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Reverse mathematics and uniformity in proofs without excluded middle

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REVERSE MATHEMATICS AND UNIFORMITY IN PROOFS WITHOUT EXCLUDED MIDDLE

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ABSTRACT. We show that when certain statements are provable in subsystems of constructive analysis using intuitionistic predicate calculus, related sequential statements are provable in weak classical subsystems. In particular, if a Π^1_2 sentence of a certain form is provable using E-HA$^\omega$ along with the axiom of choice and an independence of premise principle, the sequential form of the statement is provable in the classical system RCA. We obtain this and similar results using applications of modified realizability and the *Dialectica* interpretation. These results allow us to use techniques of classical reverse mathematics to demonstrate the unprovability of several mathematical principles in subsystems of constructive analysis.

1. Introduction

We study the relationship between systems of intuitionistic arithmetic in all finite types (without the law of the excluded middle) and weak subsystems of classical second order arithmetic. Our theorems give precise expressions of the informal idea that if a sentence $\forall X\, \exists Y\, \Phi(X, Y)$ is provable without the law of the excluded middle, then the proof should be sufficiently direct that the stronger *sequential form*

$$\forall \langle X_n \mid n \in \mathbb{N}\rangle\, \exists \langle Y_n \mid n \in \mathbb{N}\rangle\, \forall n\, \Phi(X_n, Y_n)$$

is provable in a weak subsystem of classical arithmetic. We call our theorems "uniformization results" because the provability of the sequential form demonstrates a kind of uniformity in the proof of the original sentence.

The subsystems of classical arithmetic of interest are RCA_0, which is well-known in Reverse Mathematics [12], and its extension RCA with additional induction axioms. These systems are closely related to computable analysis. In particular, both subsystems are satisfied in the model REC that has the set ω of standard natural numbers as its first order part and the collection of all computable subsets of ω as its second order part. When the conclusions of our uniformization results are viewed as statements about REC, they provide a link between constructive analysis and computable analysis.

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Moreover, because RCA_0 is the base system most often employed in Reverse Mathematics, our results also provide a link between the fields of Reverse Mathematics and constructive analysis. Full definitions of the subsystems of intuitionistic and classical arithmetic that we study are presented in section 2.

In section 3, we prove uniformization results using modified realizability, a well-known tool in proof theory. In particular, we show there is a system I_0 of intuitionistic arithmetic in all finite types such that whenever an $\forall\exists$ statement of a certain syntactic form is provable in I_0, its sequential form is provable in RCA_0 (Theorem 3.10). Moreover, the system I_0 contains the full scheme for the axiom of choice in all finite types, which is classically much stronger than RCA_0. We have attempted to make section 3 accessible to a general reader who is familiar with mathematical logic but possibly unfamiliar with modified realizability.

In section 4, we give several examples of theorems in classical mathematics that are provable in RCA_0 but not provable in I_0. These examples demonstrate empirically that the syntactic restrictions within our uniformization theorems are not excessively tight. Moreover, our uniformization theorems allow us to obtain these unprovability results simply by showing that the sequential versions of the statements are unprovable in RCA_0, which can be done using classical techniques common in Reverse Mathematics. In this way, we obtain results on unprovability in intuitionistic arithmetic solely through a combination of our uniformization theorems and the study of classical arithmetic. A reader who is willing to accept the results of section 3 should be able to skim that section and then proceed directly to section 4.

In section 5, we prove uniformization results for RCA_0 and RCA using the *Dialectica* interpretation of Gödel. These results allow us to add a Markov principle to the system of intuitionistic arithmetic in exchange for shrinking the class of formulas to which the theorems apply.

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2. Axiom systems

Our results make use of subsystems of intuitionistic and classical arithmetic in all finite types. The definitions of these systems rely on the standard type notation in which the type of a natural number is 0 and the type of a function from objects of type ρ to objects of type τ is $\rho \to \tau$. For example, the type of a function from numbers to numbers is $0 \to 0$. As is typical in the literature, we will use the types 1 and $0 \to 0$ interchangeably, essentially identifying sets with their characteristic functions. We will often write superscripts on quantified variables to indicate their type.

Full definitions of the following systems are given by Kohlenbach [8, section 3.4].

Definition 2.1. The system $\widehat{\mathsf{E\text{-}HA}}^{\omega}_{\restriction}$ is a theory of intuitionistic arithmetic in all finite types first defined by Feferman [2].

The language $\mathcal{L}(\widehat{\mathsf{E\text{-}HA}}^{\omega}_{\restriction})$ includes the constant 0; the successor, addition, and multiplication operations; terms for primitive recursion on variables of type 0; and the projection and substitution combinators (often denoted $\Pi_{\rho,\tau}$ and $\Sigma_{\delta,\rho,\tau}$ [8]) which allow terms to be defined using λ abstraction. For example, given $x \in \mathbb{N}$ and an argument list t, $\widehat{\mathsf{E\text{-}HA}}^{\omega}_{\restriction}$ includes a term for $\lambda t.x$, the constant function with value x.

The language includes equality as a primitive relation only for type 0 objects (natural numbers). Equality for higher types is defined pointwise in terms of equality of lower types, using the following extensionality scheme

$$\mathsf{E}\colon \forall x^{\rho} \forall y^{\rho} \forall z^{\rho\to\tau}\, (x =_{\rho} y \to z(x) =_{\tau} z(y)).$$

The axioms of $\widehat{\mathsf{E\text{-}HA}}^{\omega}_{\restriction}$ consist of this extensionality scheme, the basic arithmetical axioms, the defining axioms for the term-forming operators, and an axiom scheme for induction on quantifier-free formulas (which may have parameters of arbitrary types).

Definition 2.2 (Troelstra [13, 1.6.12])**.** The subsystem $\mathsf{E\text{-}HA}^{\omega}$ is an extension of $\widehat{\mathsf{E\text{-}HA}}^{\omega}_{\restriction}$ with additional terms and stronger induction axioms. Its language contains additional term-forming recursors R_{σ} for all types σ. Its new axioms include the definitions of these recursors and the full induction scheme

$$\mathsf{IA}\colon A(0) \to (\forall n(A(n) \to A(n+1)) \to \forall n A(n)),$$

in which A may have parameters of arbitrary types.

The following class of formulas will have an important role in our results. These are, informally, the formulas that have no existential commitments in intuitionistic systems.

Definition 2.3. A formula of $\mathcal{L}(\widehat{\mathsf{E\text{-}HA}}^{\omega}_{\restriction})$ is $\exists$-*free* if it is built from prime (that is, atomic) formulas using only universal quantification and the connectives $\wedge$ and $\to$. Here the symbol $\perp$ is treated as a prime formula, and a negated formula $\neg A$ is treated as an abbreviation for $A \to \perp$; thus $\exists$-free formulas may include both $\perp$ and $\neg$.

We will consider extensions of $\widehat{\mathsf{E\text{-}HA}}^{\omega}_{\restriction}$ and $\mathsf{E\text{-}HA}^{\omega}$ that include additonal axiom schemes. The following schemes have been discussed by Kohlenbach [8] and by Troelstra [13].

Definition 2.4. The following axiom schemes are defined in $\mathcal{L}(\mathsf{E\text{-}HA}^\omega)$. When we adjoin a scheme to $\widehat{\mathsf{E\text{-}HA}}^\omega_\restriction$, we implicitly restrict it to $\mathcal{L}(\widehat{\mathsf{E\text{-}HA}}^\omega_\restriction)$. The formulas in these schemes may have parameters of arbitrary types.

- *Axiom of Choice.* For any x and y of finite type,
$$\mathsf{AC}\colon \forall x\, \exists y A(x,y) \to \exists Y\, \forall x\, A(x, Y(x)).$$
- *Independence of premise for $\exists$-free formulas.* For x of any finite type, if A is $\exists$-free and does not contain x, then
$$\mathsf{IP}^\omega_{\mathrm{ef}}\colon (A \to \exists x B(x)) \to \exists x(A \to B(x)).$$
- *Independence of premise for universal formulas.* If A_0 is quantifier free, $\forall x$ represents a block of universal quantifiers, and y is of any type and is not free in $\forall x A_0(x)$, then
$$\mathsf{IP}^\omega_\forall\colon (\forall x A_0(x) \to \exists y B(y)) \to \exists y(\forall x A_0(x) \to B(y)).$$
- *Markov principle for quantifier-free formulas.* If A_0 is quantifier-free and $\exists x$ represents a block of existential quantifiers in any finite type, then
$$\mathsf{M}^\omega\colon \neg\neg\exists x A_0(x) \to \exists x A_0(x).$$

2.1. **Classical subsystems.** The full scheme AC for the axiom of choice in all finite types, which is commonly included in subsystems of intuitionistic arithmetic, becomes extremely strong in the presence of the law of the excluded middle. For this reason, we will be interested in the restricted choice scheme
$$\mathsf{QF\text{-}AC}^{\rho,\tau}\colon \forall x^\rho\, \exists y^\tau A_0(x,y) \to \exists Y^{\rho\to\tau}\, \forall x^\rho A_0(x, Y(x)),$$
where A_0 is a quantifier-free formula that may have parameters.

We obtain subsystems of classical arithmetic by adjoining forms of this scheme, along with the law of the excluded middle, to systems of intuitionistic arithmetic. Because these systems include the law of the excluded middle, they also include all of classical predicate calculus.

Definition 2.5. The system RCA^ω_0 consists of $\widehat{\mathsf{E\text{-}HA}}^\omega_\restriction$ plus $\mathsf{QF\text{-}AC}^{1,0}$ and the law of the excluded middle.

The system RCA^ω consists of $\mathsf{E\text{-}HA}^\omega$ (which includes full induction) plus $\mathsf{QF\text{-}AC}^{1,0}$ and the law of the excluded middle.

We are also interested in the following second order restrictions of these subsystems. Let $\widehat{\mathsf{E\text{-}HA}}^2_\restriction$ represent the restriction of $\widehat{\mathsf{E\text{-}HA}}^\omega_\restriction$ to formulas in which all variables are type 0 or 1, and let $\mathsf{E\text{-}HA}^2$ be the similar restriction of $\mathsf{E\text{-}HA}^\omega$ in which variables are limited to types 0 and 1 and the recursor constants are limited to those of type 0.

Definition 2.6. The system RCA_0 consists of $\widehat{\mathsf{E\text{-}HA}}^2_\restriction$ plus $\mathsf{QF\text{-}AC}^{0,0}$ and the law of the excluded middle.

The system RCA consists of $\mathsf{E\text{-}HA}^2$ (which includes the full induction scheme for formulas in its language) plus $\mathsf{QF\text{-}AC}^{0,0}$ and the law of the excluded middle.

The system RCA_0 (and hence also RCA_0^ω) is able to prove the induction scheme for Σ^0_1 formulas using $\mathsf{QF\text{-}AC}^{0,0}$ and primitive recursion on variables of type 0, as noted by Kohlenbach [7].

The following conservation results show that the second order subsystems RCA and RCA_0 have the same deductive strength for sentences in their restricted languages as the corresponding higher-type systems RCA^ω and RCA_0^ω, respectively.

Theorem 2.7. [7, Proposition 3.1] *For every sentence Φ in $\mathcal{L}(\mathsf{RCA}_0)$, if $\mathsf{RCA}_0^\omega \vdash \Phi$ then $\mathsf{RCA}_0 \vdash \Phi$.*

The proof of this theorem is based on a formalization of the extensional model of the hereditarily continuous functionals (ECF), as presented in section 2.6.5 of Troelstra [13]. The central notion is that continuous objects of higher type can be encoded by lower type objects. For example, if α is a functional of type $1 \to 0$ and α is continuous in the sense that the value of $\alpha(X)$ depends only on a finite initial segment of the characteristic function of X, then there is an *associated function* [5] of type $0 \to 0$ that encodes all the information needed to calculate values of α. Generalizing this notion, with each higher-type formula Φ we can associate a second order formula Φ_{ECF} that encodes the same information. The proof sketch for the following result indicates how this is applied to obtain conservation results.

Theorem 2.8. *For each sentence Φ in $\mathcal{L}(\mathsf{RCA})$, if $\mathsf{RCA}^\omega \vdash \Phi$ then $\mathsf{RCA} \vdash \Phi$.*

Proof. The proof proceeds in two steps. First, emulating section 2.6.5 and Theorem 2.6.10 of Troelstra [13], show that if $\mathsf{RCA}^\omega \vdash \Phi$ then $\mathsf{RCA} \vdash \Phi_{\mathsf{ECF}}$. Second, following Theorem 2.6.12 of Troelstra [13], prove that if Φ is in the language of RCA then $\mathsf{RCA} \vdash \Phi \leftrightarrow \Phi_{\mathsf{ECF}}$. $\square$

The classical axiomatization of RCA_0, presented by Simpson [12], uses the set-based language L_2 with the membership relation symbol $\in$, rather than the language based on function application used in $\widehat{\mathsf{E\text{-}HA}}^\omega_\restriction$. The systems defined above as RCA_0 is sometimes denoted RCA_0^2 to indicate it is a restriction of RCA_0^ω. As discussed by Kohlenbach [7], set-based RCA_0 and function-based RCA_0^2 are each included in a canonical definitional extension of the other, and the same holds for set-based RCA and function-based RCA^2. Throughout this paper, we use the functional variants of RCA_0 and RCA for convenience, knowing that our results apply equally to the traditionally axiomatized systems.

3. Modified realizability

Our most broadly applicable uniformization theorems are proved by an application of modified realizability, a technique introduced by Kreisel [9].

Excellent expositions on modified realizability are given by Kohlenbach [8] and Troelstra [13, 14]. Indeed, our proofs make use of only minute modifications of results stated in these sources.

Modified realizability is a scheme for matching each formula A with a formula $t \,\mathsf{mr}\, A$ with the intended meaning "the sequence of terms t realizes A."

Definition 3.1. Let A be a formula in $\mathcal{L}(\mathsf{E\text{-}HA}^\omega)$, and let x denote a possibly empty tuple of terms whose variables do not appear free in A. The formula $x \,\mathsf{mr}\, A$ is defined inductively as follows:

(1) $x \,\mathsf{mr}\, A$ is A, if x is empty and A is a prime formula.
(2) $x, y \,\mathsf{mr}(A \wedge B)$ is $x \,\mathsf{mr}\, A \wedge y \,\mathsf{mr}\, B$.
(3) $z^0, x, y \,\mathsf{mr}(A \vee B)$ is $(z = 0 \to x \,\mathsf{mr}\, A) \wedge (z \neq 0 \to y \,\mathsf{mr}\, B)$.
(4) $x \,\mathsf{mr}(A \to B)$ is $\forall y(y \,\mathsf{mr}\, A \to xy \,\mathsf{mr}\, B)$.
(5) $x \,\mathsf{mr}(\forall y^\rho A(y))$ is $\forall y^\rho(xy \,\mathsf{mr}\, A(y))$.
(6) $z^\rho, x \,\mathsf{mr}(\exists y^\rho A(y))$ is $x \,\mathsf{mr}\, A(z)$.

Note that if A is a prime formula then A and $t \,\mathsf{mr}\, A$ are identical; this is even true for $\exists$-free formulas if we ignore dummy quantifiers.

We prove each of our uniformization results in two steps. The first step shows that whenever an $\forall\exists$ statement is provable in a particular subsystem of intuitionistic arithmetic, we can find a sequence of terms that realize the statement. The second step shows that a classical subsystem is able to leverage the terms in the realizer to prove the sequential version of the original statement.

We begin with systems containing the full induction scheme. For the first step, we require the following theorem.

Theorem 3.2 ([8, Theorem 5.8])**.** *Let A be a formula in $\mathcal{L}(\mathsf{E\text{-}HA}^\omega)$. If*

$$\mathsf{E\text{-}HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_{\mathrm{ef}} \vdash A$$

then there is a tuple t of terms of $\mathcal{L}(\mathsf{E\text{-}HA}^\omega)$ such that $\mathsf{E\text{-}HA}^\omega \vdash t \,\mathsf{mr}\, A$.

For any formula A, $\mathsf{E\text{-}HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_{\mathrm{ef}}$ is able to prove $A \leftrightarrow \exists x(x \,\mathsf{mr}\, A)$. However, the deduction of A from $(t \,\mathsf{mr}\, A)$ directly in $\mathsf{E\text{-}HA}^\omega$ is only possible for some formulas.

Definition 3.3. Γ_1 is the collection of formulas in $\mathcal{L}(\mathsf{E\text{-}HA}^\omega)$ defined inductively as follows.

(1) All prime formulas are elements of Γ_1.
(2) If A and B are in Γ_1, then so are $A \wedge B$, $A \vee B$, $\forall x A$, and $\exists x A$.
(3) If A is $\exists$-free and B is in Γ_1, then $(\exists x A \to B)$ is in Γ_1, where $\exists x$ may represent a block of existential quantifiers.

The class Γ_1 is sometimes defined in terms of "negative" formulas [13, Definition 3.6.3], those which can be constructed from negated prime formulas by means of $\forall$, $\wedge$, $\to$, and $\perp$. In all the systems studied in this paper, every $\exists$-free formula is equivalent to the negative formula obtained by replacing each prime formula with its double negation. Thus the distinction between negative and $\exists$-free will not be significant.

The next lemma is proved by Kohlenbach [8, Lemma 5.20] and by Troelstra [13, Lemma 3.6.5]

Lemma 3.4. *For every formula A in $\mathcal{L}(\mathsf{E\text{-}HA}^\omega)$, if A is in Γ_1, then* $\mathsf{E\text{-}HA}^\omega \vdash (t \,\mathsf{mr}\, A) \to A$.

Applying Theorem 3.2 and Lemma 3.4, we now prove the following term extraction lemma, which is similar to the main theorem on term extraction via modified realizability (Theorem 5.13) of Kohlenbach [8]. Note that $\forall x\, \exists y\, A$ is in Γ_1 if and only if A is in Γ_1.

Lemma 3.5. *Let $\forall x^\rho\, \exists y^\tau A(x,y)$ be a sentence of $\mathcal{L}(\mathsf{E\text{-}HA}^\omega)$ in Γ_1, where ρ and τ are arbitrary types. If*

$$\mathsf{E\text{-}HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_{\mathrm{ef}} \vdash \forall x^\rho\, \exists y^\tau A(x,y),$$

then $\mathsf{RCA}^\omega \vdash \forall x^\rho A(x, t(x))$, where t is a suitable term of $\mathcal{L}(\mathsf{E\text{-}HA}^\omega)$.

Proof. Assume that $\mathsf{E\text{-}HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_{\mathrm{ef}} \vdash \forall x^\rho \exists y^\tau A(x,y)$ where $A(x,y)$ is in Γ_1. By Theorem 3.2, there is a tuple t of terms of $\mathcal{L}(\mathsf{E\text{-}HA}^\omega)$ such that $\mathsf{E\text{-}HA}^\omega$ proves $t \,\mathsf{mr}\, \forall x^\rho \exists y^\tau A(x,y)$. By clause (5) of Definition 3.1, $\mathsf{E\text{-}HA}^\omega \vdash \forall x^\rho (t(x) \,\mathsf{mr}\, \exists y^\tau A(x,y))$. By clause (6) of Definition 3.1, t has the form t_0, t_1 and $\mathsf{E\text{-}HA}^\omega \vdash \forall x^\rho [t_1(x) \,\mathsf{mr}\, A(x, t_0(x))]$. Because $A(x,y)$ is in Γ_1, Lemma 3.4 shows that $\mathsf{E\text{-}HA}^\omega \vdash \forall x^\rho A(x, t_0(x))$. Because RCA^ω is an extension of $\mathsf{E\text{-}HA}^\omega$, we see that $\mathsf{RCA}^\omega \vdash \forall x^\rho A(x, t_0(x))$. $\square$

We are now prepared to prove our first uniformization theorem.

Theorem 3.6. *Let $\forall x \exists y A(x,y)$ be a sentence of $\mathcal{L}(\mathsf{E\text{-}HA}^\omega)$ in Γ_1. If*

$$\mathsf{E\text{-}HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_{\mathrm{ef}} \vdash \forall x\, \exists y\, A(x,y),$$

then

$$\mathsf{RCA}^\omega \vdash \forall \langle x_n \mid n \in \mathbb{N}\rangle\, \exists \langle y_n \mid n \in \mathbb{N}\rangle\, \forall n\, A(x_n, y_n).$$

Furthermore, if x and y are both type 1 (set) variables, and the formula $\forall x\, \exists y A(x,y)$ is in $\mathcal{L}(\mathsf{RCA})$, then RCA^ω may be replaced by RCA in the implication.

Proof. Assume that $\mathsf{E\text{-}HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_{\mathrm{ef}} \vdash \forall x^\rho \exists y^\tau A(x,y)$. We may apply Lemma 3.5 to extract the term t such that $\mathsf{RCA}^\omega \vdash \forall x^\rho A(x, t(x))$. Working in RCA^ω, fix any sequence $\langle x_n \mid n \in \mathbb{N}\rangle$. This sequence is a function of type $0 \to \rho$, so by λ abstraction we can construct a function of type $0 \to \tau$ defined by $\lambda n.t(x_n)$. Taking $\langle y_n \mid n \in \mathbb{N}\rangle$ to be this sequence, we obtain $\forall n\, A(x_n, y_n)$. The final sentence of the theorem follows immediately from the fact that RCA^ω is a conservative extension of RCA for formulas in $\mathcal{L}(\mathsf{RCA})$. $\square$

We now turn to a variation of Theorem 3.6 that replaces $\mathsf{E\text{-}HA}^\omega$ and RCA^ω with $\widehat{\mathsf{E\text{-}HA}}^\omega_\restriction$ and RCA^ω_0, respectively. Lemmas 3.7 and 3.8 are proved by imitating the proofs of Theorem 3.2 and Lemma 3.4, respectively, as described in the first paragraph of section 5.2 of Kohlenbach [8].

Lemma 3.7. *Let A be a formula in $\mathcal{L}(\widehat{\textsf{E-HA}}^\omega_\restriction)$. If $\widehat{\textsf{E-HA}}^\omega_\restriction + \textsf{AC} + \textsf{IP}^\omega_{\text{ef}} \vdash A$, then there is a tuple t of terms of $\mathcal{L}(\widehat{\textsf{E-HA}}^\omega_\restriction)$ such that $\widehat{\textsf{E-HA}}^\omega_\restriction \vdash t \,\textsf{mr}\, A$.*

Lemma 3.8. *Let A be a formula of $\mathcal{L}(\widehat{\textsf{E-HA}}^\omega_\restriction)$. If A is in Γ_1, then $\widehat{\textsf{E-HA}}^\omega_\restriction \vdash (t \,\textsf{mr}\, A) \to A$.*

Lemma 3.9. *Let $\forall x^\rho \,\exists y^\tau A(x,y)$ be a sentence of $\mathcal{L}(\widehat{\textsf{E-HA}}^\omega_\restriction)$ in Γ_1, where ρ and τ are arbitrary types. If*

$$\widehat{\textsf{E-HA}}^\omega_\restriction + \textsf{AC} + \textsf{IP}^\omega_{\text{ef}} \vdash \forall x^\rho \exists y^\tau A(x,y),$$

then $\textsf{RCA}^\omega_0 \vdash \forall x^\rho A(x, t(x))$, where t is a suitable term of $\mathcal{L}(\widehat{\textsf{E-HA}}^\omega_\restriction)$.

Proof. Imitate the proof of Lemma 3.5, substituting Lemma 3.7 for Theorem 3.2 and Lemma 3.8 for Lemma 3.4. □

We now obtain our second uniformization theorem. This is the theorem discussed in the introduction, where I_0 refers to the theory $\widehat{\textsf{E-HA}}^\omega_\restriction + \textsf{AC} + \textsf{IP}^\omega_{\text{ef}}$.

Theorem 3.10. *Let $\forall x \,\exists y A(x,y)$ be a sentence of $\mathcal{L}(\widehat{\textsf{E-HA}}^\omega_\restriction)$ in Γ_1. If*

$$\widehat{\textsf{E-HA}}^\omega_\restriction + \textsf{AC} + \textsf{IP}^\omega_{\text{ef}} \vdash \forall x \exists y \, A(x,y),$$

then

$$\textsf{RCA}^\omega_0 \vdash \forall \langle x_n \mid n \in \mathbb{N} \rangle \,\exists \langle y_n \mid n \in \mathbb{N} \rangle \,\forall n \, A(x_n, y_n).$$

Furthermore, if x and y are both type 1 (set) variables, and the formula $\forall x \,\exists y A(x,y)$ is in $\mathcal{L}(\textsf{RCA}_0)$, then $\textsf{RCA}^\omega_0$ may be replaced by $\textsf{RCA}_0$ in the implication.

The proof is parallel to that of Theorem 3.6, which did not make use of induction or recursors on higher types. Theorem 2.7 serves as the conservation result to prove the final claim.

4. Unprovability results

We now demonstrate several theorems of core mathematics which are provable in $\textsf{RCA}_0$ but have sequential versions that are not provable in $\textsf{RCA}$. In light of Theorem 3.6, such theorems are not provable in $\textsf{E-HA}^\omega + \textsf{AC} + \textsf{IP}^\omega_{\text{ef}}$. Where possible, we carry out proofs using restricted induction, as this gives additional information on the proof-theoretic strength of the principles being studied. The terminology in the following theorem is well known; we give formal definitions as needed later in the section.

Theorem 4.1. *Each of the following statements is provable in $\textsf{RCA}_0$ but not provable in $\textsf{E-HA}^\omega + \textsf{AC} + \textsf{IP}^\omega_{\text{ef}}$.*

1. *Every 2×2 matrix has a Jordan decomposition.*
2. *Every quickly converging Cauchy sequence of rational numbers can be converted to a Dedekind cut representing the same real number.*
3. *Every enumerated filter on a countable poset can be extended to an unbounded enumerated filter.*

There are many other statements that are provable in RCA_0 but not $\mathsf{E\text{-}HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_{\mathrm{ef}}$; we have chosen these three to illustrate the what we believe to be the ubiquity of this phenomenon in various branches of core mathematics.

We will show that each of the statements (4.1.1)–(4.1.3) is unprovable in $\mathsf{E\text{-}HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_{\mathrm{ef}}$ by noting that each statement is in Γ_1 and showing that the sequential form of each statement implies a strong comprehension axiom over RCA_0. Because these strong comprehension axioms are not provable even with the added induction strength of RCA, we may apply Theorem 3.6 to obtain the desired results. The stronger comprehension axioms include weak König's lemma and the arithmetical comprehension scheme, which are discussed thoroughly by Simpson [12].

We begin with statement (4.1.1). We consider only finite square matrices whose entries are complex numbers represented by quickly converging Cauchy sequences. In RCA_0, we say that a matrix M *has a Jordan decomposition* if there are matrices (U, J) such that $M = UJU^{-1}$ and J is a matrix consisting of Jordan blocks. We call J the *Jordan canonical form* of M. The fundamental definitions and theorems regarding the Jordan canonical form are presented by Halmos [3, Section 58]. Careful formalization of (4.1.1) shows that this principle can be expressed by a Π^1_2 formula in Γ_1; the key point is that the assumptions on M, U, J, and U^{-1} can be expressed using only equality of real numbers, which requires only universal quantification.

Lemma 4.2. RCA_0 *proves that every* 2×2 *matrix has a Jordan decomposition.*

Proof. Let M be a 2×2 matrix. RCA_0 proves that the eigenvalues of M exist and that for each eigenvalue there is an eigenvector. (Compare Exercise II.4.11 of Simpson [12], which notes that the basics of linear algebra, including fundamental properties of Gaussian elimination, are provable in RCA_0.) If the eigenvalues of M are distinct, then the Jordan decomposition is trivial to compute from the eigenvalues and eigenvectors. If there is a unique eigenvalue and there are two linearly independent eigenvectors then the Jordan decomposition is similarly trivial to compute.

Suppose that M has a unique eigenvalue λ but not two linearly independent eigenvectors. Let u be any eigenvector and let $\{u, v\}$ be a basis. It follows that $(M - \lambda I)v = au + bv$ is nonzero. Now $(M - \lambda I)(au + bv) = b(M - \lambda I)v$, because u is an eigenvector of M with eigenvalue λ. This shows $(M - \lambda I)$ has eigenvalue b, which can only happen if $b = 0$, that is, if $(M - \lambda I)v$ is a scalar multiple of u. Thus $\{u, v\}$ is a chain of generalized eigenvectors of M; the Jordan decomposition can be computed directly from this chain. $\square$

It is not difficult to see that the previous proof makes use of the law of the excluded middle.

Remark 4.3. Proofs similar to that of Lemma 4.2 can be used to show that for each standard natural number n the principle that every $n \times n$ matrix has a Jordan decomposition is provable in RCA_0. We do not know whether the principle that every finite matrix has a Jordan decomposition is provable in RCA_0.

The next lemma is foreshadowed by previous research. It is well known that the function that sends a matrix to its Jordan decomposition is discontinuous. Kohlenbach [7] has shown that, over the extension RCA_0^ω of RCA_0 to all finite types, the existence of a higher-type object encoding a non-sequentially-continuous real-valued function implies the principle $\exists^2$. In turn, $\mathsf{RCA}^\omega + \exists^2$ proves every instance of the arithmetical comprehension scheme.

Lemma 4.4. *The following principle implies arithmetical comprehension over* RCA_0 *(and hence over* RCA*). For every sequence* $\langle M_i \mid i \in \mathbb{N}\rangle$ *of* 2×2 *real matrices, such that each matrix* M_i *has only real eigenvalues, there are sequences* $\langle U_i \mid i \in \mathbb{N}\rangle$ *and* $\langle J_i \mid i \in \mathbb{N}\rangle$ *such that* (U_i, J_i) *is a Jordan decomposition of* M_i *for all* $i \in \mathbb{N}$.

Proof. We first demonstrate a concrete example of the discontinuity of the Jordan form. For any real z, let $M(z)$ denote the matrix

$$M(z) = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}.$$

The matrix $M(0)$ is the identity matrix, and so is its Jordan canonical form. If $z \neq 0$ then $M(z)$ has the following Jordan decomposition:

$$M(z) = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix}^{-1}.$$

The crucial fact is that the entry in the upper-right-hand corner of the Jordan canonical form of $M(z)$ is 0 if $z = 0$ and 1 if $z \neq 0$.

Let h be an arbitrary function from $\mathbb{N}$ to $\mathbb{N}$. We will assume the principle of the theorem and show that the range of h exists; this is sufficient to establish the desired result. It is well known that RCA_0 can construct a function $n \mapsto z_n$ that assigns each n a quickly converging Cauchy sequence z_n such that, for all n, $z_n = 0$ if and only n is not in the range of h. Form a sequence of matrices $\langle M(z_n) \mid n \in \mathbb{N}\rangle$; according to the principle, there is an associated sequence of Jordan canonical forms. The upper-right-hand entry of each of these canonical forms is either 0 or 1, and it is possible to effectively decide between these two cases. Thus, in RCA_0, we may form the range of h using the sequence of Jordan canonical forms as a parameter. □

We now turn to statement (4.1.2). Recall that the standard formalization of the real numbers in RCA_0, as described by Simpson [12], makes use of quickly converging Cauchy sequences of rationals. Alternative formalizations of the real numbers may be considered, however. We define a *Dedekind* *cut* to be a subset Y of the rational numbers such that both Y and $\mathbb{Q} \setminus Y$ are nonempty, and if $p \in Y$ and $q < p$ then $q \in Y$. We say that a Dedekind cut Y is *equivalent* to a quickly converging Cauchy sequence $\langle a_i \mid i \in \mathbb{N}\rangle$ if any only if the equivalence

$$q \in Y \Leftrightarrow q \leq \lim_{i\to\infty} a_i$$

holds for every rational number q. Formalization of (4.1.2) shows that it is in Γ_1.

Hirst [4] has proved the following results that relate Cauchy sequences with Dedekind cuts. Together with Theorem 3.6, these results show that statement (4.1.2) is provable in RCA_0 but not $\mathsf{E\text{-}HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_{\mathrm{ef}}$.

Lemma 4.5 (Hirst [4, Corollary 4]). *The following is provable in* RCA_0. *For each quickly converging Cauchy sequence x there is an equivalent Dedekind cut.*

Lemma 4.6 (Hirst [4, Corollary 9]). *The following principle is equivalent to weak König's lemma over* RCA_0 *(and hence over* RCA*). For each sequence $\langle X_i \mid i \in \mathbb{N}\rangle$ of quickly converging Cauchy sequences there is a sequence $\langle Y_i \mid i \in \mathbb{N}\rangle$ of Dedekind cuts such that X_i is equivalent to Y_i for each $i \in \mathbb{N}$.*

Statement (4.1.3), which is our final application of Theorem 3.6, is related to countable posets. In RCA_0, we define a *countable poset* to be a set $P \subseteq \mathbb{N}$ with a coded binary relation $\preceq$ that is reflexive, antisymmetric, and transitive. A function $f\colon \mathbb{N} \to P$ is called an *enumerated filter* if for every $i, j \in \mathbb{N}$ there is a $k \in \mathbb{N}$ such that $f(k) \preceq f(i)$ and $f(k) \preceq f(j)$, and for every $q \in P$ if there is an $i \in \mathbb{N}$ such that $f(i) \preceq q$ then there is a $k \in \mathbb{N}$ such that $f(k) = q$. An enumerated filter is called *unbounded* if there is no $q \in P$ such that $q \prec f(i)$ for all $i \in \mathbb{N}$. An enumerated filter f *extends* a filter g if the range of g (viewed as a function) is a subset of the range of f. If we modify the usual definition of an enumerated filter to include an auxiliary function $h\colon \mathbb{N}^2 \to \mathbb{N}$ such that for all i and j, $f(h(i,j)) \preceq f(i)$ and $f(h(i,j)) \preceq f(j)$, then (4.1.3) is in Γ_1.

Mummert has proved the following two lemmas about extending filters to unbounded filters (see Lempp and Mummert [10] and the remarks after Lemma 4.1.1 of Mummert [11]). These lemmas show that (4.1.3) is provable in RCA_0 but not $\mathsf{E\text{-}HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_{\mathrm{ef}}$.

Lemma 4.7 (Lempp and Mummert [10, Theorem 3.5]). RCA_0 *proves that any enumerated filter on a countable poset can be extended to an unbounded enumerated filter.*

Lemma 4.8 (Lempp and Mummert [10, Theorem 3.6]). *The following statement is equivalent to arithmetical comprehension over* RCA_0 *(and hence over* RCA*). Given a sequence $\langle P_i \mid i \in \mathbb{N}\rangle$ of countable posets and a sequence $\langle f_i \mid i \in \mathbb{N}\rangle$ such that f_i is an enumerated filter on P_i for each $i \in \mathbb{N}$, there is a sequence $\langle g_i \mid i \in \mathbb{N}\rangle$ such that, for each $i \in \mathbb{N}$, g_i is an unbounded enumerated filter on P_i extending f_i.*

We close this section by noting that the proof-theoretic results of section 3 are proved by finitistic methods. Consequently, constructivists might accept arguments like those presented here to establish the non-provability of certain theorems from systems of intuitionistic arithmetic.

5. The *Dialectica* interpretation

In the proofs of section 3, applications of Gödel's *Dialectica* interpretation can replace the applications of modified realizability. One advantage of this substitution is that the constructive axiom system can be expanded to include the scheme M^ω, which formalizes a restriction of the Markov principle.

This gain has associated costs. First, the class of formulas for which the uniformization results hold is restricted from Γ_1 to the smaller class Γ_2 defined below. Second, the independence of premise principle $\mathsf{IP}^\omega_{\mathrm{ef}}$ is replaced with the weaker principle $\mathsf{IP}^\omega_\forall$. Finally, the extensionality scheme E is replaced with a weaker rule of inference

$$\mathsf{QF\text{-}ER}\colon \text{From } A_0 \to s =_\rho t \text{ deduce } A_0 \to r[s/x^\rho] =_\tau r[t/x^\rho],$$

where A_0 is quantifier free and $r[s/x^\rho]$ denotes the result of replacing the variable x of type ρ by the term s of type ρ in the term r of type τ. We denote the systems based on this rule of inference as $\widehat{\mathsf{WE\text{-}HA}}^\omega_{\restriction}$ and $\mathsf{WE\text{-}HA}^\omega$.

Extended discussions of Gödel's *Dialectica* interpretation are given by Avigad and Feferman [1], Kohlenbach [8], and Troelstra [13]. The interpretation assigns to each formula A a formula A^D of the form $\exists x \forall y\, A_D$, where A_D is quantifier free and each quantifier may represent a block of quantifiers of the same kind. The blocks of quantifiers in A^D may include variables of any finite type.

Definition 5.1. We follow Avigad and Feferman [1] in defining the *Dialectica* interpretation inductively via the following six clauses, in which $A^D = \exists x \forall y\, A_D$ and $B^D = \exists u \forall v\, B_D$.

(1) If A a prime formula then x and y are both empty and $A^D = A_D = A$.
(2) $(A \wedge B)^D = \exists x \exists u \forall y \forall v\, (A_D \wedge B_D)$.
(3) $(A \vee B)^D = \exists z \exists x \exists u \forall y \forall v\, ((z = 0 \wedge A_D) \vee (z = 1 \wedge B_D))$.
(4) $(\forall z\, A(z))^D = \exists X \forall z \forall y\, A_D(X(z), y, z)$.
(5) $(\exists z\, A(z))^D = \exists z \exists x \forall y\, A_D(x, y, z)$.
(6) $(A \to B)^D = \exists U \exists Y \forall x \forall v\, (A_D(x, Y(x, v)) \to B_D(U(x), v))$.

A negated formula $\neg A$ is treated as an abbreviation of $A \to \bot$.

We begin our derivation of the uniformization results with a soundness theorem of Gödel that is analogous to Theorem 3.2. A detailed proof is given by Kohlenbach [8, Theorem 8.6].

Theorem 5.2. *Let A be a formula in $\mathcal{L}(\mathsf{WE\text{-}HA}^\omega)$. If*

$$\mathsf{WE\text{-}HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_\forall + \mathsf{M}^\omega \vdash \forall x\, \exists y A(x, y),$$

then $\mathsf{WE\text{-}HA}^\omega \vdash \forall x A_D(x, t(x))$, *where* t *is a suitable term of* $\mathsf{WE\text{-}HA}^\omega$.

To prove our uniformization result, we will need to convert A^D back to A. Unfortunately, RCA^ω can only prove $A^D \to A$ for certain formulas. The class Γ_2, as found in (for example) Definition 8.10 of Kohlenbach [8], is a subset of these formulas.

Definition 5.3. Γ_2 is the collection of formulas in $\mathcal{L}(\mathsf{WE\text{-}HA}^\omega)$ defined inductively as follows.

(1) All prime formulas are elements of Γ_2.
(2) If A and B are in Γ_2, then so are $A \wedge B$, $A \vee B$, $\forall x A$, and $\exists x A$.
(3) If A is purely universal and $B \in \Gamma_2$, then $(\exists x A \to B) \in \Gamma_2$, where $\exists x$ may represent a block of existential quantifiers.

Kohlenbach [8, Lemma 8.11] states the following result for $\mathsf{WE\text{-}HA}^\omega$. Since RCA^ω is an extension of $\mathsf{WE\text{-}HA}^\omega$, this suffices for the proof of the uniformization result, where it acts as an analog of Lemma 3.4.

Lemma 5.4. *Let* A *be a formula of* $\mathcal{L}(\mathsf{WE\text{-}HA}^\omega)$ *in* Γ_2. *Then* $\mathsf{WE\text{-}HA}^\omega \vdash A^D \to A$. *This result also holds for* $\widehat{\mathsf{WE\text{-}HA}}{}^\omega_\restriction$ *for formulas in* $\mathcal{L}(\widehat{\mathsf{WE\text{-}HA}}{}^\omega_\restriction)$.

Proof. The proof is carried out by an external induction on formula complexity with cases based on the clauses in the definition of Γ_2. For details, see the proof of part (iii) of Lemma 3.6.5 in Troelstra [13]. The proof of each clause depends only on the definition of the *Dialectica* interpretation and intuitionistic predicate calculus. Consequently, the same argument can be carried out in $\widehat{\mathsf{WE\text{-}HA}}{}^\omega_\restriction$. $\square$

We can adapt our proof of Lemma 3.5 to obtain the following term extraction result.

Lemma 5.5. *Let* $\forall x^\rho \exists y^\tau A(x, y)$ *be a sentence of* $\mathcal{L}(\mathsf{WE\text{-}HA}^\omega)$ *in* Γ_2 *with arbitrary types* ρ *and* τ. *If* $\mathsf{WE\text{-}HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_\forall + \mathsf{M}^\omega \vdash \forall x^\rho \exists y^\tau A(x, y)$, *then* $\mathsf{RCA}^\omega \vdash \forall x^\rho A(x, t(x))$, *where* t *is a suitable term of* $\mathsf{WE\text{-}HA}^\omega$.

Substituting Lemma 5.5 for the use of Lemma 3.5 in the proof of Theorem 3.6, we obtain a proof of the *Dialectica* version of our uniformization result.

Theorem 5.6. *Let* $\forall x \exists y A(x, y)$ *be a sentence of* $\mathcal{L}(\mathsf{WE\text{-}HA}^\omega)$ *in* Γ_2. *If*

$$\mathsf{WE\text{-}HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_\forall + \mathsf{M}^\omega \vdash \forall x \exists y\, A(x, y),$$

then

$$\mathsf{RCA}^\omega \vdash \forall \langle x_n \mid n \in \mathbb{N}\rangle\, \exists \langle y_n \mid n \in \mathbb{N}\rangle \forall n\, A(x_n, y_n).$$

Furthermore, if x *and* y *are both type* 1 *(set) variables, and* $\forall x \exists y A(x, y)$ *is in* $\mathcal{L}(\mathsf{RCA})$, *then* RCA^ω *may be replaced by* RCA *in the implication.*

As was the case in section 3, these results can be recast in settings with restricted induction. As noted by Kohlenbach [8, section 8.3], Theorem 5.2

also holds with $\mathsf{WE\text{-}HA}^\omega$ replaced by $\widehat{\mathsf{WE\text{-}HA}}^\omega_\restriction$. Applying the restricted-induction version of Lemma 5.4 leads to the restricted form of Lemma 5.5. Combining this with the conservation result for RCA_0^ω over RCA_0 (Theorem 2.7) leads to a proof of the following version of Theorem 5.6.

Theorem 5.7. *Let $\forall x \exists y A(x,y)$ be a sentence of $\mathcal{L}(\widehat{\mathsf{WE\text{-}HA}}^\omega_\restriction)$ in Γ_2. If*

$$\widehat{\mathsf{WE\text{-}HA}}^\omega_\restriction + \mathsf{AC} + \mathsf{IP}^\omega_\forall + \mathsf{M}^\omega \vdash \forall x\, \exists y\, A(x,y),$$

then

$$\mathsf{RCA}_0^\omega \vdash \forall \langle x_n \mid n \in \mathbb{N}\rangle\, \exists \langle y_n \mid n \in \mathbb{N}\rangle \forall n\, A(x_n, y_n).$$

Furthermore, if x and y are both type 1 (set) variables, and $\forall x \exists y A(x,y)$ is in $\mathcal{L}(\mathsf{RCA}_0)$, then RCA_0^ω may be replaced by RCA_0 in the implication.

Uniformization results obtained by the *Dialectica* interpretation are less broadly applicable than those obtained by modified realizability, due to the fact that Γ_2 is a proper subset of Γ_1. In practice, however, the restriction to Γ_2 may not be such a serious impediment. Examination of the statements in Theorem 4.1 shows that the hypotheses in their implications are purely universal, and consequently each of the statements is in Γ_2. Thus an application of Theorem 5.6 shows that Theorem 4.1 holds with $\mathsf{E\text{-}HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_{\mathrm{ef}}$ replaced by $\mathsf{WE\text{-}HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_\forall + \mathsf{M}^\omega$.

While Γ_2 may not be the largest class of formulas for which an analog of Theorem 5.7 can be obtained, any class substituted for Γ_2 must omit a substantial collection of formulas. For example, imitating the proof of Kohlenbach [6], working in $\widehat{\mathsf{WE\text{-}HA}}^\omega_\restriction + \mathsf{AC}$ one can deduce the Π^0_n collection schemes, also known as $\mathsf{B}\Pi^0_n$. These schemes contain formulas that are not provable in RCA_0, and any class of formulas for which Theorem 5.7 holds must omit such formulas. The same observation holds for Theorem 3.10.

References

[1] Jeremy Avigad and Solomon Feferman, *Gödel's functional ("Dialectica") interpretation*, Handbook of proof theory, Stud. Logic Found. Math., vol. 137, North-Holland, Amsterdam, 1998, pp. 337–405. MR1640329 (2000b:03204)

[2] S. Feferman, *Theories of finite type related to mathematical practice*, Handbook of mathematical logic, North-Holland, Amsterdam, 1977, pp. 913–971.

[3] Paul R. Halmos, *Finite-dimensional vector spaces*, The University Series in Undergraduate Mathematics, D. Van Nostrand Co., Inc., Princeton-Toronto-New York-London, 1958. 2nd ed. MR0089819 (19,725b)

[4] Jeffry L. Hirst, *Representations of reals in reverse mathematics*, Bull. Pol. Acad. Sci. Math. **55** (2007), no. 4, 303–316. MR2369116

[5] S. C. Kleene, *Countable functionals*, Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting), Studies in Logic and the Foundations of Mathematics, North-Holland Publishing Co., Amsterdam, 1959, pp. 81–100. MR0112837 (22 #3686)

[6] Ulrich Kohlenbach, *A note on Goodman's theorem*, Studia Logica **63** (1999), no. 1, 1–5. MR1742380 (2000m:03150)

[7] ______, *Higher order reverse mathematics*, Reverse mathematics 2001, Lect. Notes Log., vol. 21, Assoc. Symbol. Logic, La Jolla, CA, 2005, pp. 281–295. MR2185441 (2006f:03109)

[8] ______, *Applied proof theory: proof interpretations and their use in mathematics*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2008. MR2445721 (2009k:03003)

[9] Georg Kreisel, *Interpretation of analysis by means of constructive functionals of finite types*, Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting), Studies in Logic and the Foundations of Mathematics, North-Holland Publishing Co., Amsterdam, 1959, pp. 101–128. MR0106838 (21 #5568)

[10] Steffen Lempp and Carl Mummert, *Filters on computable posets*, Notre Dame J. Formal Logic **47** (2006), no. 4, 479–485. MR2272083 (2007j:03084)

[11] Carl Mummert, *On the reverse mathematics of general topology*, Ph.D. Thesis, The Pennsylvania State University, 2005.

[12] Stephen G. Simpson, *Subsystems of second order arithmetic*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1999. MR1723993 (2001i:03126)

[13] A. S. Troelstra (ed.), *Metamathematical investigation of intuitionistic arithmetic and analysis*, Lecture Notes in Mathematics, Vol. 344, Springer-Verlag, Berlin, 1973. MR0325352 (48 #3699)

[14] ______, *Realizability*, Handbook of proof theory, Stud. Logic Found. Math., vol. 137, North-Holland, Amsterdam, 1998, pp. 407–473. MR1640330 (99f:03084)

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