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Intensional Logic and Topology

A thesis

Presented to the

Department of Mathematics

and the

Faculty of the Graduate College

University of Nebraska

In Partial Fulfillment

of the Requirements for the Degree

Masters of Arts in Mathematics

University of Nebraska at Omaha

by

Andrew Scott Buchan

April 2003

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THESIS ACCEPTANCE

Acceptance for the faculty of the Graduate College, University of

Nebraska, in partial fulfillment of the

Requirements for the degree Master of Arts,

University of Nebraska at Omaha.

Committee

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Date: *4/21/03*

ABSTRACT

Intensional Logic and Topology

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University of Nebraska at Omaha, 2003

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This thesis is concerned with mathematical logic, in particular it is an investigation of a branch of mathematical logic called modal logic. This branch of mathematical logic extends the propositional calculus by adding two unary operators \Box and \Diamond to the standard set of logical operators. This extension of classical logic has many interpretations; traditionally it is said to be the logic of necessity, denoted by the box operator, and possibility, denoted by the diamond operator. The notion of necessity within modal logic is ubiquitous and lends itself to a vast sea of metaphysics. For example, if X is necessarily true, denoted $\Box X$, then it is said to be true in all possible worlds. This way of understanding modalities gave impetus for a semantics

that provided fodder for the first completeness proofs in modal logic.

Modalities in logic have its roots in philosophy and dates back as far as Aristotle's *Metaphysics*, but was brought into the limelight with the work of the philosopher mathematician Saul Kripke who in 1959, as a high school student, published the first completeness proof for a class of modal logics [Kripke]. His method used the so-called *semantic-tableaux* which was introduced by Beth's *The foundations of mathematics* to obtain quick completeness proof for the propositional and predicate calculus. In this thesis, we are also interested in completeness for modal logics, but will use a more modern method known in the literature as *canonical model constructions*. Moreover, we wish to provide a semantics for modal logics that is not the traditional possible world semantics. Our models will be topological in nature. Our goal is to provide a completeness proof for a particular modal logic called S4 which interprets the modal operators as the interior and closure operators on topological spaces. We will also prove that the logic S4 is complete with respect to the class of transitive and reflexive trees. This gives us two new completeness proof for the modal logic S4.

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1. THE BASICS

This introduction provides the basic language for our work. We shall construct modal logic by building model structures from a set of primitive objects, boolean connectives and basic rules articulating the constructions of well-formed formulas. The construction of models will be dominant in the following chapter and is expositional in character. Our exposition will, for the sake of brevity, be dense. For further details on model theory for intensional logic the reader should consult [Goldblatt 1][Goldblatt 2][Chellas]. To begin, we shall introduce the language of modal logic and then their model structure.

1.1 *The Language $\mathcal{L}(\Box)$*

Modal logic is an extension of the propositional calculus together with unary operators \Box and \Diamond that are defined on all atomic and non-atomic formula. In fact, modal logic is a fragment of first-order logic. This will become more

clear when we introduce our relational semantics below. We introduce the language $\mathcal{L}(\Box)$ which denotes the extension just described.

Let $\mathbb{PV} := \{p_0, p_1, \dots\}$ be a denumerably infinite set of propositional variables and add to \mathbb{PV} the logical constant \perp as an abbreviation for any logical contradiction such as $(\phi \wedge \neg\phi)$, and \top which corresponds to any tautology. It should be clear that \top and \perp are dual logical constants. The set of formulas Φ for propositional modal logic is defined inductively on atomic formulas as follows:

- $\mathbb{PV} \cup \{\perp, \top\} \subseteq \Phi$
- if $\alpha, \beta \in \Phi$, then $\alpha \star \beta \in \Phi$ where \star is a binary operator in $\{\wedge, \vee, \rightarrow\}$
- if $\alpha \in \Phi$, then $\neg\alpha \in \Phi$
- if $\alpha \in \Phi$, then $\Box\alpha \in \Phi$, and $\Diamond\alpha \in \Phi$
- no other string of symbols is in Φ

As is customary in mathematical logic, the elements of Φ are called well-formed-formulas, denoted *wff*'s. Our construction is not the most economical since the elements of \star are inter-definable and the so-called Box and Diamond

operators are dual operators¹. That is we can define $\neg\Box\neg\alpha$ as $\Diamond\alpha$. It is well known in the literature that all wff's have an equivalent representation in which the only logical connectives are \wedge and \neg . By a similar argument, given the duality between \Box and \Diamond , any wff of $\mathcal{L}(\Box)$ has an equivalent formula using only \Box , \neg and \wedge . This observation confirms the *completeness* of our logical connectives which should not be confused with the completeness of a logical system². Unless otherwise stated, we let lowercase Greek letters range over modal formulas save λ which will stand for any arbitrary *normal modal logic*.

Definition 1: A modal logic is a set $\lambda \subseteq \Phi$ which; (1) contains all tautologies of the propositional calculus, (2) satisfies *modus ponens*, and (3) satisfies universal substitution, i.e., if $\alpha \in \lambda$ and β is obtained from α by uniformly replacing some variable by some other wff, then $\beta \in \lambda$

¹ In modal literature the operators, \Box and \Diamond have been interpreted to mean *it is necessary that* and *it is possible that*, respectively. However, for this thesis we will not concern ourselves with any metaphysical interpretations, instead we will provide restrictions on the types of relations and functions we define on our models.

² We say that a set of boolean connectives is complete for a logical system iff for any assignment of truth values to ϕ , the same value can be given to a wff that contains only the boolean connectives in question.

A logic is called *normal* iff it satisfies the definition of a modal logic and is closed under *necessitation* and *monotonicity* of the box operator. That is, $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi) \in \lambda$ and $\Box\top \in \lambda$.

Remark 1: It follows from the definition of normality that in every normal logic λ we have,

$$(\Box(\phi \wedge \psi) \leftrightarrow \Box\phi \wedge \Box\psi) \in \lambda$$

Proof Indeed, $(\phi \wedge \psi \rightarrow \phi) \leftrightarrow \top \in \lambda$. So, $\Box(\phi \wedge \psi \rightarrow \phi) \leftrightarrow \Box\top \in \lambda$. Then it follows that $\Box(\phi \wedge \psi \rightarrow \phi) \in \lambda$. And by monotonicity, $\Box(\phi \wedge \psi) \rightarrow \Box\phi \in \lambda$. So, $\Box(\phi \wedge \psi) \rightarrow \Box\phi \wedge \Box\psi \in \lambda$.

Now, $(\phi \rightarrow (\psi \rightarrow \phi \wedge \psi)) \leftrightarrow \top \in \lambda$ which allow us to conclude that $\Box\phi \rightarrow (\Box\psi \rightarrow \Box(\phi \wedge \psi)) \in \lambda$, which is equivalent to saying $\Box(\phi \wedge \psi) \rightarrow \Box\phi \wedge \Box\psi \in \lambda$ ■

2. MODAL LOGIC: KRIPKE FRAMES AND MODELS

In this section, the model signature for modal logics and algebras is established, frames and models will be defined, and the concepts of modal *completeness* and modal *soundness* are presented. To begin we shall give the traditional motivation for modal logic and intuitive foundations for \Box and \Diamond are given via possible world semantics.

2.1 Possible World Semantics

The semantical part of meta-mathematics is essentially concerned with the assignment of meanings or interpretations to wff's and providing conditions under which a formula can be said to be *true*. In philosophy, the semantics of modal logic can be traced back to Leibnitz who claimed that a proposition is necessary if and only if it is true in all possible worlds. To get of sense of his metaphysics imagine a set of possible worlds which can have some alternatives. Denoting the *alternativeness relation* as R , we write xRy to

say that y is alternative, or possible with respect to x . Every world x lives under the laws of classical logic. A modal formula $\Box\phi$ is thus said to be true in a world x if ϕ is true in all worlds that are possible with respect to x . It is an easy observation to see that $\Diamond\phi$ is true in x if ϕ is true in at least one world y such that xRy

The most commonly studied models of modal logic are **Kripke Models**. Kripke models, or K-models, are first-order relational structures equipped with a binary relation $\prec_r \subseteq X \times X$ where X is some non-empty set. Traditionally, the elements of X in K-models are thought of as *possible worlds*, nodes, or points and the binary relation \prec_r is a relation that holds between possible worlds, nodes or points, however, the only important thing to understand is that worlds are collections of propositions. If w_1, w_2 are possible worlds and \prec_R is a relation, then $w_1 \prec_R w_2$ if w_2 is *accessible* from w_1 . This intuitive way of speaking about the relations between worlds lends itself to mathematical structures such as directed graphs which we will use later in this thesis. For the moment, it is convenient to think of traversing a tree from node to node in an upward traversal to capture the mathematical intuition necessary for working in modal logic. The following series of definitions will be relevant throughout the this thesis; they are standard in the literature

and can be found in [Goldblatt 1].

Definition 2: A **K-frame** is a structure $\mathcal{F} = \langle X, \prec \rangle$ where X is a non-empty set and \prec is a binary relation between ordered pairs in $X \times X$.

We call the set X the carrier or the base-set of \mathcal{F} . In this thesis, the relation \prec is a transitive and reflexive relation, such relations are called pre-orders or quasi-orders. Other restrictions on \prec correspond to stronger and weaker modal algebras and logics. A model for a basic modal language is a K-frame which is equipped with valuation function.

Definition 3: A **K-model** is a structure $\mathcal{M} = \langle \mathcal{F}, v \rangle$ where \mathcal{F} is a frame and $v : \Phi \rightarrow \mathcal{P}(X)$ is a valuation function, i.e., v is a function that assigns to each $p \in \Phi$ a subset $v(p)$ of X .

Informally, we may think of $v(p)$ as the set of points in our model where p is true. Given a model $\mathcal{M} = \langle \mathcal{F}, v \rangle$, we say that \mathcal{M} is based on the frame \mathcal{F} or that \mathcal{F} is the frame underlying \mathcal{M} .

2.1.1 Satisfaction and Validity

As mentioned above, we may understand the function v as saying which propositions are *true* in which world. That is to say, given some wff ϕ , the

function v provides a set of possible world in which ϕ is considered to hold. However, we have no notion of truth in a model as of yet. To do this we use the machinery we have just established to define the standard Tarskian satisfaction relation on modal formulas.

Let $\mathcal{M} = \langle \mathcal{F}, v \rangle$ be a K-model, $x \in X \in \mathcal{F}$ and $\phi \in \Phi$. We define the relation $\mathcal{M}, x \Vdash \phi$, read ϕ is true at x in \mathcal{M} , recursively on ϕ as follows:

$$\mathcal{M}, x \Vdash p_i \quad \text{iff} \quad x \in v(p_i)$$

$$\mathcal{M}, x \Vdash \perp \quad \text{iff} \quad \text{never} \quad \text{i.e., no contradiction is true in any model.}$$

$$\mathcal{M}, x \Vdash \neg\phi \quad \text{iff} \quad \mathcal{M}, x \not\Vdash \phi$$

$$\mathcal{M}, x \Vdash \phi \wedge \psi \quad \text{iff} \quad \mathcal{M}, x \Vdash \phi \quad \text{and} \quad \mathcal{M}, x \Vdash \psi$$

$$\mathcal{M}, x \Vdash \Box\phi \quad \text{iff} \quad (\forall y \in X)(x \prec y)(\mathcal{M}, y \Vdash \phi).$$

Instead of writing $\mathcal{M}, x \Vdash \phi$ we may write $x \Vdash \phi$ if the model is clear. The *truth-set* of ϕ in \mathcal{M} is defined as $v(\phi) = \{x \in X : x \Vdash \phi\}$.

We say that a formula is *valid* if it is true in all models and *satisfiable* if it is true in at least one models. Moreover, a formula is *valid with respect to a class of models* if it is valid with respect to all models on the class. The case for satisfiability is similar. It is an easy exercise to show that ϕ is valid in \mathcal{M} iff $\neg\phi$ is not satisfiable in \mathcal{M} .

We now want to say something about the meta-logical results that will

be important throughout this thesis.

Definition 4: Let \mathcal{C} be a class of modal frames. A logic is said to be strongly complete with respect to \mathcal{C} if for any set of formulas $\Gamma \cup \{\phi\}$, if $\Gamma \Vdash_{\mathcal{C}} \phi$, then $\Gamma \vdash_{\lambda} \phi$, read ϕ is a theorem of λ . That is to say, if Γ semantically entails ϕ on \mathcal{C} , then ϕ is λ -deducible¹. We say that a logic is sound with respect to \mathcal{C} if for any $\phi \in \lambda$, if ϕ is a theorem then ϕ is valid.

For every frame $\mathcal{F} \in \mathcal{C}$, if each instance of the axiom schemes are \mathcal{F} -valid, and \mathcal{F} validates the rules of inference, in the sense that for all models \mathcal{M} over \mathcal{F} , and every instance of a rule, if $\mathcal{M} \Vdash \psi_i$ for each of the premises ψ_i of the rule, then $\mathcal{M} \Vdash \phi$, where ϕ is the consequence of that instance of the rule. If such a situation is the case, then this establishes the following:

Theorem 1: The Soundness theorem of λ with respect to \mathcal{C} . If $\vdash_{\lambda} \phi$ ², then $\Vdash_{\mathcal{C}} \phi$, where \mathcal{C} is the class of frames over λ

The converse this theorem is called completeness and it states:

Theorem 2: Completeness theorem for λ with respect to \mathcal{C} . If $\Vdash_{\mathcal{C}} \phi$, then $\vdash_{\lambda} \phi$

¹ We will define what deducibility theoremhood means below in definition 5.

² The notation $\vdash_{\lambda} \phi$ is shorthand for saying that ϕ is derivable in the logic λ

The completeness theorem is a non-trivial result in mathematical logic. Together, soundness and completeness say that \mathcal{C} -validity can be characterized by theoremhood or provability in λ . This thesis is concerned with completeness with respect to topological interpretations of modal logic and modal algebra so it will be advantageous for the reader to become comfortable with these notions.

2.2 Proof Theory

Recall from basic logic that a proof system is a purely syntactical method for illustrating logical inferences, characterizing validity and logical consequence relations. In modal logic, this is done with respect to classes of frames. In this thesis, the logical system in which our modal logic shall be axiomatized is the standard Hilbert system.

2.3 Maximal Sets and The Canonical Model

In this section we shall introduce the most important concept of this thesis, namely the *canonical model* for modal logic. The canonical model is the main technical tool that we shall use to prove our completeness results. The method is derived from Henkin's construction for completeness of the pred-

icate calculus and proceeds by demonstrating that if λ is a normal modal logic and $\alpha \notin \lambda$, then there is a model of λ in which α fails in some state or world. This result follows from a more general results that says that any λ -consistent set of formulas Γ is simultaneously satisfiable in a model \mathcal{M} of λ . In traditional modal terms, this means that there is a world in the model at which every member of Γ is true, i.e., from each consistent normal modal logic there is a model called the *canonical model* for that logic, in which all and only theorems of the logic are valid. We will need the following definitions.

Definition 5: Let λ be a normal modal logic and Φ the set of wff's of λ , $\Gamma \subseteq \Phi$ and $\alpha \in \Phi$. We say that α is λ -derivable from the set Γ , written $\Gamma \vdash_{\lambda} \alpha$ iff for some finite set of wff's $\{\alpha_0 \dots \alpha_n\} \subseteq \Gamma$, we have $(\bigwedge_{i=1}^n \alpha_i) \rightarrow \alpha$. We say that α is a *theorem* of λ if α is derivable from no premises, i.e., if it follow from the empty set. Clearly, this is equivalent to saying that $\alpha \in \Gamma$. Γ is said to be λ -consistent provided there exist a formula not derivable from Γ , and λ -inconsistent otherwise. Γ is λ -*maximal* iff Γ is λ -consistent and for each wff α , either α or $\neg\alpha$ is contained in the set Γ . If we write X_{λ} , we shall mean the class of λ -maximal subsets of Φ .

We use maximally consistent sets in our completeness results for several reasons. Observe that every point x in every model \mathcal{M} for a normal modal logic λ is associated with the set of wff's $\{\phi : \mathcal{M}, x \Vdash \phi\}$. One can easily prove that $\{\phi : \mathcal{M}, x \Vdash \phi\}$ is in fact a maximally consistent set; if ϕ is true in some model \mathcal{M} for λ , then ϕ belongs to a λ -maximally consistent set. Also, if x and x' stand in some relation in \mathcal{M} , then the information embodied in the maximally consistent sets associated with x and x' is *coherently related*. This means that models give rise to collections of *coherently related* maximally consistent sets. The main program for the canonical model is to attempt to turn these observations around, that is, to take a class of coherently related maximally consistent sets to the model we want. Below, we will prove the statement that claims that if ϕ is an element of a maximally consistent set, then ϕ is true in some model \mathcal{M} whose points are all maximal consistent sets of the logic in which we are interested. This is achieved by building the canonical model which we will do presently. First, we want to prove that if a set is consistent then it can be extended to a maximally consistent set. The following theorem is sometimes called the Lindenbaum theorem for Boolean algebras. Here we provide a modal version.

Theorem 3 (Goldblatt 1): If Σ is a λ -consistent set of formulas, then there

is some $x \in X_\lambda$, so that $\Sigma \in x$.

Proof: Since our language is countable so is the set of wff's, thus we can enumerate them. Consider the following sequence of sets,

$$\begin{aligned}\Sigma_0 &= \Sigma, \\ \Sigma_{n+1} &= \begin{cases} \Sigma_n \cup \{\phi_n\} & \text{if this is } \lambda\text{-consistent,} \\ \Sigma_n \cup \{\neg\phi_n\} & \text{otherwise.} \end{cases} \\ \Sigma^+ &= \bigcup_{n \geq 0} \Sigma_n\end{aligned}$$

It is clear that Σ_0 is a λ -consistent set and so is Σ_{n+1} provided that Σ_n is λ -consistent. Now let $x = \Sigma^+$. We claim that x is λ -consistent. Suppose toward contradiction otherwise, then there is a sequence of formulas $\gamma_0, \dots, \gamma_k$ so that the set $\{\gamma_0, \dots, \gamma_k\}$ is λ -inconsistent. Because the cardinality of this set is k , for some integer k , the set has a maximum with respect to the enumeration of wff's. Let us call this maximum α_n . Then $\{\gamma_0, \dots, \gamma_k\} \subseteq \Sigma_n$, and consequently Σ_n would be λ -inconsistent. This is a contradiction.

We claim that x is maximally consistent. Indeed; for suppose that for some α_n that neither $\neg\alpha$ nor α is in x . Then by the construction above, if $\alpha_n \notin x$, then $\Sigma_n \cup \{\alpha_n\}$ is inconsistent, otherwise $\Sigma_{n+1} = \Sigma_n \cup \{\alpha_n\}$ and thus $\Sigma_n \cup \{\alpha_n\} \subseteq x$ and $\alpha_n \in x$ and since $\neg\alpha_n \notin x$ then $\Sigma_m \cup \{\neg\alpha_n\}$ is inconsistent where $\neg\alpha$ is

the m^{th} wff in the enumeration. Since n and m are integers, either $n < m$ or $m < n$ and by construction either $\Sigma_m \subseteq \Sigma_n$, or $\Sigma_n \subseteq \Sigma_m$. Now consider the follow chain of implications: We have

$$\Sigma_n \vdash \neg\alpha_n$$

$$\text{and } \Sigma_m \vdash \neg\neg\alpha_n$$

$$\text{and thus, } \Sigma_{\max(n,m)} \vdash \neg\alpha_n$$

$$\text{and } \Sigma_{\max(n,m)} \not\vdash \neg\neg\alpha_n.$$

But this means that $\Sigma_{\max(n,m)} \vdash p \wedge \neg p$, thus $x \vdash p \wedge \neg p$, but we have shown that x is λ -consistent. This shows that x is maximal and the $x \in X_\lambda$ ■

2.3.1 The Canonical Model Construction

We are now ready to build the canonical model for a general normal modal logic. Let λ be a consistent normal logic, $\mathcal{F} = \langle X_\lambda, \prec_\lambda \rangle$ be a frame, where X_λ is a maximally consistent set of well formed formulas of λ and \prec_λ is a binary relations such that, for any $x, y \in X_\lambda$, $x \prec_\lambda y$ if and only if for every ϕ of $\mathcal{L}(\Box)$, if $\Box\phi \in x$ then $\phi \in y$. Let $\mathcal{M} = \langle \mathcal{F}, v_\lambda \rangle$ denote the canonical model with carrier \mathcal{F} and the function v_λ , where $v_\lambda(p)$ is $\{x \in X_\lambda : p \in x\}$. We will often drop the use of the subscript λ when the logic we are working with is clear.

Let us take a moment to comment on the constituents of this construction. First, note that the function v equates the truth of a propositional variable at a point x with its membership in x . Below we will prove a generalization of this fact that will equate truth with membership for arbitrary formulas. We will prove:

Theorem 4: Let \mathcal{M} be the canonical model for a logic λ . For any well formed formula α and $x \in X_\lambda$, $\mathcal{M}, x \Vdash \alpha$ iff $\alpha \in x$

Secondly, the set of points or worlds of \mathcal{M}_λ are all λ -maximally consistent sets. **Theorem 3** shows us that any λ -consistent set is a subset of some point in \mathcal{M}_λ , and by **Theorem 4** proved below, any λ -consistent set of formulas is true at some point in this model. Lastly, the canonical relation says that if $\Box\phi$ is in x and $x \prec y$, then y contains all the information that is contained in ϕ . This transfer of information captures the intuitive claim above about worlds being *coherently related*.

We now want to prove **Theorem 4**, but before we provide our proof we will prove the following proposition.

Proposition 1: If λ is a normal modal logic and Γ is a λ -consistent set of wff's, then for any wff β such that $\neg\Box\beta \in \Gamma$, then $I(\Gamma) \cup \{\neg\beta\}$ is λ -consistent,

where $I(\Gamma) = \{\alpha : \Box\alpha \in \Gamma\}$

Proof: Suppose that the proposition is false, then $I(\Gamma) \cup \{\neg\beta\}$ is inconsistent.

Thus there is a sequence $\alpha_i \in I(\Gamma)$, $0 \leq i \leq n$, so that

$$(\alpha_i \rightarrow \beta) \in \lambda.$$

But this means that

$$(\Box(\bigwedge_{i=1}^n \alpha_i \rightarrow \beta)) \in \lambda,$$

and thus since \Box may be distributed across implications in any normal logic

we get,

$$(\Box \bigwedge_{i=1}^n \alpha_i \rightarrow \Box\beta) \in \lambda.$$

By **Remark 1**, we get

$$(\bigwedge_{i=1}^n \Box\alpha_i \rightarrow \Box\beta) \in \lambda,$$

and thus the set $\{\Box\alpha_i, \neg\Box\beta\}$ is inconsistent. However, this is a subset of Γ

and Γ is consistent so we have our contradiction. ■

Let's return to the proof of **Theorem 4**.

Proof of Theorem 4: We proceed by induction on the degree of wff's. By definition of the canonical valuation function, we have for atomic proposition p and $x \in X_\lambda$, that

$$\mathcal{M}, x \Vdash p \quad \text{iff} \quad p \in x$$

The cases for the boolean operators is obvious. Since we are working with maximally consistent sets, each $x \in X_\lambda$ assures us that $\phi \in x$ iff $\neg\phi$ is not and that $(\phi \wedge \psi) \in x$ iff $\phi \in x$ and $\psi \in x$. The only interesting case is for the modal operators which we shall now consider. The induction hypothesis states:

$$\mathcal{M}, x \Vdash \psi \quad \text{iff} \quad \psi \in x$$

Let $\psi = \Box\phi$, we wish to establish that

$$\mathcal{M}, x \Vdash \Box\phi \quad \text{iff} \quad \Box\phi \in x$$

Suppose that $\Box\phi \in x$ and that $x \prec y$. Then, by the definition for the relation ' \prec ', we have $\phi \in y$ and thus,

$$\mathcal{M}, y \Vdash \phi$$

Since this is the case for all y such that $x \prec y$ we have shown

$$\mathcal{M}, x \Vdash \Box\phi$$

Now suppose that $\Box\phi \notin x$. Then by the maximal consistency of x we have that

$$\neg\Box\phi \in x$$

Because x is consistent we have $I(x) \cup \{\neg\phi\}$ is also consistent and thus there is a state $y \in X_\lambda$ so that $I(x) \cup \{\neg\phi\} \subseteq y$. Further, because $I(x) \subseteq y$ implies $x \prec y$, and since $\neg\phi \in y$, then $\phi \notin y$, thus $\mathcal{M}, y \not\Vdash \phi$ and because $x \prec y$, $\mathcal{M}, x \not\Vdash \Box\phi$. This proves the inductive step and the theorem. ■

The theorem just proved provides means to establish all of main results of this thesis. We have show that a modal formula ϕ is true at a point x iff it is an element of that point. From this fact it follow that there is a derivation of ϕ at x as well. See definition 5. Before we begin to present results, we shall need one more fact which is an easy corollary of the previous theorem.

Corollary 1 (Goldblatt 1): Any normal modal logic is complete with respect to its canonical model.

Proof: Let $\mathcal{M}_\lambda = \langle X_\lambda, \prec_\lambda, v_\lambda \rangle$ be the canonical model for a normal modal logic λ . Let Σ be a consistent subset of λ . We have shown that any such set has a maximal extension, so let Σ^+ be that extension. By the proof of the theorem above, we see that $\mathcal{M}_\lambda, \Sigma^+ \Vdash \Sigma$, thus $\Sigma \in \Sigma^+$ which is enough to prove the corollary. ■

What is significant about this corollary is that it allows us to get completeness quickly by constructing the canonical model. We now wish to use these facts to establish some theorems about the logic S4. **Theorem 5** is standard in the literature, but we will give a detailed proof for the reader. **Theorem 6**, however is an extension and provides a stronger proof than **theorem 5**.

2.3.2 The Logic S4

Let S4 denote the smallest class of normal modal logics that consist of all instances of propositional tautologies together with the following modal axioms schemes:

$$(R) \quad \Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$$

$$(N) \quad \Box\top$$

(T) $\Box\phi \rightarrow \phi$

(4) $\Box\phi \rightarrow \Box\Box\phi$

We also have the following rules of inference:

$$[\text{Modus Ponens}] - \frac{\phi \rightarrow \psi, \phi}{\psi} \quad [\text{Monotonicity}] - \frac{\phi \rightarrow \psi}{\Box\phi \rightarrow \Box\psi}$$

A formula $\phi \in \mathcal{L}(\Box)$ is an S4-deductive consequence of a set of formulas $\Gamma \subseteq \mathcal{L}(\Box)$, denoted $\Gamma \vdash_{S4} \phi$ if there is a finite sequence of formulas $\phi_0 \dots \phi_n$ in $\mathcal{L}(\Box)$ so that $\phi = \phi_n$ and each $\phi_i, i \leq n$, is either an instance of an axiom scheme of S4, or in Γ , or the conclusion of an instance of a rule of S4 whose premises are contained in $\{\phi_0 \dots \phi_{n-1}\}$. That is, $\Gamma \vdash_{S4} \phi$ iff $\phi \in \Gamma$

Let us use this fact to show that S4 is complete with respect to the class of pre-orders.

2.3.3 Completeness Proof for S4 on Pre-ordered Sets

Theorem 5: S4 is complete with respect to the class of pre-orders

Proof: By corollary [1], to prove this result it is enough to find an S4-consistent set Γ , a model $\mathcal{M} = \langle \mathcal{F}, v \rangle$ based on a pre-ordered set $\mathcal{F} = \langle X, \prec \rangle$, and a state $x \in X$ such that $\mathcal{M}_{S4, x} \Vdash \Gamma$. To this end, let's choose the

canonical model for S4, $\mathcal{M}_{S4} = \langle \mathcal{F}_{S4}, v_{S4} \rangle$, and let Γ^+ be any S4-maximally consistent set extending Γ . It follows that $\mathcal{M}_{S4}, \Gamma^+ \Vdash \Gamma$ by **theorem 4**. All that is left to show is that \mathcal{F}_{S4} is a pre-order.

The frame is transitive. Indeed: the x, y, z be states in X_{S4} , so that $x \prec y$ and $y \prec z$. Then for any wff ϕ , if $\Box\phi \in x$, then $\phi \in y$ and for any ψ , if $\Box\psi \in y$, then $\psi \in z$. Now suppose that $\Box\phi \in x$, then it follows that $\Box\Box\phi \in x$ as well, and thus $\Box\phi \in y$ which give us $\phi \in z$. That is for any ϕ , if $\Box\phi \in x$, then $\phi \in z$, which is to say that $x \prec z$. All that is left to establish is that the frame is reflexive as well. To show that \prec is reflexive it is enough to show that for all wff ϕ and $x \in X_{S4}$, if $\Box\phi \in x$, then $\phi \in x$, but this follow immediately by the substitution of ϕ for p in $\Box p \rightarrow p$ and this completes the proof. ■

An obvious question now arises: What is the modal logic for partially-ordered sets? This is not obvious, and in fact quite novel since modal logic cannot tell the difference between posets and pre-sets. We will now prove that S4 is indeed complete with respect to poset and use this fact to establish a new completeness results, namely S4 is complete to the class of transitive reflexive trees. Our plan is to transform the standard S4 frame in such a way that the new frame is partially ordered. To do this, we need the following definition

of *p-morphism* from [Goldblatt 1] and *unwinding* from [Mints 1].

2.3.4 P-morphisms and Unwinding

Definition 6 (Goldblatt 1): **P-morphism** Given two models

$$\mathcal{M} = \langle X, \prec, v \rangle \text{ and } \mathcal{M}' = \langle X', \prec', v' \rangle,$$

a map f from \mathcal{M} to \mathcal{M}' is called a p-morphism from \mathcal{M} into \mathcal{M}' if it satisfies the following conditions, for all $x, y \in X$

- (a) x and $f(x)$ satisfy the same propositional variables
- (b) if $x \prec y$, then $f(x) \prec' f(y)$,
- (c) if $f(x) \prec' y'$, then there is a $y \in X$ such that $x \prec y$ and $f(y) = y'$

Recall from algebra that a function f satisfying condition (b) is a homomorphism. Further, if f is onto, then we say that \mathcal{M}' is a *p-morphic image* of \mathcal{M} . It is not hard to show by induction that for all wff's ϕ and all $x \in X$,

$$\mathcal{M}, x \Vdash \phi \quad \text{iff} \quad \mathcal{M}', f(x) \Vdash \phi$$

In other words: Modal satisfaction is invariant under p-morphisms. Our goal is to show that models of S4 can be transformed into *trees* which will make

a new partially ordered model such that the original model is a p-morphic image of the new model. Subsequently, we can, as a quick corollary, obtain a new proof of completeness for S4 on the class of reflexive and transitive trees. Below, we will illustrate how we can transform a pre-ordered set into a tree³.

Remark 2: From this point on we assume that all of the frames under consideration are rooted, i.e., $\langle X, \prec \rangle$ has a \prec -least element.

Definition 7: Unwinding Let $\mathcal{F} = \langle X, \prec \rangle$ be a frame generated from some point $x \in X$. We define the unwinding of \mathcal{F} around x as the frame $\langle \vec{X}, \vec{\prec} \rangle$ where:

1. \vec{X} is the set of all finite sequences (x, x_1, \dots, x_n) such that $x, x_1, \dots, x_n \in X$ and $x \prec x_1 \dots x_{n-1} \prec x_n$, and
2. If \vec{y}_1 and $\vec{y}_2 \in \vec{X}$ then $\vec{y}_1 \vec{\prec} \vec{y}_2$ if there is some $z \in X$ such that $\vec{y}_1 + (z) = \vec{y}_2$. Here $+$ denotes the concatenation of sequences.

If $\mathcal{M} = \langle X, \prec, v \rangle$ is a model and $\langle \vec{X}, \vec{\prec} \rangle$ is the unwinding around x , then we define the function \vec{v} as follows:

³ The following definition comes from a forthcoming paper by Mints where he uses the method below to establish a completeness proof for our modal logic on \mathbb{R} .

$$\vec{v}(p) = \{(x, x_1 \dots x_n) \in \vec{X} : x_n \in v(p)\}$$

The unwinding clearly is a tree where the root is the sequence (x) , and the relation $\vec{\prec}$ is simply the family of immediate successor relation on trees. The relation obtained by the unwinding is partially ordered since $\vec{x} \vec{\prec} \vec{y}$ iff \vec{x} is an initial segment of \vec{y} , so if $\vec{x} \vec{\prec} \vec{y}$ and $\vec{y} \vec{\prec} \vec{x}$, then clearly, $\vec{x} = \vec{y}$. Consequently, the unwinding of a transitive and reflexive structure is also antisymmetric. More precisely, the unwinding has the property that:

1. There is a point x with no predecessors, namely the root.
2. The relation $\vec{\prec}$ has no cycles.
3. Every point under the relation $\vec{\prec}$, with the exception of the root, has a unique predecessor.

Now consider the following theorem.

Theorem 6: Let $\vec{\mathcal{M}} = \langle \vec{X}, \vec{\prec}, \vec{v} \rangle$ be the unwinding of $\mathcal{M} = \langle X, \prec, v \rangle$ around x . Then $\langle X, \prec \rangle$ is a p-morphic image of $\langle \vec{X}, \vec{\prec} \rangle$ and \mathcal{M} is a p-morphic image of $\vec{\mathcal{M}}$.

Proof: Consider the function $f : \vec{X} \rightarrow X$ so that $f(x, x_1 \dots x_n) = x_n$. It is routine to check the function f is onto and satisfies condition homomorphism. This is clear by construction. Also, for any $\vec{x} \in \vec{X}$, \vec{x} and $f(\vec{x})$ satisfy the same propositional variables. This is all we need to establish the theorem. ■

2.3.5.- S4 is Complete on Partially-Ordered Trees

What does this mean for S4? It means that we can construct a tree that is partially ordered and get a stronger completeness proof than the one given above. To this end we take the reflexive transitive closure of the *unwound* model; this new model will also be a model of S4. By the reflexive transitive closure on a frame $\langle X, \prec \rangle$, we mean the smallest reflexive transitive relation \prec^* on X containing \prec . That is, we unwind $\mathcal{M} = \langle X, \prec, v \rangle$ around a point x to get a new model $\vec{\mathcal{M}} = \langle \vec{X}, \vec{\prec}, \vec{v} \rangle$. Next we look at the model and consider the reflexive transitive closure \prec^* . Call this model $\mathcal{M}^* = \langle \vec{X}, \prec^*, \vec{v} \rangle$. This model will clearly be a model of S4 since the reflexive and transitive closure is still reflexive and transitive. Moreover, the new model will be antisymmetric since it is a tree where \prec^* become the *ancestor-of-relation*. We still need to show that \mathcal{M} is a p-morphic image of \mathcal{M}^* .

Theorem 7: Let $\mathcal{M} = \langle X, \prec, v \rangle$ be a reflexive transitive model generated by $x \in X$, and let $\vec{\mathcal{M}} = \langle \vec{X}, \vec{\prec}, \vec{v} \rangle$ be the unwinding of \mathcal{M} around x . Let \prec^* be the reflexive transitive closure of \prec . Let $\mathcal{M}^* = \langle \vec{X}, \prec^*, \vec{v} \rangle$, then \mathcal{M} is a p-morphic image of \mathcal{M}^*

Proof: Let f be as in **Theorem 6**. It is clear that the function is still a p-morphism, moreover the fact that f is onto and satisfies condition (b) remains unchanged. The only thing we need to be concerned with is that the relation \prec^* is reflexive and transitive. However, since \prec is already reflexive and transitive and the transitive reflexive closure is also transitive and reflexive, we have nothing to worry about. ■

Theorem 8: S4 is complete with respect to the class of partially ordered reflexive and transitive trees

Proof: Let Γ be an S4-consistent set of formulas and Γ^+ its maximal extension. Let \mathcal{M}_{S4} be the canonical model. Then $\mathcal{M}_{S4}, \Gamma^+ \Vdash \Gamma$, and thus $\Gamma \in \Gamma^+$. Now let \mathcal{M}_s be a sub-model of \mathcal{M}_{S4} generated by Γ^+ . Clearly \mathcal{M}_s is reflexive and transitive since \mathcal{M}_{S4} is. Moreover, we have $\mathcal{M}_s, \Gamma^+ \Vdash \Gamma$.

Now let $\mathcal{M}^* = \langle \vec{X}, \prec^*, \vec{v} \rangle$ be the reflexive and transitive closure of the unwinding of \mathcal{M}_s around Γ^+ . Let $f : \mathcal{M}^* \rightarrow \mathcal{M}_s$ be as before. Then, by

the previous result, \mathcal{M}_s is a p-morphic image \mathcal{M}^* under f . Thus for all sequences $\vec{x} \in f^{-1}[\Gamma]$, we have $\mathcal{M}^*, \vec{x} \Vdash \Gamma$. Since f is onto there is at least one such \vec{x} . Thus, we have satisfied Γ on a reflexive and transitive tree.

■

3. TOPOLOGICAL INTERPRETATIONS FOR MODALITY

3.1 *Topological Spaces*

Let us recall what a topology is and some facts about topological spaces.

Definition 8: A topology on a set X is a collection $\tau \in \mathcal{P}(X)$ denoted by the pair $\langle X, \tau \rangle$ and satisfying the following conditions:

1. \emptyset and X are in τ
2. τ is closed under arbitrary unions
3. τ is closed under finite intersections

We say that elements of τ are open sets. Further, if A is an open set, then $X \setminus A$ is closed, i.e., the complement of open sets results in closed sets and the complement of closed sets results in open sets. If a set is both open and closed, then the set is call *clopen*. The empty set and the whole space of a

topological space are *clopen*.

We now want to give a semantics for modal logic that allows us to use the tools of general topology. In particular, we want to impose an interpretation on the intensional operators \Box and \Diamond , which are topological. To do this, let the atomic formulas points in a topological space, wff's range over sets of points, boolean operators become their obvious set-theoretical counterparts, \top is taken to be the whole space, \perp the empty set, and the modal operator \Box and \Diamond get mapped to the interior and closure operators respectively. The *cl* operator is called the closure operator. It is a unary operator which assigns to each subset A of X a subset $cl(A)$ of X . We say that the closure of a set A is the smallest closed set containing A . The dual to this operator for a subset A of X is the interior operator denoted $int(A)$, which is defined to be the union of all open set included in A , or, the largest open subset of A . The following are the so-called *Kuratowski* closure axioms:

$$int(\alpha \cap \beta) = int(\alpha) \cap int(\beta) \quad cl(\alpha \cup \beta) = cl(\alpha) \cup cl(\beta)$$

$$int(\alpha) \subseteq \alpha \quad cl(\alpha) \supseteq \alpha$$

$$int(int(\alpha)) = int(\alpha) \quad clcl(\alpha) = cl(\alpha)$$

$$int(X) = X \quad cl(\emptyset) = \emptyset$$

Note the correspondence between the axioms for S4 and the axioms above.

There is, however, no obvious connection to S4's axiom R. We will show in the next section that we can overcome this problem; we will show that any topological model makes S4 valid, turning S4 into a modal logic for topological spaces.

3.1.1 Topological Semantics

Let $\mathcal{X} = \langle X, \tau \rangle$ be a topological space. The topological model for a modal logic is the structure $M = \langle \mathcal{X}, v \rangle$, where v is a valuation on \mathcal{X} ; i.e. $v : \Phi \rightarrow \mathcal{P}(X)$ is a function from the set of propositional variables of our modal logic to the powerset of X . We can now say what it means for a formula ϕ to be true at a point x .

1. $x \Vdash p$ iff $v(p) \subseteq x$, where p is a propositional variable.
2. $x \Vdash \neg\phi$ iff $x \not\Vdash \phi$

-
3. $x \Vdash \phi \wedge \psi$ iff $x \Vdash \phi$ and $x \Vdash \psi$
4. $x \Vdash \Box\phi$ iff $(\exists U \in \tau)(x \in U)(\forall y \in U)(y \Vdash \phi)$

The last part of the definition is the only place where the topological semantics differs from the previous definition. It says that $\Box\phi$ is true just in case ϕ is true at all points in some neighborhood of x . This is consistent with taking the \Box as the interior operator on topological spaces. We now wish to demonstrate the connection between modal logic and topology by inducing a topology on pre-orders. Recalling that our frames have a transitive and reflexive binary relation, we call a subset $Y \subseteq X$ in a frame $\langle X, \prec \rangle$ upward closed if $x \in Y$ and $x \prec y$ implies $y \in Y$. The following theorem is easily established.

Theorem 9: Every S4-frame $\langle X, \prec \rangle$ induces a topological space $\langle X, \tau_{\prec} \rangle$ where τ_{\prec} is the set of all upward closed subsets of $\langle X, \prec \rangle$

Proof Obvious. ■

The topology induced by τ_{\prec} has the property that arbitrary intersections of open set are open. Here the least open set around a point x is $\{y \in X :$

$x \prec y$. The topology induced by upward closed sets is known in computer science as Alexandroff topologies. See [Vickers] for its application there.

Definition 9: A topological space X is said to be an Alexandroff space iff

$$B_\tau(x) = \bigcap \{U \in \tau : x \in U\} \text{ is open } (\forall x \in X).$$

This definition is equivalent to the claim that a space is Alexandroff iff arbitrary intersections of open sets are open.

Lemma 1: The family $\{B_\tau(x)\}_{x \in X}$ is a base for the topology τ .

Proof: Let $U \in \tau$ and $x \in U$ then $B_\tau(x) \subseteq U$ and $x \in B_\tau(x)$ so $\{B_\tau(x)\}_{x \in X}$ is a base for τ ■

Theorem 10: There is a one-to-one correspondence between pre-orders on X and Alexandroff topologies on X .

Proof: To show this, let $\mathcal{X} = \langle X, \tau \rangle$ be a topological space where $\tau \in \mathcal{X}$.

Define a relation \prec_τ by:

$$(1) \quad x \prec_\tau y \text{ iff } (\forall U \in \tau)(x \in U \rightarrow y \in U)$$

By construction, (1) is reflexive and transitive and thus a pre-order. Indeed:

$(\forall U \in \tau)(x \in U \rightarrow x \in U)$, and thus, the relation is reflexive. To show transitivity,

assume that $x \prec_\tau y$ and $y \prec_\tau z$. Then $(\forall U \in \tau)(x \in U \rightarrow y \in U)$

and $(\forall U \in \tau)(y \in U \rightarrow z \in U)$. Clearly this entails that $(\forall U \in \tau)(x \in U \rightarrow z \in U)$

Now let \prec be a pre-order on X . Call a subset U of X open in τ_{\prec} if it is upward closed. That is whenever $x \in U$ we have as our basic open sets:

$$(2) \quad B_{\prec}(x) = \{y \in X : x \prec y\} \subseteq U$$

Claim 1: τ_{\prec} as defined in (2) is an Alexandroff topology on X .

Indeed. It is clear that both the empty-set and X are in τ_{\prec} . We require proof that τ_{\prec} is closed under unions and intersections.

1. Let U_i be an open set for each $i \in \omega$. Suppose that $x \in \bigcup_{i \in \omega} U_i$. Then $x \in U_j$ for some $j \in \omega$ so that,

$$B_{\prec}(x) \subseteq U_j \subseteq \bigcup_{i \in \omega} U_i \quad \text{and thus} \quad \bigcup_{i \in \omega} U_i \in \tau_{\prec}$$

2. Now we want to look at intersections. Let $V_i \in \tau_{\prec}$ and $x \in \bigcap_{i \in \omega} V_i$, then for every $i \in \omega$

$$B_{\prec}(x) \subseteq V_i$$

i.e.,

$$B_{\prec}(x) \subseteq \bigcap_{i \in \omega} V_i \quad \text{so} \quad \bigcap_{i \in \omega} V_i \in \tau_{\prec}$$

We now will show that applying (1) and (2) in turn bring us back to our starting point, and thus provides the one-to-one correspondence we require for the proof of the theorem.

We want to show that starting with a pre-order \prec on X , that the order induced by the topology by that pre-order are the same pre-order. This shows that, $\prec_{\tau_{\prec}} = \prec$.

Suppose that \prec is a pre-order on X and that τ_{\prec} is the topology induced by it from (2) above. Assume that $x \prec y$ (we want to show that $x \prec_{\tau_{\prec}} y$), then $y \in B_{\prec}(x)$. Thus every open set containing x also contains y by (2). So, from (1) it follows that $x \prec_{\tau_{\prec}} y$.

Conversely, suppose that $x \prec_{\tau} y$. That is, suppose that every open set containing x also contains y . Now the set $B_{\prec}(x) = \{y \in X : x \prec y\}$ is open.

Indeed. Assume $z \in B_{\prec}(x)$, then $x \prec z$. We need to show that $B_{\prec}(z) \subseteq B_{\prec}(x)$. Let $w \in B_{\prec}(z)$, then $z \prec w$ implies $x \prec w$. By transitivity, this means that $w \in B_{\prec}(x)$ and by (2), $B_{\prec}(z)$ is open which is what we wanted to demonstrate. Thus, $y \in B_{\prec}(x)$, (since $x \in B_{\prec}(x)$), $x \prec y$. What this show is that the pre-order derived from τ by (1) coincides with \prec . This is equivalent to showing $\prec_{\tau} = \prec$.

Now we want to show that starting with an Alexandroff topology τ on X that the topology induced by the order induced by that topology coincide, that is $\tau_{\prec_{\tau}} = \tau$.

Let τ be an Alexandroff topology on X and \prec_{τ} the pre-order induced by it from (1). If $U \in \tau$ and $x \in U$, then $B_{\prec_{\tau}}(x) \subseteq U$. So U is open in the sense of (2).

Conversely, suppose that U is open in the sense of (2). Then $x \in U$ implies that $B_{\prec_\tau}(x) \subseteq U$. Then from the definition of \prec_τ , each set $B_{\prec_\tau}(x)$ is in the intersection of all members of τ containing x . But, because τ is Alexandroff, each $B_{\prec_\tau}(x)$ is open. But by assumption, U is the union of the sets $B_{\prec_\tau}(x)$ for all $x \in U$, and the union of open sets is open. Consequently, $U \in \tau$ as required. ■

Corollary 2: Any finite topology is Alexandroff.

Proof: Let $\langle X, \tau \rangle$ be a finite topology. Then there are only a finite number of open sets in τ and since in any topology finite intersections of open sets is open it follows that for any $x \in X$ there will be a least open set containing it. ■

Corollary 3: If $\mathcal{X} = \langle X, \tau \rangle$ is a topological space and X is finite, then \mathcal{X} is Alexandroff.

Proof: Suppose that the cardinality of X is n for some integer n . Then the cardinality of τ is less than or equal to 2^n which is finite, and by the previous corollary we see that \mathcal{X} is Alexandroff. ■

It is interesting to see what sort of topologies we get when we impose separation axioms on Alexandroff spaces. Recall that a topological space is T_0 if, given two distinct points x and y , there is an open set containing one of x or y but not the other. A topology is T_1 if there is an open set containing one x but not y and an open set containing y but not x . A topology is T_2 for any distinct points x, y if there is a pair of disjoint open sets such that x is in one and y is in the other.

Proposition 2: If an Alexandroff space X is T_1 , then it is a discrete space.

Proof Indeed, observe that for any $y \in X$, the singleton $\{y\}$ is closed, thus for every $x \in X$, $\{x\} = \bigcap_{y \neq x} (X \setminus \{y\})$ is open. If the topology τ on X is T_1 , then for any $x, y \in X$, $x \prec_\tau y$ iff $x = y$ so $B(x) = \{x\}$ this is consistent with τ_\prec being discrete. ■

Since $T_1 \subset T_2$, it follows that imposing stronger separation axioms on Alexandroff spaces will always be discrete.

The topological completeness proof now comes almost for free. But, before we give the proof, we want to establish the topological soundness theorem.

3.2 The Proof of Soundness

The soundness theorem is the claim that for any formula $\phi \in \mathcal{L}(\Box)$, if ϕ is deducible from the axioms, then it has a topological model. Subsequently, we require a proof that the axioms are valid on any topological space, and that our rules of inference preserve validity.

We now want to establish the soundness theorem for S4 with respect to topological spaces:

Theorem 11: If $S4 \vdash \phi$, then ϕ is topologically valid.

Proof Assume the antecedent of the theorem. We will examine each axiom and show that it is valid topologically. To begin, let's look at axiom **K**: $\Box \top$

This axiom is equivalent to saying that the interior of a space is open. That is, $\text{int}(X) = X$. But is true for any topological space $\langle X, \tau \rangle$ since the whole space is open.

Now consider the axiom **T**: $\Box\phi \rightarrow \phi$

We need to show that $\text{int}(\phi) \subseteq (\phi)$. Again, this is topologically valid and

follows from the properties of the interior operator.

Next we consider axiom 4: $\Box\phi \rightarrow \Box\Box\phi$.

This axiom requires that we demonstrate that $\text{int}(\phi) \subseteq \text{int}(\text{int}(\phi))$. Recall that for any set $\alpha \subset X$ in $\langle X, \tau \rangle$, that $\text{int}(\alpha)$ coincides with $\text{int}(\text{int}(\alpha))$ which establishes the case for 4.

Axiom **R**: $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$ is the least trivial. When we translate logical connectives into their set theoretical analogue, we set the arrow equal to the subset relation, but since in topology it does not make sense to distribute the interior operator across the subset relation, we must do some work to establish the result we desire.

Recall that $(p \rightarrow q)$ is equivalent to $(\neg p \vee q)$. With this in mind, we can get the result we need by showing the $\text{int}(\phi \cup \psi) \subseteq (\text{int}((\phi^c)^c \cup \text{int}(\psi)))$, is topologically valid, where c is the set-theoretical complement.

Claim: For D and B arbitrary subsets of a topological space $\mathcal{X} = \langle X, \tau \rangle$, we have $\text{int}(D \cup B) \subseteq (\text{int}((D^c)^c \cup \text{int}(B)))$

If $x \in \text{int}(D \cup B)$, then $x \in B \cup D$ then we are done

If $x \notin \text{int}(D^c)$, then $x \in (\text{int}(D^c))^c \subseteq (\text{int}(D^c))^c \cup \text{int}(B)$ then we are done.

If $x \in \text{int}(D^c)$, then x is in the intersection of two open sets: $\text{int}(D^c) \cap \text{int}(D \cup B)$. This subset is clearly a subset of $D \cup B$. But is any $y \in \text{int}(D^c) \cap \text{int}(D)$, then also $y \in \text{int}(D^c) \subset D^c$. So $y \notin D$, so $y \in B$. This open set $\text{int}(D^c) \cap \text{int}(D \cup B) \subseteq B$. So $x \in \text{int}(B)$ and so $x \in (\text{int}(D^c))^c \cup \text{int}(B)$. Now take $D = A^c$ to get $\text{int}(A^c \cup B) \subseteq (\text{int}(A))^c \cup \text{int}(B)$. This proves the claim and the soundness of the axiom.

All that is left to show is that the rules of inference preserve validity. The fact that modus ponens has this property is established for the propositional calculus and is consequently invariant for modal logics. The interesting case is for monotonicity. The soundness of this rule is nevertheless immediate. Indeed, assume $\phi \rightarrow \psi$ is topologically valid, then by monotonicity of the interior operator we get

$$\text{int}(\phi) \subseteq \text{int}(\psi)$$

and thus $\Box\phi \rightarrow \Box\psi$ is also valid in \mathcal{F} . This completes the proof of the soundness theorem. ■

Now we shall provide a proof for the main theorem of this thesis. By the proof of theorem 10 we observed that pre-ordered sets and the corresponding Alexandroff spaces are essentially the same. We need to show that this correspondence remains in modal models. To show this, we need to prove the next theorem.

Theorem 12: Let $\mathcal{M} = \langle X, \prec, v \rangle$ be a model on the pre-ordered frame $\langle X, \prec \rangle$ and $\mathcal{T} = \langle X, \tau, v \rangle$ be the corresponding topological model based on the Alexandroff space $\langle X, \tau \rangle$. Then for any valuation function v on X , for any point $x \in X$, and for any formula ϕ , it follows that

$$\mathcal{M}, x \Vdash \phi \quad \text{iff} \quad \mathcal{T}, x \Vdash \phi$$

Proof: We proceed by induction on the complexity of the wff ϕ . The cases for the boolean connectives is trivial, so we consider only the modal case. Let $\phi = \Box\psi$. Observe that the relation \prec has the property that \prec -successors

of a point x are exactly the smallest open set containing x . Moreover, in any Alexandroff space, $\Box\psi$ is true at x if and only if ψ is true in the smallest open set containing x . Consequently, we have shown that $\mathcal{M}, x \Vdash \Box\psi$ iff $\mathcal{T}, x \Vdash \Box\psi$ ■

Now the topological completeness theorem we are after follow readily.

Theorem 13: S4 is complete with respect to all Alexandroff topological models.

Proof: Consider any S4-consistent set of formulas Γ . Let Γ^+ be the maximally consistent extension of Γ . **Theorem 5** proves that Γ^+ is satisfied on pre-ordered frames. **By Theorem 10**, we know that there is an isomorphic relation between pre-orders on X and Alexandroff spaces on X . Thus, to prove this theorem it is enough to take the corresponding Alexandroff space as our model. This model will satisfy Γ and thus completes the proof. ■

Here we want to give a proof of Alexandroff's one-point Compactification using the properties of our new topological interpretation of modal logic.

Recall that a topological space $\langle X', \tau' \rangle$ is a subspace of a topological space $\langle X, \tau \rangle$ if, (1) $X' \subseteq X$ and, (2) $\tau' = \tau \upharpoonright X'$, read the restriction of τ to X' . We say that a topological space is *compact* if every open covering \mathcal{A} of X contains a finite subcover. That a space is compact is equivalent to saying that for every indexed family of closed subsets $\{\tau_i\}_{i \in I}$ of $\langle X, \tau \rangle$, if for every finite $J \subseteq I$, $\bigcap_{j \in J} \tau_j \neq \emptyset$, then $\bigcap_{i \in I} \tau_i \neq \emptyset$. The if part of the conditional is called the *finite intersection property* and the consequent of the conditional is simply called the *intersection property*. Thus, compactness may be stated as requiring that every family of closed sets which has the finite intersection property also has the intersection property. We will call a topological space *strongly compact* iff for each for every indexed family of open subsets $\{O_i\}_{i \in I}$, whenever $\bigcup_{i \in I} O_i = X$, there exists an $i \in I$ such that $O_i = X$. The dual for closed sets is the statement that for every indexed family of closed subsets $\{C_i\}_{i \in I}$, if for every $i \in I$, $C_i \neq \emptyset$, then $\bigcap_{i \in I} C_i \neq \emptyset$. [Rasiow 1]

3.3 One-Point-Compactification

Any boolean algebra with a unary operator that satisfies Kuratowski's axioms is called a *closure-algebra*. If one wants to put the emphasis on the interior operator, the algebra is called an *interior algebra*. Boolean algebra like those described were introduced by Tarski and Mckinsey. It should be clear that there is a direct correspondence between the axioms of S4 and the interior axioms above. Given this correspondence, we wish to prove two topological theorems which have nice modal analogues. The first is a closure algebraic version of the so-called *one-point Compactification* theorem.

Theorem 14 (One-point-Compactification): Every topological space is an open subspace of a strongly compact space X_0 such that $X_0 \setminus X$ is a one point set and the class of all open subsets of X_0 is composed of X_0 and all open subsets of X

Proof 1: Let \mathcal{X} be a topological space. Let x_0 be a arbitrary element such that $x_0 \notin X$. Set $X_0 = X \cup \{x_0\}$ and let

- (1) $int_0(X_0) = X_0$
- (2) $int_0(Y \cup \{x_0\}) = int(Y)$, for every $Y \subseteq X$,

$$(3) \quad \text{int}_0(Y) = \text{int}(Y), \text{ for every } Y \subseteq X$$

where int is the interior operator on the space X . It is clear that int_0 is an interior operation on X_0 and that the open sets in X_0 are the open subsets of X and the set X_0 itself. In particular, the X is an open subset of X_0 . Thus, if the union of an arbitrary family of open set in X_0 is equal to X_0 , then x_0 belongs to this union and hence x_0 belongs the at least one of the sets in the union. However, there is only one open set containing x_0 which is the whole space X_0 , Subsequently, it follows that at least one of the elements of the union must equal X_0 and thus X_0 is strongly compact. ■

The modal version is the following.

Theorem 15: Let $\mathcal{F} = \langle X, cl \rangle$ be a k-frame where X is non-empty and cl is a the closure operator. $\mathcal{F} = \langle X, cl \rangle$ is a topological space and $\langle X, cl \rangle$ is a subspace of a strongly compact topological space $\langle X', cl' \rangle$ such that $X' \setminus X$ is a one-element set and is closed.

Proof 2: Let x_0 be any point not in X , and let $X' = X \cup \{x_0\}$. Define $cl'(X) = cl(X) \cup \{x_0\}$. By the way we defined cl' . it is clear $\langle X, cl \rangle$ is a subspace of $\langle X', cl' \rangle$ and is closed. ■

3.4 Conclusion

Our aim in this essay was to establish several completeness results on the modal logic S4. We wanted to avoid the traditional possible world semantics that has dominated intensional logics. With this in mind, we striped away the philosophical nuances inherited by the metaphysics of possible worlds and investigated only the underlying pre-order frame of S4. We then extended the standard completeness proof for this logic by showing that any pre-ordered frame can be transformed into a partially-ordered tree and that this new model is a model for S4. Consequently, we were able to establish that S4 is complete with respect to the class of partially ordered trees. The second extension was the observation that there is a one-to-one relationship between pre-orders on a set X and Alexandroff topologies on X . Having established this relationship, it becomes possible to take any S4 frame whose underlying frame is a pre-ordered set and substitute it for its corresponding Alexandroff space. We then construct a model on Alexandroff space which gives us a topological model for S4 and thus a new proof of completeness for S4 with respect to the class of all Alexandroff topological models.

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