1-21-2009

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Peter Hyland

Brent Follin

Emory F. Bunn
University of Richmond, ebunn@richmond.edu

Recommended Citation

Hyland, Peter, Brent Follin, and Emory F. Bunn. "Phase Shift Sequences for an Sdding Interferometer." *Monthly Notices of the Royal Astronomical Society* 393, no. 2 (January 21, 2009): 531-37. doi:10.1111/j.1365-2966.2008.14194.x.

Mon. Not. R. Astron. Soc. **393,** 531–537 (2009) doi:10.1111/j.1365-2966.2008.14194.x

Phase shift sequences for an adding interferometer

Peter Hyland,[1★†] Brent Follin[2] and Emory F. Bunn[2]
[1]*Physics Department, University of Wisconsin - Madison, Madison, WI 53706, USA*
[2]*Physics Department, University of Richmond, Richmond, VA 23173, USA*

Accepted 2008 October 28. Received 2008 October 28; in original form 2008 August 25

ABSTRACT
Cosmic microwave background (CMB) polarimetry has the potential to provide revolutionary advances in cosmology. Future experiments to detect the very weak B-mode signal in CMB polarization maps will require unprecedented sensitivity and control of systematic errors. Bolometric interferometry may provide a way to achieve these goals. In a bolometric interferometer (or other adding interferometer), phase shift sequences are applied to the inputs in order to recover the visibilities. Noise is minimized when the phase shift sequences corresponding to all visibilities are orthogonal. We present a systematic method for finding sequences that produce this orthogonality, approximately minimizing both the length of the time sequence and the number of discrete phase shift values required. When some baselines are geometrically equivalent, we can choose sequences that read out those baselines simultaneously, which has been shown to improve the signal-to-noise ratio.

Key words: techniques: interferometric – techniques: polarimetric – cosmic microwave background.

1 INTRODUCTION

The field of observational cosmology has been advancing quickly in recent years. Observations of the cosmic microwave background (CMB) radiation have been leading the way, as evidenced by the *Wilkinson Microwave Anisotropy Probe* (*WMAP*)'s highly successful mapping of CMB anisotropy (Hinshaw et al. 2008), the Degree Angular Scale Interferometer's (DASI's) detection of the polarized component of the CMB (Leitch et al. 2005) and the 2006 Nobel Prize in Physics awarded to John Mather and George Smoot. Momentum is building for experiments that characterize the CMB polarization in detail (Bock et al. 2006).

A linear polarization map can always be expressed as the sum of two component maps, denoted as E and B (Kamionkowski, Kosowsky & Stebbins 1997; Seljak & Zaldarriaga 1997). CMB experiments to date have detected only the 'curl-free' E component, which is produced primarily by (scalar) density perturbations. The 'divergence-free' B component is not produced by scalar perturbations at linear order, and is therefore a clean probe of other, smaller effects. In particular, inflationary models predict a B-mode signal produced by gravitational wave (tensor) perturbations in the early Universe. These B modes promise to hold key information about the process of inflation and particle physics above the Grand Unification scale. The challenge of finding the B modes is no small task, however: the B component is expected to be at least an order of magnitude weaker than the E component (which is itself small compared to the temperature anisotropy) over all angular scales. Experiments to search for B modes will require unprecedented sensitivity and control of systematic errors.

Bolometric interferometry is one proposed method for achieving these goals (Korotkov et al. 2006; Timbie et al. 2006; Polenta et al. 2007; Charlassier 2008; Tucker et al. 2008). A bolometric interferometer is a marriage between highly sensitive, incoherent bolometric detectors and the phase-sensitive, systematic-error-reducing observing technique of interferometry. Hamilton et al. (2008) have shown that a bolometric interferometer can achieve sensitivities comparable to traditional technologies. The question of whether bolometric interferometry is useful for CMB polarimetry will thus depend on the method's ability to control systematic errors. Systematic errors in interferometers are certainly different from those in imaging experiments (Bunn 2007); it can be argued that interferometers are superior in this regard, although this question requires further research.

In a bolometric interferometer, the signals from a set of input feedhorns are combined with either a Butler combiner or a quasi-optical (Fizeau) combiner. In either case, bolometers measure the total power in the combined beam – that is, each bolometer is illuminated by signals from all of the input horns. Since the signal in each detector is the sum of all the inputs, a bolometric interferometer is an example of an 'adding' interferometer (as opposed to traditional radio interferometers, which are 'multiplying' interferometers). One of the keys to making this method work is to

★Present address: Physics Department, McGill University, Montréal, QC, Canada H3A 2T8.
†E-mail: peter.o.hyland@gmail.com

arrange for the phase information to be encoded in the bolometer signals, so that individual pairwise visibilities can be extracted. To achieve this goal, a sequence of phase shifts can be applied to each of the input horns in such a way that each visibility is phase shifted in an independent fashion. The resulting time series can be solved for the individual visibilities. The phase shift sequences should be chosen so that this inversion can be done with minimal noise.

We would like the length of the phase shift sequence to be as short as possible, to avoid error due to $1/f$ noise in the detectors. Clearly, the number of phase shifts must be at least as large as the number of visibilities to be recovered. In the most general case, an N-horn interferometer has $N(N-1)/2$ distinct visibilities, requiring long phase shift sequences for interferometers with many inputs. On the other hand, if the input horns are arranged in a regular pattern, such as a square array, then many antenna pairs correspond to identical visibilities. These can be given identical phase shifts and read out together. This coherent treatment of equivalent (or redundant) baselines has two advantages. First, it allows for shorter phase shift sequences. Secondly, by co-adding equivalent signals, the signal-to-noise ratio is improved (Charlassier et al. 2008, hereafter C08).

C08 gave an excellent overview of how a bolometric interferometer works and considered the choice of phase shift sequences in detail. For the case of a square array of horns, the paper presented a method of phase modulating the inputs that gives equivalent baselines identical phase shift sequences. In this paper, we present independently developed work on phase modulation and coherent addition of baselines that complements the methods of C08. We consider general horn arrangements as well as a regular square lattice. In the general case, we present a method for finding the optimal phase shift sequence assuming no baselines are redundant. In the case of a square array, we present a refinement of the method of C08. Unlike the original method, which achieves optimal noise performance only in the limit as the number of time-steps tends to infinity, our method is optimal for sequences of nearly or exactly the minimum possible length.

In Section 2, we present our formalism for denoting sequences of phase shifts and consider the criterion for an optimal phase shift sequence. Section 3 introduces a shorthand notation and applies this to a method for constructing bases for phase sequences and selecting optimal sequences for interferometers without equivalent baselines. In Section 4, we consider a square array of horns, accounting for redundant baselines. For simplicity, we consider only one linear polarization state in these sections, but in Section 5 we generalize to an array with two polarizations. Section 6 contains a brief concluding discussion, and Appendix A contains a useful mathematical result.

2 FORMALISM

Suppose that our interferometer has N input horns, each of which receives one electric field component. (We will consider the case where both x and y components are received in Section 5.) Assuming monochromatic radiation of frequency ν for simplicity, the signal entering the jth horn can be written as $E_j e^{2\pi i\nu t}$. This signal is presumed to have already been averaged over the antenna pattern, so no integral over the sky position will be explicitly shown in the equations below. We apply a time-dependent phase shift $\phi_j(t)$ to each of these inputs. Since bolometers measure the total power, the signal detected by the mth detector is

$$S_m(t) \propto \left| \sum_{j=1}^{N} E_j\, e^{i(\Delta_{jm}+\phi_j(t))} \right|^2 = \sum_{j,k=1}^{n} E_j E_k^* e^{i(\Delta_{jm}-\Delta_{km}+\phi_j(t)-\phi_k(t))}, \tag{1}$$

where the phase shifts Δ_{jm} are fixed by the geometry of the system and do not vary in time. In this expression, the detector can correspond either to a single point on the focal plane of a quasi-optical combiner or to a single output of a Butler combiner.

We are assuming here that all inputs contribute to all detectors with equal amplitude. If this assumption is relaxed, then an additional real factor A_{jm} would need to be included in each term of the sum. The presence of these factors would affect the overall sensitivity of the detector to the various visibilities, but we do not expect it to influence the optimal choice of phase shifts, so we omit it.

Let us assume for the moment that we wish to recover the visibility associated with each pair of horns separately; we will return below to the case in which redundant baselines are coherently added before detection. We wish to choose the phase shifts $\phi_j(t)$ to enable the recovery of all of the cross-terms $E_j E_k^*$ from each detector. In principle, we could aim for a weaker goal, namely to ensure that each cross-term be recoverable from the full set of detector outputs $S_1(t), S_2(t), \ldots$; however, to avoid systematic errors resulting from subtracting signals in different detectors, it is preferable to insist that each visibility be recovered from each detector separately.

Since we are focusing on one detector at a time, we suppress the subscript m in equation (1). Furthermore, the time-independent phase shifts Δ_{jm} do not affect the problem of visibility recovery, so we suppress these as well. Finally, we assume that the phase shifts ϕ_j are changed in discrete time-steps, so we replace the functions $\phi_j(t)$ with sequences ϕ_{jt}. Here $t = 1, 2, \ldots, M$, where M is the number of steps in the phase shift sequence at each detector. With these changes, equation (1) becomes

$$S_t \propto \sum_{j,k=1}^{N} E_j E_k^*\, e^{i(\phi_{jt}-\phi_{kt})} \equiv V_{jk}\, e^{i(\phi_{jt}-\phi_{kt})}. \tag{2}$$

Finally, we assume that the phase shifts can take on P equally spaced values from 0 to 2π:

$$\phi_{jt} \in \{2\pi p/P \mid p = 0, 1, 2, \ldots, P-1\}. \tag{3}$$

Given the time sequence of measurements $S_1, S_2, \ldots, S_M$, we wish to recover all $N(N-1)$ complex visibilities $V_{jk} \equiv E_j E_k^*$ with $j \neq k$ (or equivalently to recover both real and imaginary parts of all pairs with $j < k$). In addition, we will always recover the total power term $\sum_j V_{jj} = \sum_j |E_j|^2$, which enters each S_t equally. Solving for the cross-terms is therefore simply inverting a linear system of M equations for $N(N-1)+1$ unknowns. Generically, we expect this to be possible as long as

$$M \geq N(N-1)+1 \equiv N_{\mathrm{vis}}. \tag{4}$$

We want to insist not just that the visibilities be recovered, but that they be recovered with minimal possible noise. To be specific, the recovery problem we wish to solve is

$$\boldsymbol{S} = \mathbf{A}\boldsymbol{V}, \tag{5}$$

where $\boldsymbol{S}$ is the M-dimensional signal vector, $\boldsymbol{V}$ is the N_{vis}-dimensional vector of visibilities to be recovered and $\mathbf{A}$ is a matrix whose elements are determined by the phase shifts:

$$A_{tm} = e^{i(\phi_{jt}-\phi_{kt})}, \tag{6}$$

where the mth visibility V_m corresponds to the horn pair jk. In this situation, where all elements of the matrix $\mathbf{A}$ have absolute value equal to one, the minimum possible noise contributions to the visibilities is achieved when all columns of A are orthogonal:

$$\sum_t A^*_{tm} A_{tm'} = 0 \qquad \text{for } m \neq m' \tag{7}$$

or equivalently for $\mathbf{A}^\dagger \mathbf{A}$ proportional to the identity matrix.

A proof of this statement is provided in Appendix A. Intuitively, it says that the visibilities are recovered with minimum noise when they are maximally independent of each other, that is, when they contribute orthogonally to the time series of signals at the detector.

Let us summarize. For any pair of horns j, k, define an M-dimensional vector

$$\mathbf{\Phi}_{jk} = \left(\mathrm{e}^{\mathrm{i}(\phi_{j1}-\phi_{k1})}, \mathrm{e}^{\mathrm{i}(\phi_{j2}-\phi_{k2})}, \ldots, \mathrm{e}^{\mathrm{i}(\phi_{jM}-\phi_{kM})}\right). \tag{8}$$

Our goal is to choose the set of phase shifts ϕ_{jt} such that the vectors $\mathbf{\Phi}_{jk}$ and $\mathbf{\Phi}_{j'k'}$ are orthogonal whenever $(jk) \neq (j'k')$. When this condition is satisfied, each visibility is recovered simply by taking the dot product of the detector signal with the corresponding vector $\mathbf{\Phi}$: the estimator of V_{jk} is

$$\hat{V}_{jk} = \frac{1}{M}\mathbf{\Phi}^\dagger_{jk}\boldsymbol{S} = \frac{1}{M}\sum_{t=1}^{M} e^{-i(\phi_{jt}-\phi_{kt})} S_t. \tag{9}$$

We will call the vector $\mathbf{\Phi}_{jk}$ the 'mask' for the baseline jk.

Note that $\mathbf{\Phi}_{jk}$ and $\mathbf{\Phi}_{kj}$ are complex conjugates of each other. The requirement that these be orthogonal, which means roughly that the elements of Φ_{jk} uniformly sample directions in the complex plane, is necessary for both the real and imaginary parts of V_{jk} to be recovered with minimum noise.

It may be instructive to compare the phase shift schemes for the bolometric interferometer with those applied in a traditional multiplying interferometer. In traditional interferometry, orthogonal patterns of square-wave phase shifts (e.g. Walsh functions) are applied to each of the input antennas in order to reduce the response of the instrument to spurious signals (e.g. Thompson, Moran & Swenson 2001). The phase shift patterns we require in the adding interferometer must obey a more stringent orthogonality requirement: rather than merely demanding orthogonality of all of the input phase shifts (i.e. demanding that the $\boldsymbol{\phi}_j$ be orthogonal), we require that the phase shifts associated with all *visibilities* (i.e. all $\mathbf{\Phi}_{jk}$) be orthogonal.

3 METHOD FOR FINDING PHASE SHIFTS

Let us suppose that the number N of horns is fixed, as is the number P of possible phase shift values. We wish to find the shortest sequence of time-steps (i.e. the minimum M) that satisfies our orthogonality criterion. Alternatively, given M, P, we can ask for the maximum number of horns that can be accommodated.

We will introduce the following shorthand notation for the possible phase factors:

$$[p] = \mathrm{e}^{\mathrm{i}2\pi p/P}, \qquad p = 0, 1, 2, \ldots, P-1. \tag{10}$$

For purposes of illustration, we will consider the case $P = 4$ in this section, so that the four possible phase shift values are

$$[0123] = (1, \mathrm{i}, -1, -\mathrm{i}). \tag{11}$$

The method we outline generalizes to other values of P.

We will present a systematic procedure for searching for optimal phase shift patterns below, but first let us motivate the procedure by considering ways of assigning phase shift patterns to the first few horns in our instrument. Suppose that no phase shift at all is applied to the first horn. We denote this by an M-dimensional vector of all zeroes: $\boldsymbol{\phi}_1 = [000\ldots000]$. Since we can always subtract any one phase shift pattern from all horns without changing the relative phase shifts, there is no loss of generality in this supposition. Then the second horn should undergo a pattern of phase shifts that equally samples the four phase factors. Again, without loss of generality, we can assume that it steps as slowly as possible through these factors. That is, the second phase shift pattern consists of $M/4$ repetitons of 0, followed by $M/4$ repetitions of 1, etc. If we then wish to add a third horn, a natural choice would be to modulate its phase shifts four times as fast as the previous horn. Specifically, for three horns, we can set $M = 16$ and choose phase shift sequences $\boldsymbol{\phi}_1 = [0000000000000000]$, $\boldsymbol{\phi}_2 = [0000111122223333]$, $\boldsymbol{\phi}_3 = [0123012301230123]$. It seems obvious and is straightforward to check that this choice does lead to orthogonal masks for all six baselines of this interferometer.

In principle, we can proceed in this way indefinitely, but such a procedure is far from optimal, since each new horn requires multiplying the phase shift sequence length M by 4. We can do much better than this, though. In particular, we can add a fourth horn to the set described above without increasing M at all. One way to do this is to set $\boldsymbol{\phi}_4 = \boldsymbol{\phi}_2\boldsymbol{\phi}_3 = [0123123023013012]$. Here multiplication of vectors is performed elementwise, and since $[j]$ is shorthand for $\mathrm{e}^{\mathrm{i}j\pi/2}$, multiplication corresponds to addition modulo 4 of the quantities in square brackets. In fact, it is possible to generalize this procedure of obtaining new phase shift patterns by starting with a small set of orthogonal modulations at various speeds, and multiplying them together in various combinations. We now describe this procedure systematically.

Let the number of time-steps M be a power of 4: $M = 4^\mu$ for some positive integer μ. We can define a set of μ mutually orthogonal M-dimensional vectors as follows: the vector $\boldsymbol{\alpha}_1$ is obtained by stepping through the four possible phase values as slowly as possible – that is, it consists of $M/4$ repetitions of [0], followed by $M/4$ repetitions of [1], etc. Each subsequent vector $\boldsymbol{\alpha}_j$ cycles through the possible phases four times faster until the last one $\boldsymbol{\alpha}_\mu$, which consists of $M/4$ repetitions of the sequence [0123]. To be explicit, the vectors $\boldsymbol{\phi}_2$, $\boldsymbol{\phi}_3$ in the example above are the vectors corresponding to the case $\mu = 2$, and here is the case $\mu = 3$:

$$\begin{aligned}\boldsymbol{\alpha}_1 = [&00000000000000001111111111111111\\ &22222222222222223333333333333333]\end{aligned} \tag{12}$$

$$\begin{aligned}\boldsymbol{\alpha}_2 = [&00001111222233330000111122223333\\ &00001111222233330000111122223333]\end{aligned} \tag{13}$$

$$\begin{aligned}\boldsymbol{\alpha}_3 = [&01230123012301230123012301230123\\ &01230123012301230123012301230123].\end{aligned} \tag{14}$$

Here is an alternative description of the construction of these vectors: the kth element of the vector $\boldsymbol{\alpha}_j$ is the jth most significant digit in the base-4 expression for $(k-1)$. In the general case with $M = P^\mu$, $\boldsymbol{\alpha}_1$ steps through the P values as slowly as possible and each subsequent $\boldsymbol{\alpha}_j$ cycles P times faster.

We now define

$$\langle j_\mu, \ldots, j_2, j_1\rangle = \boldsymbol{\alpha}_\mu^{j_\mu}\cdots\boldsymbol{\alpha}_2^{j_2}\boldsymbol{\alpha}_1^{j_1} \tag{15}$$

for integers $j_\mu, \ldots j_2, j_1$ between 0 and 3. As noted above, multiplication and exponentiation are performed elementwise in each vector, and multiplication corresponds to addition modulo 4 on the values in square brackets. For instance, in the case $\mu = 2$,

$$\begin{aligned}\langle 2, 1\rangle = \boldsymbol{\alpha}_2^2\boldsymbol{\alpha}_1 &= [0123012301230123]^2[0000111122223333]\\ &= [0202020202020202][0000111122223333]\\ &= [0202131320203131]. \qquad (16)\end{aligned}$$

It is straightforward to check that the vectors $\langle j_\mu, \ldots, j_2, j_1\rangle$ are all mutually orthogonal. Since there are 4^μ distinct vectors, they are a maximal set of orthogonal vectors. We can therefore search among this set for the optimal set of N phase shift patterns to apply to our input horns.

As an example, consider the case $\mu = 2$, that is, let the number of time-steps be $M = 4^2 = 16$. We will determine the maximum value of N that can be accommodated. We proceed by assigning phase shift sequences to the horns one at a time. Without loss of generality, we can assume that the first horn has no phase shift at all (since any phase shift sequence can be subtracted from all inputs without altering the solution):

$$\boldsymbol{\phi}_0 = \langle 0, 0\rangle = \boldsymbol{\alpha}_2^0\boldsymbol{\alpha}_1^0 = \boldsymbol{\alpha}_0 = [0000000000000000]. \qquad (17)$$

Here $\boldsymbol{\phi}_j$ refers to the M-dimensional vector $(\mathrm{e}^{\mathrm{i}\phi_{j1}}, \mathrm{e}^{\mathrm{i}\phi_{j2}}, \ldots, \mathrm{e}^{\mathrm{i}\phi_{jM}})$. We can accommodate two more inputs by choosing

$$\boldsymbol{\phi}_1 = \langle 0, 1\rangle = \boldsymbol{\alpha}_2^0\boldsymbol{\alpha}_1^1 = \boldsymbol{\alpha}_1 = [0000111122223333], \qquad (18)$$

$$\boldsymbol{\phi}_2 = \langle 1, 0\rangle = \boldsymbol{\alpha}_2^1\boldsymbol{\alpha}_1^0 = \boldsymbol{\alpha}_2 = [0123012301230123]. \qquad (19)$$

For these three input horns, we have six distinct baselines, with masks (equation 8)

$$\boldsymbol{\Phi}_{01} = \langle 0, 3\rangle, \quad \boldsymbol{\Phi}_{02} = \langle 3, 0\rangle, \quad \boldsymbol{\Phi}_{12} = \langle 3, 1\rangle, \qquad (20)$$

$$\boldsymbol{\Phi}_{10} = \langle 0, 1\rangle, \quad \boldsymbol{\Phi}_{20} = \langle 1, 0\rangle, \quad \boldsymbol{\Phi}_{21} = \langle 1, 3\rangle. \qquad (21)$$

These are obtained by subtracting the values in angle brackets for the two horns modulo 4, for instance $\boldsymbol{\Phi}_{12} = \langle 0, 1\rangle - \langle 1, 0\rangle = \langle -1, 1\rangle = \langle 3, 1\rangle$. These masks are all distinct, and hence mutually orthogonal, and furthermore are all orthogonal to the vector $\langle 0, 0\rangle$, which is sensitive to the total power.

This construction shows that we can accommodate three horns with a sequence of 16 time-steps. We next ask whether it is possible to accommodate a fourth vector $\boldsymbol{\phi}_3$ in such a way that the new masks $\boldsymbol{\Phi}_{03}$, $\boldsymbol{\Phi}_{13}$, etc. are independent of the ones we have already found. A search of the $16 - 7 = 9$ candidates reveals an affirmative answer: $\boldsymbol{\phi}_3 = \langle 3, 3\rangle = \boldsymbol{\alpha}_2^3\,\boldsymbol{\alpha}_1^3$ works.

The value $N = 4$ is the maximum that can be achieved for the case of $M = 4^2$ time-steps, as is clear from a counting argument: $N = 5$ horns would require at least $M = N(N - 1) + 1 = 21$ steps.

Table 1 shows the maximum number of horns that can be accommodated for various values of M. These were found by recursively searching the space of possible phase shifts in the manner described above. The last column shows the maximum value that would be possible according to the simple counting argument that the number of time-steps must exceed the number of baselines. We have repeated this analysis for the case $P = 2$, where the phase shifters are capable of only 0° and 180° shifts, and found very similar results for the relationship between M and N.

As noted in the Introduction, extremely large values of M are impractical. This is one reason that a bolometric interferometer with a large number of horns should surely be designed with a high degree of symmetry, so that there are many equivalent baselines that can be read out coherently. (The other reason is the signal-to-noise ratio advantage.)

Table 1. The number of horns N that can be accommodated with a given number of time-steps M. We assume $P = 4$ distinct phase shift values. The second column shows the maximum number that can be accommodated, while the third column shows the number found by the simple counting argument $N(N - 1) + 1 \le M$.

M	$N_{\max}$ (actual)	$N_{\max}$ (counting)
4	2	2
$4^2 = 16$	4	4
$4^3 = 64$	8	8
$4^4 = 256$	15	16
$4^5 = 1024$	24	32
$4^6 = 4096$	40	64

In summary, this section has presented a procedure for selecting $\boldsymbol{\phi}_j$ that yields fully orthogonal masks. This means that the result of applying the mask for a given baseline will only be sensitive to the signal from the desired baseline, or equivalently that the reconstruction of all visibilities is accomplished with minimal noise.

4 SQUARE ARRAY

We now consider the case where the input horns are arranged in a square array with $N_{\rm side}$ horns on a side. In this case, many different baselines (i.e. pairs of horns) sample the same visibility. We wish to apply identical phase shifts to such equivalent baselines, so that a single mask reads out their sum. Naturally, we also require that inequivalent baselines have orthogonal masks. This is the case considered in detail by Charlassier et al. (C08). Our method parallels theirs in many respects but refines it in some ways.

Following the notation of C08, we parametrize the position of horns in the array in units of the minimum horn separation as a vector $\boldsymbol{d}_j = (l_j, m_j)$. Here l_j, m_j are integers running from 0 to $N_{\rm side} - 1$, labelling the position of the horn along the x and y directions. The index j runs from 0 to $N_{\rm side}^2 - 1$ according to $j = l_j + N_{\rm side}\, m_j$. We can construct a set of phase shifts for all horns that satisfy the desired criteria using the basis described in the previous section with $\mu = 2$. We let the phase shift sequence for horn j be

$$\boldsymbol{\phi}_j = \langle l_j, m_j\rangle. \qquad (22)$$

Below, we have explicitly written out the modulations for each horn in a 6×6 array.

$$\begin{array}{cccccc}
\langle 0,0\rangle & \langle 1,0\rangle & \langle 2,0\rangle & \langle 3,0\rangle & \langle 4,0\rangle & \langle 5,0\rangle\\
\langle 0,1\rangle & \langle 1,1\rangle & \langle 2,1\rangle & \langle 3,1\rangle & \langle 4,1\rangle & \langle 5,1\rangle\\
\langle 0,2\rangle & \langle 1,2\rangle & \langle 2,2\rangle & \langle 3,2\rangle & \langle 4,2\rangle & \langle 5,2\rangle\\
\langle 0,3\rangle & \langle 1,3\rangle & \langle 2,3\rangle & \langle 3,3\rangle & \langle 4,3\rangle & \langle 5,3\rangle\\
\langle 0,4\rangle & \langle 1,4\rangle & \langle 2,4\rangle & \langle 3,4\rangle & \langle 4,4\rangle & \langle 5,4\rangle\\
\langle 0,5\rangle & \langle 1,5\rangle & \langle 2,5\rangle & \langle 3,5\rangle & \langle 4,5\rangle & \langle 5,5\rangle.
\end{array}$$

In this case, the mask for the visibility corresponding to any pair of horns is simply $\langle \Delta l, \Delta m\rangle$. This means that all pairs with the same relative spacing get the same mask. Furthermore, as long as the number of phase shift steps P is large enough, all inequivalent visibilities correspond to orthogonal masks as desired. The minimum value of P is set by the fact that phases are only defined modulo P. Since Δl, Δm can range from $-(N_{\rm side} - 1)$ to $N_{\rm side} - 1$ we need at least $P = 2(N_{\rm side} - 1) + 1$ distinct phase shifts. (As we

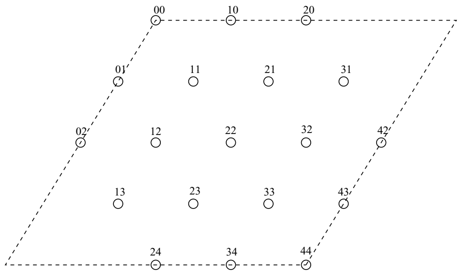

Figure 1. A hexagonal array of 19 horns can be seen as a subset of a 5×5 parallelogram-shaped array. The phase shifting scheme described in Section 4 with $N_{\rm side} = 5$ can be applied to this array.

will see in the next section, it may be desirable for P to be a multiple of 3, in which case we simply round up to the nearest such value.) If P is smaller than this, then distinct visibilities will be mapped on to the same phase shift sequence. For the above case, for example, we require $P \geq 11$. If we tried a smaller value, say $P = 10$, then the visibility corresponding to horns (0,0) and (5,1), for example, would get the same phase shift sequence as (5,0) and (0,1), namely $\langle 5, 1\rangle = \langle -5, 1\rangle$.

It is instructive to compare this scheme with the very similar one of C08. In both methods, the phase shift sequence for horn (l, m) is expressed in the form $l\boldsymbol{h} + m\boldsymbol{v}$ for two basis shift patterns $\boldsymbol{h}$, $\boldsymbol{v}$. In order to achieve the desired orthogonality properties, Charlassier et al. choose $\boldsymbol{h}$, $\boldsymbol{v}$ to be independent random vectors of phase shifts. The randomness ensures approximate orthogonality, up to errors of order $M^{-1/2}$, where M is the length of the phase shift sequence. In contrast, we choose $\boldsymbol{h} = \langle 1, 0\rangle$ and $\boldsymbol{v} = \langle 0, 1\rangle$. This results in strict orthogonality, as opposed to approximate orthogonality.

The number P of distinct phase shift values required is essentially the same in the two methods. As in our method, C08 found that $P \simeq 2N_{\rm side}$ was required in order to produce orthogonal phase shifts using random basis vectors.

Using either method, the length M of the modulation sequences is greatly reduced compared to the case of inequivalent baselines. The number of required phase shifts is $M = P^2 = (2N_{\rm side} - 1)^2$ when redundant baselines are tagged equivalently. If we instead used the methods of the previous section, we would require $M > N_{\rm side}^2(N_{\rm side}^2 - 1) + 1 \approx N_{\rm side}^4$. For the 6×6 array denoted above, this is the difference between a 121-step sequence and a 1261-step sequence. Even more important is the signal-to-noise ratio benefit of co-adding equivalent baselines.

Although we have described this procedure as applying to a square array, it is in fact more general. It applies whenever the horn positions can be expressed as integer multiples of any two basis vectors, even if the two are not orthogonal, or in other words, to any parallelogram-shaped array. Furthermore, it can be applied to any subset of such a parallelogram-shaped array, since we can simply ignore the parts of the array with no horns in them. In particular, this means that the method can be applied to a hexagonal close-packed array of horns, as shown in Fig. 1.

5 TWO POLARIZATIONS

Thus far, for simplicity we have been considering only one polarization state of the incoming radiation field. We now imagine that two orthogonal linear polarizations (x, y) are measured at each horn. In this case, we can in principle recover visibilities for all four Stokes parameters I, Q, U, V. To be specific, if visibilities V_{xx}, V_{xy}, V_{yx}, V_{yy} are measured for a particular baseline, then

$$V_I = V_{xx} + V_{yy}, \tag{23}$$

$$V_Q = V_{xx} - V_{yy}, \tag{24}$$

$$V_U = V_{xy} + V_{yx}, \tag{25}$$

$$V_V = -\mathrm{i}(V_{xy} - V_{yx}). \tag{26}$$

However, as C08 have pointed out, it is impossible to recover all visibilities while taking full advantage of the noise reduction resulting from co-adding redundant baselines. C08 describe two schemes for recovering some of the Stokes parameters with full accuracy, one of which (mode 2 of C08) involves measuring the visibilities for Stokes I, U, V but not Q. Stokes Q can then be measured by rotating the instrument 45°. In this section, we show how to implement this mode of operation using our phase shifting scheme.

Aside from the phase shifting scheme, there is another reason for adopting an observing scheme in which Stokes Q is measured only by rotating the instrument. As equation (24) indicates, the visibility for Stokes Q is obtained by subtracting two measured visibilities, each of which contains a contribution proportional to the much larger Stokes parameter I. As a result, this visibility is likely to be subject to much larger errors than the other linear polarization (Stokes U).

As in the previous section, we assume an $N_{\rm side} \times N_{\rm side}$ array of horns, but now we introduce an orthomode transducer for each horn, doubling the number of signals to be interfered. We can represent each of these $2N_{\rm side}^2$ signals with a triple of labels (l_j, m_j, n_j) where (l_j, m_j) labels the position of the horn as in the previous section, and $n_j = 0, 1$ labels the polarization state. In the previous section, we identified the horn (l_j, m_j) with a phase shift sequence $\langle l_j, m_j\rangle$. In the present case, we can define phase shift sequences similarly in terms of the triple $\langle l_j, m_j, n_j\rangle$, each of which represents a sequence of $3P^2$ time-steps, where $P \geq 2N_{\rm side} - 1$ as in the previous section. For one of the two polarization states, the phase shift sequences are simply three repetitions of the sequences we found previously:

$$\langle l, m, 0\rangle = (\langle l, m\rangle, \langle l, m\rangle, \langle l, m\rangle)\,. \tag{27}$$

For the other polarization state, we apply a slow three-phase modulation to this sequence: the first P^2 steps are unchanged, the next P^2 steps are multiplied by $\mathrm{e}^{2\pi\mathrm{i}/3}$ and the final block is multiplied by $\mathrm{e}^{4\pi\mathrm{i}/3}$:

$$\langle l, m, 1\rangle = \left(\langle l, m\rangle, \mathrm{e}^{2\pi\mathrm{i}/3}\langle l, m\rangle, \mathrm{e}^{4\pi\mathrm{i}/3}\langle l, m\rangle\right). \tag{28}$$

Note that this scheme is most natural to apply when P is a multiple of 3 so that for every phase state p there is another whose phase is $p + 2\pi/3$. Otherwise, the set of phase shifts involved in the sequences $\langle l, m, 1\rangle$ will be larger than that involved in $\langle l, m, 0\rangle$. In implementing this scheme, one would surely round P up to the nearest multiple of 3.

We choose an additional three-phase modulation to distinguish the two polarization states, because this is the smallest amount of additional modulation that allows full recovery of all of the visibilities. In particular, this scheme distinguishes between the xy and yx visibilities for a given horn pair, which have phase shifts $\langle l, m, 1\rangle - \langle l', m', 0\rangle = \langle l - l', m - m', 1\rangle$ and $\langle l, m, 0\rangle - \langle l', m', 1\rangle = \langle l - l', m - m', 2\rangle$, respectively. (The 2 here arises from the fact that $-1 \equiv 2$ mod 3.) A two-phase scheme would not make this distinction, giving $\langle l - l', m - m', 1\rangle$ for both visibilities.

It is straightforward to check that all equivalent baselines have identical phase shift sequences as desired. All pairs that interfere x and y polarization have independent, orthogonal phase shift sequences, allowing optimal reconstruction of Stokes U, V visibilities (equations 25 and 26). Those that interfere x and x have identical sequences to those that interfere y and y. Applying these phase shift masks therefore allows recovery of the sum of these visibilities, which is V_I.

As in the previous section, this method is similar to that of C08, except that our method imposes strict orthogonality on distinct baselines, as opposed to relying on the approximate statistical orthogonality that results from choosing random phase shift sequences.

As an example, consider a square array with $N_{\rm side} = 8$. The number of different phase shift values must satisfy $P \geq 2N_{\rm side} - 1 = 15$. The length of the phase shift sequences is $M = 3P^2 = 675$. The shortest sequence of phase shifts we could possibly hope for would have M equal to the number of unknowns we are trying to solve for. In this arrangement, there are 112 inequivalent baselines, each of which has three complex visibilities that are measured, and in addition the total power in I, Q, U are measured, resulting in a total of $6 \times 112 + 3 = 675$ unknowns. Our phase shift sequence is therefore as short as possible. For comparison, according to fig. 4 of C08, the optimal noise levels in the C08 scheme are obtained only when the phase shift sequence is nearly three times the minimum length. In these comparisons, we only consider the length of mode 2 in C08. It should be noted that when $2N_{\rm side} - 1$ is divisible by 3 we recover visibilities with maximum efficiency. When this is not the case the ratio of our length to the minimum approaches 1 for large $N_{\rm side}$. For arrays of reasonable size (8×8 or larger) the maximum ratio is 1.22 and occurs for a 10×10 array.

6 CONCLUSIONS

Optimal recovery of visibilities in a bolometric interferometer depends on the proper choice of phase shift sequences. We have laid out a method for finding such sequences that lead to fully orthogonal masks for all visibilities and introduced a compact notation for describing such phase shift sequences.

In the case of an array with a regular lattice structure, equivalent baselines can be read out simultaneously, reducing the length of the required phase shift sequence and improving the signal-to-noise ratio. This method refines that of C08. The method applies to arrays that are based on replication of any parallelogram-shaped fundamental cell, including, for example, hexagonal arrays.

For the case of rectangular arrays, the method described herein is very similar to that of C08, although our method imposes strict orthogonality on the masks for distinct baselines, rather than relying on approximate orthogonality resulting from random sequences. As a result, our method leads to optimal recovery of visibilities for Stokes I, U, V, with shorter time sequences than that of C08.

The ability to shorten the sequence of phase shifts is likely to be important in instrument design, because it reduces the degree to which $1/f$ noise must be controlled. For example, suppose that we can shift phase states at a rate of one state per 10 ms (because of either the design of the phase shifters or the bolometer time constants). As we saw in the previous section, an 8×8 array requires $\sim$1000 phase shifts, which would take 10 s. We therefore require the $1/f$ noise knee to be below $\sim$0.1 Hz. An alternative scheme involving a longer phase shift sequence would require correspondingly tighter control of the $1/f$ knee.

The modulation schemes discussed here and in C08 show distinct advantages over traditional Walsh functions for this application. Walsh functions require an increase in modulation length as more elements are added, while this scheme trades some of that length for an increase in the number of phase states, thus reducing the period of the modulation and relaxing $1/f$ requirements. This allows for the co-adding of redundant baselines, which is not possible with Walsh functions.

These phase shifting schemes have promise to be useful in many applications of interferometers. For interferometers that do pairwise combination, this approach allows all pairs to be combined at once in a single process. Since the power in each element does not need to be divided, this produces a higher signal-to-noise ratio upon detection. Conventional heterodyne interferometers may be able to apply phase shifts to each element and then combine all the signals together in a single data stream. While the values for visibilities are by far the most compact way to store data from an interferometer, being able to condense multiple data streams into a single one would reduce the band width for data transfer and storage size for raw data.

ACKNOWLEDGMENTS

EFB is supported by National Science Foundation Award AST-0507395. During the development of this paper, POH was supported by NASA awards NAG5-12758 and NNX07AG82G and a grant from the Wisconsin Space Grant Consortium. We thank Peter Timbie, Andrei Korotkov, Greg Tucker and the other members of the MBI collaboration, as well as Roman Charlassier, Jean-Christophe Hamilton and Jean Kaplan for valuable discussions.

REFERENCES

Bock J. et al., 2006, preprint (astro-ph/0604101)
Bunn E. F., 2007, Phys. Rev. D, 75, 083517
Charlassier R., 2008, preprint (astro-ph/0805.4527)
Charlassier R., Hamilton J., Bréelle É., Ghribi A., Giraud-Héraud Y., Kaplan J., Piat M., Prêle D., 2008, preprint (astro-ph/0807.0438)
Hamilton J., Charlassier R., Cressiot C., Kaplan J., Piat M., Rosset C., 2008, A&A, 491, 924
Hinshaw G. et al., 2008, preprint (astro-ph/0803.0732)
Kamionkowski M., Kosowsky A., Stebbins A., 1997, Phys. Rev. Lett., 78, 2058
Korotkov A. L. et al., 2006, in Zmuidzinas J., Holland W. S., Withington S., Duncan W. D. eds, Millimeter and Submillimeter Detectors and Instrumentation for Astronomy III. Proceedings of the SPIE, Vol. 6275, p. 62750X [Vol. 6275 of Presented at the Society of Photo-Optical Instrumentation Engineers (SPIE) Conference. SPIE, Orlando
Leitch E. M., Kovac J. M., Halverson N. W., Carlstrom J. E., Pryke C., Smith M. W. E., 2005, ApJ, 624, 10
Polenta G. et al., 2007, New Astron. Rev., 51, 256
Seljak U., Zaldarriaga M., 1997, Phys. Rev. Lett., 78, 2054
Thompson A. R., Moran J. M., Swenson G. W. Jr, 2001, Interferometry and Synthesis in Radio Astronomy, 2nd Edn. Wiley, New York
Timbie P. T. et al., 2006, New Astron. Rev., 50, 999
Tucker G. S. et al., 2008, Vol. 7020, The Millimeter-wave Bolometric Interferometer (MBI). SPIE, Bellingham, p. 70201M

APPENDIX A: PROOF OF MINIMUM-NOISE CONDITION

In this section, we provide a proof of the assertion that orthogonal phase shift patterns minimize the noise in the recovered visibilities.

Assume that the visibilities are arranged in an $N_{\rm vis}$-dimensional vector $\boldsymbol{V}$ and the observed signals in an M-dimensional vector $\boldsymbol{S}$.

The two are related by an $M \times N_{\rm vis}$ matrix $\mathbf{A}$:

$$\boldsymbol{S} = \mathbf{A}\boldsymbol{V}. \tag{A1}$$

All entries of $\mathbf{A}$ are complex numbers with absolute value 1. We assume that $M \geq N_{\rm vis}$ and that the matrix $\mathbf{A}$ has maximal rank, so that it is possible to solve for the unknown visibilities.

Assuming that the signals are contaminated with white noise with variance σ^2, the optimal reconstruction of the visibilities is the least-squares vector

$$\hat{\boldsymbol{V}} = (\mathbf{A}^\dagger\mathbf{A})^{-1}\mathbf{A}^\dagger\boldsymbol{S}. \tag{A2}$$

The noise covariance matrix for $\hat{\boldsymbol{V}}$ is

$$\mathbf{N} = \sigma^2(\mathbf{A}^\dagger\mathbf{A})^{-1}. \tag{A3}$$

The noise in the jth recovered visibility has variance $\mathbf{N}_{jj}$. We wish to show that this noise is minimized when the matrix $\mathbf{A}$ has orthogonal columns.

The diagonal elements of the inverse noise matrix are

$$(\mathbf{N}^{-1})_{jj} = \sigma^{-2}(\mathbf{A}^\dagger\mathbf{A})_{jj} = \sigma^{-2}\sum_{m=1}^{M} A^*_{mj}A_{mj} = \frac{M}{\sigma^2}. \tag{A4}$$

We can therefore write

$$\mathbf{N}^{-1} = \frac{M}{\sigma^2}(\mathbf{I} + \mathbf{D}), \tag{A5}$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{D}$ is a Hermitian matrix with zeroes along the diagonal.

The noise covariance matrix is

$$\begin{aligned}\mathbf{N} &= \frac{\sigma^2}{M}(\mathbf{I} - \mathbf{D} + \mathbf{D}^2 - \mathbf{D}^3 + \cdots)\\ &= \frac{\sigma^2}{M}[\mathbf{I} - \mathbf{D} + \mathbf{D}(1 - \mathbf{D} + \mathbf{D}^2 - \cdots)\mathbf{D}]\\ &= \frac{\sigma^2}{M}(\mathbf{I} - \mathbf{D} + \mathbf{DND}).\end{aligned} \tag{A6}$$

Since $\mathbf{D}$ has no diagonal elements, an arbitrary diagonal element of the noise covariance matrix is

$$\mathbf{N}_{jj} = \frac{\sigma^2}{M}[1 + (\mathbf{DND})_{jj}] = \frac{\sigma^2}{M}[1 + \boldsymbol{v}^\dagger\mathbf{N}\boldsymbol{v}], \tag{A7}$$

where $\boldsymbol{v}_k = D_{kj}$. Since $\mathbf{N}$ is a positive definite matrix, we conclude that

$$\mathbf{N}_{jj} \geq \frac{\sigma^2}{M}. \tag{A8}$$

That is, the minimum noise variance achievable on any one visibility is σ^2/M. This value is achieved when the matrix $\mathbf{A}$ is column orthogonal, since in this case $\mathbf{A}^\dagger\mathbf{A} = (M/\sigma^2)\mathbf{I}$ and $\mathbf{N} = (\sigma^2/M)\mathbf{I}$.

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