N	$k=1$		$k=2$		$k=3$	
	$\|\|e_u\|\|^*$	order	$\|\|e_u\|\|^*$	order	$\|\|e_u\|\|^*$	order
4	1.9715e-3		9.3203e-6		7.8796e-8	
8	2.5042e-4	2.9769	6.3148e-7	3.8836	2.5906e-9	4.9268
12	7.4586e-5	2.9872	1.2793e-7	3.9377	3.4660e-10	4.9609
16	3.1547e-5	2.9911	4.0979e-8	3.9572	8.2930e-11	4.9714
20	1.6177e-5	2.9931	1.6907e-8	3.9675	2.7304e-11	4.9787
	$\|\|e_q\|\|^*$	order	$\|\|e_q\|\|^*$	order	$\|\|e_q\|\|^*$	order
4	2.0226e-3		1.0128e-5		7.6217e-8	
8	2.6055e-4	2.9566	6.5687e-7	3.9466	2.5101e-9	4.9243
12	7.7907e-5	2.9775	1.3106e-7	3.9753	3.3897e-10	4.9379
16	3.3030e-5	2.9828	4.1730e-8	3.9781	8.1569e-11	4.9515
20	1.6958e-5	2.9877	1.7154e-8	3.9839	2.6974e-11	4.9590
	$\|\|e_p\|\|^*$	order	$\|\|e_p\|\|^*$	order	$\|\|e_p\|\|^*$	order
4	2.2667e-3		8.9319e-6		7.2285e-8	
8	3.1968e-4	2.8259	6.0879e-7	3.8750	2.2931e-9	4.9783
12	9.8371e-5	2.9067	1.2655e-7	3.8742	3.0420e-10	4.9819
16	4.2138e-5	2.9470	4.0313e-8	3.9765	7.2239e-11	4.9975
20	2.1887e-5	2.9356	1.6513e-8	3.9998	2.3756e-11	4.9840
	$\|\|e_r\|\|^*$	order	$\|\|e_r\|\|^*$	order	$\|\|e_r\|\|^*$	order
4	2.0359e-3		1.1301e-5		1.1618e-7	
8	2.6030e-4	2.9674	7.1385e-7	3.9847	3.6639e-9	4.9868
12	7.7720e-5	2.9811	1.4151e-7	3.9912	4.8364e-10	4.9941
16	3.2914e-5	2.9867	4.4861e-8	3.9933	1.1485e-10	4.9976
20	1.6891e-5	2.9896	1.8398e-8	3.9944	3.8549e-11	4.8924

TABLE 3. Global effectivity indices at $t = 1$ for Example 5.1 on uniform meshes having $N = 4,\ 8,\ 12,\ 16,\ 20$ elements using P^k, $k = 1$ to 3.

N	$k=1$		$k=2$		$k=3$	
	θ_u	θ_q	θ_u	θ_q	θ_u	θ_q
4	0.9738	1.0304	0.9794	1.0210	0.9845	1.0157
8	0.9857	1.0154	0.9896	1.0105	0.9922	1.0078
12	0.9902	1.0103	0.9931	1.0070	0.9948	1.0052
16	0.9925	1.0077	0.9948	1.0052	0.9961	1.0039
20	0.9940	1.0062	0.9958	1.0042	0.9969	1.0031
	θ_p	θ_r	θ_p	θ_r	θ_p	θ_r
4	0.9618	1.0408	0.9790	1.0457	0.9842	1.0267
8	0.9823	1.0265	0.9895	1.0184	0.9923	0.9885
12	0.9887	1.0173	0.9930	1.0110	0.9946	1.0539
16	0.9917	1.0124	0.9948	1.0088	0.9911	1.1962
20	0.9934	1.0104	0.9958	1.0065	0.9713	1.1550

[3] S. Adjerid, M. Baccouch, The discontinuous Galerkin method for two-dimensional hyperbolic problems. II: *A posteriori* error estimation, Journal of Scientifc Computing 38 (2009) 15–49.
[4] S. Adjerid, M. Baccouch, Asymptotically exact a posteriori error estimates for a one-dimensional linear hyperbolic problem, Applied Numerical Mathematics 60 (2010) 903–914.

TABLE 4. $||e_u||$, $||e_q||$, $||e_p||$ and $||e_r||$ errors at $t = 5$ and orders of convergence for Example 5.2 on uniform meshes having $N = 4,\ 8,\ 12,\ 16,\ 20$ elements using P^k, $k = 1$ to 3.

N	$k=1$		$k=2$		$k=3$	
	$\|\|e_u\|\|$	order	$\|\|e_u\|\|$	order	$\|\|e_u\|\|$	order
4	4.0976e+1		4.9826		4.7344e-1	
8	9.9604	2.0405	6.2229e-1	3.0012	2.9922e-2	3.9839
12	4.4054	2.0120	1.8431e-1	3.0010	5.9219e-3	3.9953
16	2.4740	2.0057	7.7746e-2	3.0004	1.8750e-3	3.9976
20	1.5821	2.0036	3.9803e-2	3.0003	7.6823e-4	3.9987
	$\|\|e_q\|\|$	order	$\|\|e_q\|\|$	order	$\|\|e_q\|\|$	order
4	4.2341e+1		4.9377		4.7298e-1	
8	1.0101e+1	2.0676	6.2193e-1	2.9890	2.9920e-2	3.9826
12	4.4352	2.0299	1.8429e-1	2.9998	5.9219e-3	3.9951
16	2.4836	2.0156	7.7743e-2	3.0002	1.8750e-3	3.9976
20	1.5861	2.0096	3.9803e-2	3.0002	7.6823e-4	3.9987
	$\|\|e_p\|\|$	order	$\|\|e_p\|\|$	order	$\|\|e_p\|\|$	order
4	3.6719e+1		4.5895		4.5778e-1	
8	9.7350	1.9153	6.1020e-1	2.9110	2.9669e-2	3.9476
12	4.3624	1.9797	1.8273e-1	2.9738	5.8996e-3	3.9836
16	2.4605	1.9906	7.7372e-2	2.9873	1.8710e-3	3.9919
20	1.5767	1.9944	3.9681e-2	2.9925	7.6719e-4	3.9952
	$\|\|e_r\|\|$	order	$\|\|e_r\|\|$	order	$\|\|e_r\|\|$	order
4	3.4230e+1		4.8173		4.6869e-1	
8	9.5236	1.8457	6.1690e-1	2.9651	2.9834e-2	3.9736
12	4.3188	1.9503	1.8353e-1	2.9900	5.9148e-3	3.9910
16	2.4469	1.9749	7.7589e-2	2.9927	1.8736e-3	3.9961
20	1.5710	1.9858	3.9746e-2	2.9977	7.6789e-4	3.9973

[5] S. Adjerid, M. Baccouch, A superconvergent local discontinuous Galerkin method for elliptic problems, Journal of Scientific Computing 52 (2012) 113–152.

[6] M. Baccouch, S. Adjerid, A Posteriori local discontinuous Galerkin error estimation for two-dimensional convection-diffusion problems. Journal of Scientific Computing (2014) 1-32.

[7] S. Adjerid, M. Baccouch, Adaptivity and error estimation for discontinuous galerkin methods, in: X. Feng, O. Karakashian, Y. Xing (eds.), Recent Developments in Discontinuous Galerkin Finite Element Methods for Partial Differential Equations, vol. 157 of The IMA Volumes in Mathematics and its Applications, Springer International Publishing, 2014, pp. 63–96.

[8] S. Adjerid, K. D. Devine, J. E. Flaherty, L. Krivodonova, A posteriori error estimation for discontinuous Galerkin solutions of hyperbolic problems, Computer Methods in Applied Mechanics and Engineering 191 (2002) 1097–1112.

[9] S. Adjerid, T. C. Massey, A posteriori discontinuous finite element error estimation for two-dimensional hyperbolic problems, Computer Methods in Applied Mechanics and Engineering 191 (2002) 5877–5897.

[10] S. Adjerid, T. C. Massey, Superconvergence of discontinuous finite element solutions for nonlinear hyperbolic problems, Computer Methods in Applied Mechanics and Engineering 195 (2006) 3331–3346.

[11] R. Agarwal, On the fourth-order boundary value problems arising in beam analysis, Differential Integral Equations 2 (1989) 91–110.

[12] R. Agarwal, M. Chow, Iterative methods for a fourth-order boundary value problem, Journal of Computational and Applied Mathematics 10 (1984) 203–207.

[13] M. Ainsworth, J. T. Oden, A posteriori Error Estimation in Finite Element Analysis, John Wiley, New York, 2000.

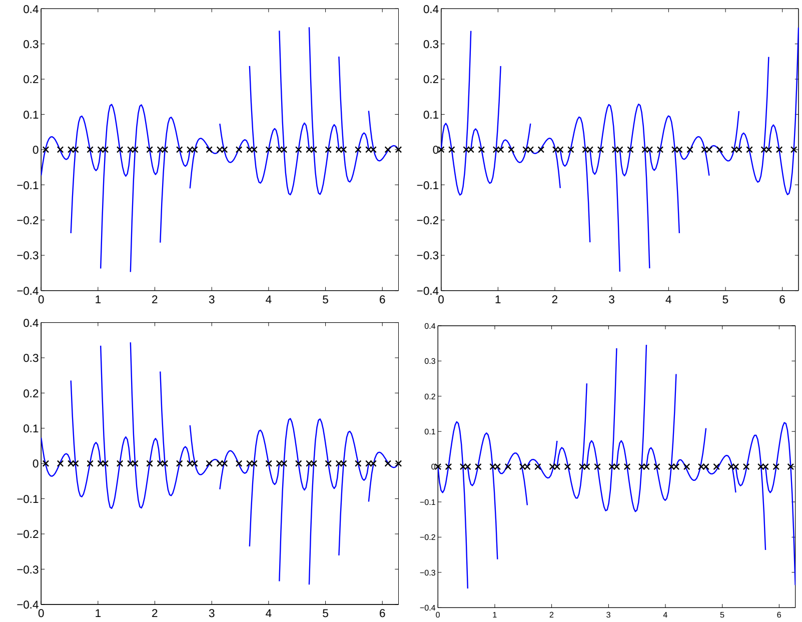

FIGURE 5. Zero-level curves of $e_u(\cdot, t=5)$, $e_q(\cdot, t=5)$, $e_p(\cdot, t=5)$, $e_r(\cdot, t=5)$ (from upper left to lower right) for Example 5.2 using P^k, $k=2$ on uniform meshes having $N=12$ elements.

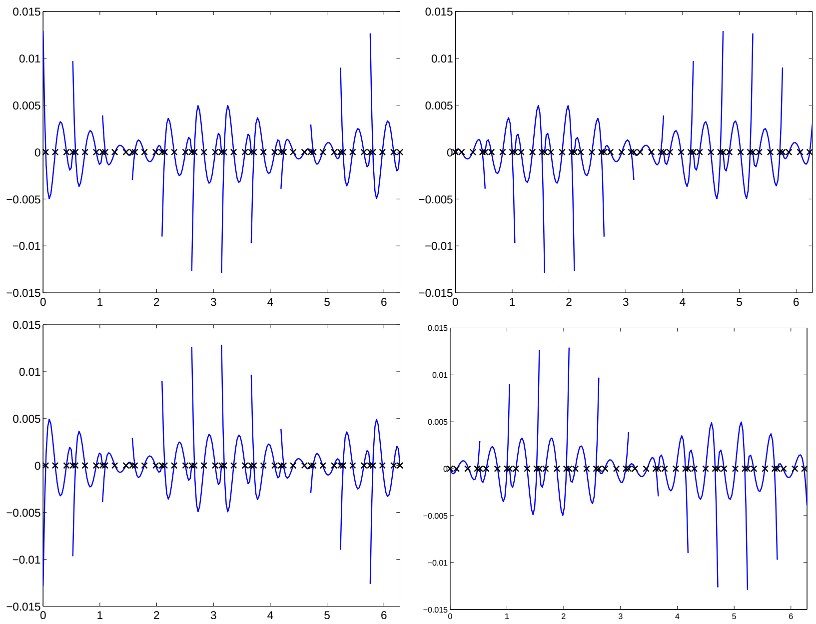

FIGURE 6. Zero-level curves of $e_u(\cdot, t=5)$, $e_q(\cdot, t=5)$, $e_p(\cdot, t=5)$, $e_r(\cdot, t=5)$ (from upper left to lower right) for Example 5.2 using P^k, $k=2$ on uniform meshes having $N=12$ elements.

[14] B. Attili, D. Lesnic, An efficient method for computing eigenelements of sturm-liouville fourth-order boundary value problems, Applied Mathematics and Computation 182 (2006) 1247–1254.

Table 5. Maximum errors and orders of convergence of e_u, e_q, e_p and e_r at Radau points and $t = 5$ for Example 5.2 on uniform meshes having $N = 4,\ 8,\ 12,\ 16,\ 20$ elements using P^k, $k = 1-3$.

N	$k=1$		$k=2$		$k=3$	
	$\|\|e_u\|\|^*$	order	$\|\|e_u\|\|^*$	order	$\|\|e_u\|\|^*$	order
4	6.0206		3.2547e-1		1.5898e-2	
8	8.5198e-1	2.8210	2.2443e-2	3.8582	5.4020e-4	4.8792
12	2.5840e-1	2.9424	4.3296e-3	4.0583	7.0849e-5	5.0100
16	1.0991e-1	2.9715	1.3537e-3	4.0414	1.6763e-5	5.0103
20	5.6485e-2	2.9832	5.5104e-4	4.0279	5.4838e-6	5.0074
	$\|\|e_q\|\|^*$	order	$\|\|e_q\|\|^*$	order	$\|\|e_q\|\|^*$	order
4	7.4938		2.6504e-1		1.1768e-2	
8	9.9104e-1	2.9187	2.0263e-2	3.7093	4.8209e-4	4.6094
12	2.9521e-1	2.9869	4.1152e-3	3.9316	6.7248e-5	4.8580
16	1.2473e-1	2.9947	1.3140e-3	3.9683	1.6273e-5	4.9321
20	6.3905e-2	2.9970	5.4046e-4	3.9813	5.3801e-6	4.9600
	$\|\|e_p\|\|^*$	order	$\|\|e_p\|\|^*$	order	$\|\|e_p\|\|^*$	order
4	6.7028		3.1762e-1		1.9194e-2	
8	8.7421e-1	2.9387	1.9592e-2	4.0190	5.3538e-4	5.1639
12	2.6134e-1	2.9781	4.0473e-3	3.8895	7.0486e-5	5.0006
16	1.1062e-1	2.9884	1.3038e-3	3.9376	1.6696e-5	5.0064
20	5.6718e-2	2.9936	5.3744e-4	3.9715	5.4725e-6	4.9987
	$\|\|e_r\|\|^*$	order	$\|\|e_r\|\|^*$	order	$\|\|e_r\|\|^*$	order
4	8.9681		8.3742e-1		5.3033e-2	
8	1.0251	3.1290	6.0824e-2	3.7832	2.0130e-3	4.7195
12	2.9913e-1	3.0377	1.2332e-2	3.9357	2.7368e-4	4.9213
16	1.2556e-1	3.0175	3.9417e-3	3.9647	6.5725e-5	4.9585
20	6.4129e-2	3.0110	1.6209e-3	3.9823	2.1657e-5	4.9751

Table 6. Global effectivity indices at $t = 5$ for Example 5.2 on uniform meshes having $N = 4,\ 8,\ 12,\ 16,\ 20$ elements using P^k, $k = 1$ to 3.

N	$k=1$		$k=2$		$k=3$	
	θ_u	θ_q	θ_u	θ_q	θ_u	θ_q
4	0.9475	0.9090	0.9707	0.9788	0.9886	0.9894
8	0.9905	0.9761	0.9930	0.9935	0.9971	0.9971
12	0.9961	0.9893	0.9969	0.9970	0.9987	0.9987
16	0.9979	0.9939	0.9983	0.9983	0.9993	0.9993
20	0.9987	0.9961	0.9989	0.9989	0.9995	0.9995
	θ_p	θ_r	θ_p	θ_r	θ_p	θ_r
4	0.9289	1.0575	1.0040	0.9823	0.9972	0.9897
8	0.9796	1.0137	1.0002	0.9957	0.9993	0.9976
12	0.9908	1.0060	1.0001	0.9983	0.9997	0.9990
16	0.9948	1.0034	1.0000	0.9989	0.9998	0.9994
20	0.9966	1.0022	1.0000	0.9994	0.9999	0.9995

[15] I. Babuška, T. Strouboulis, The Finite Element Method and Its Reliability, Numerical mathematics and scientific computation, Clarendon Press, 2001.

[16] M. Baccouch, A local discontinuous Galerkin method for the second-order wave equation, Computer Methods in Applied Mechanics and Engineering 209–212 (2012) 129–143.

[17] M. Baccouch, Asymptotically exact local discontinuous Galerkin error estimates for the linearized Korteweg-de Vries equation in one space dimension, International Journal of Numerical Analysis and Modeling, under review, 2014

[18] M. Baccouch, The local discontinuous Galerkin method for the fourth-order Euler-Bernoulli partial differential equation in one space dimension. Part I: Superconvergence error analysis, Journal of Scientific Computing 59 (2014) 795-840.

[19] M. Baccouch, The local discontinuous Galerkin method for the fourth-order Euler-Bernoulli partial differential equation in one space dimension. Part II: *A posteriori* error estimation, Journal of Scientific Computing 60 (2014) 1-34.

[20] M. Baccouch, Superconvergence of the local discontinuous Galerkin method applied to the one-dimensional second-order wave equation, Numerical methods of partial differential equations 30 (2014) 862-901.

[21] M. Baccouch, Asymptotically exact a posteriori LDG error estimates for one-dimensional transient convection-diffusion problems, Applied Mathematic and Computation 226 (2014) 455 – 483.

[22] M. Baccouch, Global convergence of a posteriori error estimates for a discontinuous Galerkin method for one-dimensional linear hyperbolic problems, International Journal of Numerical Analysis and Modeling 11 (2014) 172–193.

[23] M. Baccouch, Superconvergence and a posteriori error estimates for the local discontinuous Galerkin method for convection-diffusion problems in one space dimension. Computers and Mathematics with applications 67 (2014) 1130-1153.

[24] M. Baccouch, S. Adjerid, Discontinuous Galerkin error estimation for hyperbolic problems on unstructured triangular meshes, Computer Methods in Applied Mechanics and Engineering 200 (2010) 162–177.

[25] P. Castillo, A superconvergence result for discontinuous Galerkin methods applied to elliptic problems, Computer Methods in Applied Mechanics and Engineering 192 (2003) 4675–4685.

[26] P. Castillo, B. Cockburn, I. Perugia, D. Schötzau, An a priori error analysis of the local discontinuous Galerkin method for elliptic problems, SIAM Journal on Numerical Analysis 38 (2000) 1676–1706.

[27] F. Celiker, B. Cockburn, Superconvergence of the numerical traces for discontinuous Galerkin and hybridized methods for convection-diffusion problems in one space dimension, Mathematics of Computation 76 (2007) 67–96.

[28] Y. Cheng, C.-W. Shu, Superconvergence of discontinuous Galerkin and local discontinuous Galerkin schemes for linear hyperbolic and convection-diffusion equations in one space dimension, SIAM Journal on Numerical Analysis 47 (2010) 4044–4072.

[29] B. Cockburn, A simple introduction to error estimation for nonlinear hyperbolic conservation laws, in: Proceedings of the 1998 EPSRC Summer School in Numerical Analysis, SSCM, volume 26 of the Graduate Student's Guide for Numerical Analysis, pages 1-46, Springer, Berlin, 1999.

[30] B. Cockburn, P. A. Gremaud, Error estimates for finite element methods for nonlinear conservation laws, SIAM Journal on Numerical Analysis 33 (1996) 522–554.

[31] B. Cockburn, G. Kanschat, I. Perugia, D. Schötzau, Superconvergence of the local discontinuous Galerkin method for elliptic problems on cartesian grids, SIAM Journal on Numerical Analysis 39 (2001) 264–285.

[32] B. Cockburn, G. E. Karniadakis, C. W. Shu, Discontinuous Galerkin Methods Theory, Computation and Applications, Lecture Notes in Computational Science and Engineering, vol. 11, Springer, Berlin, 2000.

[33] B. Cockburn, C. W. Shu, The local discontinuous Galerkin method for time-dependent convection-diffusion systems, sinum 35 (1998) 2440–2463.

[34] M. Delfour, W. Hager, F. Trochu, Discontinuous Galerkin methods for ordinary differential equation, Mathematics of Computation 154 (1981) 455–473.

[35] L. Greenberg, M. Marletta, Oscillation theory and numerical solution of fourth-order sturm-liouville problem, IMA Journal of Numerical Analysis 15 (1995) 319–356.

[36] L. Greenberg, M. Marletta, Algorithm 775: the code sleuth for solving fourth order sturm-liouville problems, ACM Transactions on Mathematical Software 23 (1997) 453–493.

[37] C. Gupta, Existence and uniqueness theorems for a bending of an elastic beam equation at resonance, Journal of Mathematical Analysis and Applications 135 (1988) 208–225.

[38] R. Hartmann, P. Houston, Adaptive discontinuous Galerkin finite element methods for nonlinear hyperbolic conservations laws, SIAM Journal on Scientific Computing 24 (2002) 979–1004.

[39] P. Houston, D. Schötzau, T. Wihler, Energy norm a posteriori error estimation of *hp*-adaptive discontinuous Galerkin methods for elliptic problems, Mathematical Models and Methods in Applied Sciences 17 (2007) 33–62.

[40] P. Lesaint, P. Raviart, On a finite element method for solving the neutron transport equations, in: C. de Boor (ed.), Mathematical Aspects of Finite Elements in Partial Differential Equations, Academic Press, New York, 1974.

[41] L. Liu, X. Zhang, Y. Wu, Positive solutions of fourth-order nonlinear singular sturm-liouville eigenvalue problems, Journal of Mathematical Analysis and Applications 326 (2007) 1212–1224.

[42] R. Ma, H. Wang, On the existence of positive solutions of fourth-order ordinary differential equation, Applicable Analysis 59 (1995) 225–231.

[43] X. Meng, C.-W. Shu, B. Wu, Superconvergence of the local discontinuous Galerkin method for linear fourth order time dependent problems in one space dimension, IMA Journal of Numerical Analysis 32 (2012) 1294–1328.

[44] W. H. Reed, T. R. Hill, Triangular mesh methods for the neutron transport equation, Tech. Rep. LA-UR-73-479, Los Alamos Scientific Laboratory, Los Alamos (1973).

[45] B. Rivière, M. Wheeler, A posteriori error estimates for a discontinuous Galerkin method applied to elliptic problems, Computational and Applied Mathematics 46 (2003) 143–163.

[46] C.-W. Shu, Discontinuous galerkin method for time-dependent problems: Survey and recent developments, in: X. Feng, O. Karakashian, Y. Xing (eds.), Recent Developments in Discontinuous Galerkin Finite Element Methods for Partial Differential Equations, vol. 157 of The IMA Volumes in Mathematics and its Applications, Springer International Publishing, 2014, pp. 25–62.

[47] M. Syam, H. Siyyam, An efficient technique for finding the eigenvalues of fourth-order sturm-liouville problems, Chaos, Solitons and Fractals 30 (2009) 659–665.

[48] Y. Xu, C.-W. Shu, Local discontinuous Galerkin methods for two classes of two dimensional nonlinear wave equations, Physica D 208 (2005) 21–58.

[49] Y. Xu, C.-W. Shu, Local discontinuous Galerkin methods for high-order timedependent partial differential equations, Communications in Computational Physics 7 (2010) 1–46.

[50] J. Yan, C.-W. Shu, A local discontinuous Galerkin method for KdV type equations, SIAM Journal on Numerical Analysis 40 (2002) 769–791.

[51] J. Yan, C.-W. Shu, Local discontinuous Galerkin methods for partial differential equations with higher order derivatives, Journal of Scientific Computing 17 (2002) 27–47.

[52] Y. Yang, C.-W. Shu, Analysis of optimal superconvergence of discontinuous Galerkin method for linear hyperbolic equations, SIAM Journal on Numerical Analysis 50 (2012) 3110–3133.

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