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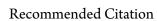
Department of Computer Science

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Yuliya Lierler
University of Nebraska at Omaha, ylierler@unomaha.edu

Esta Erdem
Sabancı University

Vladimir Lifschitz
University of Texas at Austin

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Fages' Theorem
and Answer Set Programming

Yuliya Babovich, Esra Erdem and Vladimir Lifschitz
Department of Computer Sciences
University of Texas at Austin
Austin, TX 78712, USA
Email: {yuliya,esra,vl}@cs.utexas.edu

Abstract

We generalize a theorem by François Fages that describes the relationship between the completion semantics and the answer set semantics for logic programs with negation as failure. The study of this relationship is important in connection with the emergence of answer set programming. Whenever the two semantics are equivalent, answer sets can be computed by a satisfiability solver, and the use of answer set solvers such as SMODELS and DLV is unnecessary. A logic programming representation of the blocks world due to Ilkka Niemelä is discussed as an example.

Introduction

This note is about the relationship between the completion semantics (Clark 1978) and the answer set ("stable model") semantics (Gelfond & Lifschitz 1991) for logic programs with negation as failure. The study of this relationship is important in connection with the emergence of answer set programming (Marek & Truszczyński 1999; Niemelä 1999; Lifschitz 1999). Whenever the two semantics are equivalent, answer sets can be computed by a satisfiability solver, and the use of "answer set solvers" such as SMODELS[1] and DLV[2] is unnecessary.

Consider a finite propositional (or grounded) program Π without classical negation, and a set X of atoms. If X is an answer set for Π then X, viewed as a truth assignment, satisfies the completion of Π. The converse, generally, is not true. For instance, the completion of

$$p \leftarrow p \qquad (1)$$

is $p \equiv p$. This formula has two models $\emptyset$, $\{p\}$; the first is an answer set for (1), but the second is not. François Fages [1994] defined a syntactic condition on logic programs that implies the equivalence between the two semantics—"positive-order-consistency," also called "tightness" (Lifschitz 1996). What he requires is the existence of a function λ from atoms to nonnegative integers (or, more generally, ordinals) such that,

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[1] http://saturn.hut.fi/pub/smodels/ .

[2] http://www.dbai.tuwien.ac.at/proj/dlv/ .

for every rule

$$A_0 \leftarrow A_1, \ldots, A_m, not\ A_{m+1}, \ldots, not\ A_n$$

in Π,

$$\lambda(A_1), \ldots, \lambda(A_m) < \lambda(A_0).$$

It is clear, for instance, that program (1) is not tight. Fages proved that, for a tight program, every model of its completion is an answer set. Thus, for tight programs, the completion semantics and the answer set semantics are equivalent.

Our generalization of Fages' theorem allows us to draw similar conclusions for some programs that are not tight. Here is one such program:

$$
\begin{aligned}
p &\leftarrow not\ q, \\
q &\leftarrow not\ p, \qquad (2) \\
p &\leftarrow p, r.
\end{aligned}
$$

It is not tight. Nevertheless, each of the two models $\{p\}$, $\{q\}$ of its completion

$$
\begin{aligned}
p &\equiv \neg q \vee (p \wedge r), \\
q &\equiv \neg p, \\
r &\equiv \bot
\end{aligned}
$$

is an answer set for (2).

The idea of this generalization is to make function λ partial. Instead of tight programs, we will consider programs that are "tight on a set of literals."

First we relate answer sets to a model-theoretic counterpart of completion introduced in (Apt, Blair, & Walker 1988), called supportedness. This allows us to make the theorem applicable to programs with both negation as failure and classical negation, and to programs with infinitely many rules.[3] Then a corollary about completion is derived, and applied to a logic programming representation of the blocks world due to Ilkka Niemelä. We show how the satisfiability solver SATO (Zhang 1997) can be used to find answer sets for that representation, and compare the performance of SMODELS and SATO on several benchmarks.

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[3] The familiar definition of completion (see Appendix) is applicable to finite programs only, unless we allow infinite disjunctions in completion formulas.

Generalized Fages' Theorem

We define a *rule* to be an expression of the form

$$Head \leftarrow L_1, \ldots, L_m, not\ L_{m+1}, \ldots, not\ L_n \quad (3)$$

($n \geq m \geq 0$) where each L_i is a literal (propositional atom possibly preceded by classical negation $\neg$), and *Head* is a literal or the symbol $\bot$. A rule (3) is called a *fact* if $n = 0$, and a *constraint* if $Head = \bot$. A *program* is a set of rules. The familiar definitions of answer sets, closed sets and supported sets for a program, as well as the definition of the completion of a program, are reproduced in the appendix.

Instead of "level mappings" used by Fages, we consider here *partial level mappings*—partial functions from literals to ordinals. A program Π is *tight* on a set X of literals if there exists a partial level mapping λ with the domain X such that, for every rule (3) in Π, if $Head, L_1, \ldots, L_m \in X$ then

$$\lambda(L_1), \ldots, \lambda(L_m) < \lambda(Head).$$

(For the constraints in Π this condition holds trivially, because the head of a constraint is not a literal and thus cannot belong to X.)

Theorem. *For any program Π and any consistent set X of literals such that Π is tight on X, X is an answer set for Π iff X is closed under and supported by Π.*

The proof below is almost unchanged from the proof of Fages' theorem given in (Lifschitz & Turner 1999, Section 7.4).

Lemma. *For any program Π without negation as failure and any consistent set X of literals such that Π is tight on X, if X is closed under and supported by Π, then X is an answer set for Π.*

Proof: We need to show that X is minimal among the sets closed under Π. Assume that it is not. Let Y be a proper subset of X that is closed under Π, and let λ be a partial level mapping establishing that Π is tight on X. Take a literal $L \in X \setminus Y$ such that $\lambda(L)$ is minimal. Since X is supported by Π, there is a rule

$$L \leftarrow L_1, \ldots, L_m$$

in Π such that $L_1, \ldots, L_m \in X$. By the choice of λ,

$$\lambda(L_1), \ldots, \lambda(L_m) < \lambda(L).$$

By the choice of L, we can conclude that

$$L_1, \ldots, L_m \in Y.$$

Consequently Y is not closed under Π, contrary to the choice of Y.

Proof of the Theorem: Left-to-right, the proof is straightforward. Right-to-left: assume that X is closed under and supported by Π. Then X is closed under and supported by Π^X. Since Π is tight on X, so is Π^X. Hence, by the lemma, X is an answer set for Π^X, and consequently an answer set for Π.

In the special case when Π is a finite program without classical negation, a set of atoms satisfies the completion of Π iff it is closed under and supported by Π. We conclude:

Corollary 1. *For any finite program Π without classical negation and any set X of atoms such that Π is tight on X, X is an answer set for Π iff X satisfies the completion of Π.*

For instance, program (2) is tight on the model $\{p\}$ of its completion: take $\lambda(p) = 0$. By Corollary 1, it follows that $\{p\}$ is an answer set for (2). In a similar way, the theorem shows that $\{q\}$ is an answer set also.

By $pos(\Pi)$ we denote the set of all literals that occur without negation as failure at least once in the body of a rule of Π.

Corollary 2. *For any program Π and any consistent set X of literals disjoint from $pos(\Pi)$, X is an answer set for Π iff X is closed under and supported by Π.*

Corollary 3. *For any finite program Π without classical negation and any set X of atoms disjoint from $pos(\Pi)$, X is an answer set for Π iff X satisfies the completion of Π.*

To derive Corollary 2 from the theorem, and Corollary 3 from Corollary 1, take $\lambda(L) = 0$ for every $L \in X$.

Consider, for instance, the program

$$
\begin{aligned}
p &\leftarrow not\ q, \\
q &\leftarrow not\ p, \\
r &\leftarrow r, \\
p &\leftarrow r.
\end{aligned}
\quad (4)
$$

The completion of (4) is

$$
\begin{aligned}
p &\equiv \neg q \vee r, \\
q &\equiv \neg p, \\
r &\equiv r.
\end{aligned}
$$

The models of these formulas are $\{p\}$, $\{q\}$ and $\{p, r\}$. The only literal occurring in the bodies of the rules of (4) without negation as failure is r. In accordance with Corollary 3, the models of the completion that do not contain r—sets $\{p\}$ and $\{q\}$—are answer sets for (4).

Planning in the Blocks World

As a more interesting example, consider a logic programming encoding of the blocks world due to Ilkka Niemelä. The main part of the encoding consists of the following schematic rules:

```
goal :- time(T), goal(T).
:- not goal.

goal(T2) :- nextstate(T2,T1), goal(T1).

moveop(X,Y,T):-
    time(T), block(X), object(Y), X != Y,
    on_something(X,T), available(Y,T),
    not covered(X,T), not covered(Y,T),
    not blocked_move(X,Y,T).
```

```
on(X,Y,T2) :-
   block(X), object(Y), nextstate(T2,T1),
   moveop(X,Y,T1).

on_something(X,T) :-
   block(X), object(Z), time(T), on(X,Z,T).

available(table,T) :- time(T).

available(X,T) :-
   block(X), time(T), on_something(X,T).

covered(X,T) :-
   block(Z), block(X), time(T), on(Z,X,T).

on(X,Y,T2) :-
   nextstate(T2,T1), block(X), object(Y),
   on(X,Y,T1), not moving(X,T1).

moving(X,T) :- time(T), block(X), object(Y),
   moveop(X,Y,T).

blocked_move(X,Y,T):-
   block(X), object(Y), time(T), goal(T).

blocked_move(X,Y,T) :-
   time(T), block(X), object(Y),
   not moveop(X,Y,T).

blocked_move(X,Y,T) :-
   block(X), object(Y), object(Z), time(T),
   moveop(X,Z,T), Y != Z.

blocked_move(X,Y,T) :-
   block(X), object(Y), time(T), moving(Y,T).

blocked_move(X,Y,T) :-
   block(X), block(Y), block(Z), time(T),
   moveop(Z,Y,T), X != Z.

:- block(X), time(T), moveop(X,table,T),
   on(X,table,T).

:- nextstate(T2,T1), block(X), object(Y),
   moveop(X,Y,T1), moveop(X,table,T2).

nextstate(Y,X) :- time(X), time(Y),
   Y = X + 1.

object(table).
object(X) :- block(X).
```

To solve a planning problem, we combine these rules
with

(i) a set of facts defining `time/1` as an initial segment of
nonnegative integers, for instance

```
time(0).   time(1).   time(2).
```

(ii) a set of facts defining `block/1`, such as

```
block(a).  block(b).  block(c).
```

(iii) a set of facts encoding the initial state, such as

```
on(a,b,0).  on(b,table,0).
```

(iv) a rule that encodes the goal, such as

```
goal(T) :- time(T), on(a,b,T), on(b,c,T).
```

The union is given as input to the "intelligent ground-
ing" program LPARSE, and the result of grounding is
passed on to SMODELS (Niemelä 1999, Section 7). The
answer sets for the program correspond to valid plans.

Concurrently executed actions are allowed in this for-
malization as long as their effects are not in conflict, so
that they can be arbitrarily interleaved.

The schematic rules above contain the variables `T`,
`T1`, `T2`, `X`, `Y`, `Z` that range over the object constants
occurring in the program, that is, over the nonnega-
tive integers that occur in the definition of `time/1`, the
names of blocks `a`, `b`,... that occur in the definition of
`block/1`, and the object constant `table`.

The expressions in the bodies of the schematic rules
that contain `=` and `!=` restrict the constants that are
substituted for the variables in the process of grounding.
For instance, we understand the schematic rule

```
nextstate(Y,X) :- time(X), time(Y),
   Y = X + 1.
```

as an abbreviation for the set of all ground instances of

```
nextstate(Y,X) :- time(X), time(Y).
```

in which `X` and `Y` are instantiated by a pair of consecu-
tive integers. The schematic rule

```
blocked_move(X,Y,T) :-
   block(X), object(Y), object(Z), time(T),
   moveop(X,Z,T), Y != Z.
```

stands for the set of all ground instances of

```
blocked_move(X,Y,T) :-
   block(X), object(Y), object(Z), time(T),
   moveop(X,Z,T).
```

in which `Y` and `Z` are instantiated by different object
constants.

According to this understanding of variables and
"built-in predicates," Niemelä's schematic program, in-
cluding rules (i)–(iv), is an abbreviation for a finite pro-
gram BW in the sense defined above.

In the proposition below we assume that schematic
rule (iv) has the form

```
goal(T) :- time(T), ...
```

where the dots stand for a list of schematic atoms with
the predicate symbol `on` and the last argument `T`.

Proposition. *Program BW is tight on each of the
models of its completion.*

Lemma. *For any atom of the form* `nextstate(Y,X)` *that belongs to a model of the completion of program* BW, $Y = X + 1$.

Proof: The completion of BW contains the formula

$$\texttt{nextstate(Y,X)} \equiv \texttt{false}$$

for all `Y`, `X` such that $Y \neq X + 1$.

Proof of the Proposition. Let X be an answer set for BW. By T_{max} we denote the largest argument of `time/1` in its definition (i). Consider the partial level mapping λ with domain X defined as follows:

$$\lambda(\texttt{time(T)}) = 0,$$
$$\lambda(\texttt{block(X)}) = 0,$$
$$\lambda(\texttt{object(X)}) = 1,$$
$$\lambda(\texttt{nextstate(Y,X)}) = 1,$$
$$\lambda(\texttt{covered(X,T)}) = 4 \cdot \texttt{T} + 3,$$
$$\lambda(\texttt{on\_something(X,T)}) = 4 \cdot \texttt{T} + 3,$$
$$\lambda(\texttt{available(X,T)}) = 4 \cdot \texttt{T} + 4,$$
$$\lambda(\texttt{moveop(X,Y,T)}) = 4 \cdot \texttt{T} + 5,$$
$$\lambda(\texttt{on(X,Y,T)}) = 4 \cdot \texttt{T} + 2,$$
$$\lambda(\texttt{moving(X,T)}) = 4 \cdot \texttt{T} + 6,$$
$$\lambda(\texttt{goal(T)}) = 4 \cdot \texttt{T} + 3,$$
$$\lambda(\texttt{blocked\_move(X,Y,T)}) = 4 \cdot \texttt{T} + 7,$$
$$\lambda(\texttt{goal}) = 4 \cdot T_{max} + 4.$$

This level mapping satisfies the inequality from the definition of a tight program for every rule of BW; the lemma above allows us to verify this assertion for the rules containing `nextstate` in the body.

According to Corollary 1, we can conclude that the answer sets for program BW can be equivalently characterized as the models of the completion of BW.

Answer Set Programming with CCALC and SATO

The equivalence of the completion semantics to the answer set semantics for program BW shows that it is not necessary to use an answer set solver, such as SMODELS, to compute answer sets for BW; a satisfiability solver can be used instead. Planning by giving the completion of BW as input to a satisfiability solver is a form of answer set programming and, at the same time, a special case of satisfiability planning (Kautz & Selman 1992).

The Causal Calculator, or CCALC[4], is a system that is capable, among other things, of grounding and completing a schematic logic program, clausifying the completion, and calling a satisfiability solver (for instance, SATO) to find a model. We have conducted a series of experiments aimed at comparing the run times of SATO, when its input is generated from BW by CCALC, with the run times of SMODELS, when its input is generated from BW by LPARSE.

Because the built-in arithmetic of CCALC is somewhat different from that of LPARSE, we had to modify BW

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[4] `http://www.cs.utexas.edu/users/tag/ccalc/` .

Problem	Blocks	Steps	Run time of SMODELS	Run time of SATO
large.c	15	7	9.86	1.82
		8	31.25	2.16
large.d	17	8	18.25	2.96
		9	62.48	4.14
large.e	19	9	27.31	5.40
		10	101.4	7.16

Figure 1: Planning with BW: SATO vs. SMODELS

slightly. Our CCALC input file uses variables of sorts `object`, `block` and `time` instead of the unary predicates with these names. The rules of BW that contain those predicates in their bodies are modified accordingly. For instance, rule

```
on_something(X,T) :-
    block(X), object(Z), time(T), on(X,Z,T).
```

turns into

```
on_something(B1,T) :- on(B1,O2,T).
```

The macro expansion facility of CCALC expands

```
nextstate(T2,T1)
```

into the expression

```
T2 is T1 + 1
```

that contains Prolog's built-in `is`.

Figure 1 shows the run times of SMODELS (Version 2.25) and SATO (Version 3.1.2) in seconds, measured using the Unix `time` command, on the benchmarks from (Niemelä 1999, Section 9, Table 3). For each problem, one of the two entries corresponds to the largest number of steps for which the problem is not solvable, and the other to the smallest number of steps for which a solution exists. The experiments were performed on an UltraSPARC with 124 MB main memory and a 167 MHz CPU.

The numbers in Figure 1 are "search times"—the grounding and completion times are not included. The computation involved in grounding and completion does not depend on the initial state or the goal of the planning problem and, in this sense, can be viewed as "preprocessing." LPARSE performs grounding more efficiently than CCALC, partly because the former is written in C++ and the latter in Prolog. The last benchmark in Figure 1 was grounded by LPARSE (Version 0.99.49) in 16 seconds; CCALC (Version 1.23) spent 50 seconds in grounding and about the same amount of time forming the completion. But the size of the grounded program is approximately the same in both cases: LPARSE generated 191621 rules containing 13422 atoms, and CCALC generated 200661 rules containing 13410 atoms.

Discussion

Fages' theorem, and its generalization proved in this note, allow us to compute answer sets for some programs by completing them and then calling a satisfiability solver. We showed that this method can be applied, for instance, to the representation of the blocks world proposed in (Niemelä 1999). This example shows that satisfiability solvers may serve as useful computational tools in answer set programming.

There are cases, on the other hand, when the completion method is not applicable. Consider computing Hamiltonian cycles in a directed graph (Marek & Truszczyński 1999). We combine the rules

```
in(U,V) :- edge(U,V), not out(U,V).
out(U,V) :- edge(U,V), not in(U,V).

:- in(U,V), in(U,W), V != W.
:- in(U,W), in(V,W), U != V.

reachable(V) :- in(v0,V).
reachable(V) :- reachable(U), in(U,V).

:- vertex(U), not reachable(U).
```

with a set of facts defining the vertices and edges of the graph; v0 is assumed to be one of the vertices. The answer sets for the resulting program correspond to the Hamiltonian cycles. Generally, the completion of the program has models different from its answer sets. Take, for instance, the graph consisting of two disjoint loops:

```
    vertex(v0).  vertex(v1).
    edge(v0,v0).  edge(v1,v1).
```

This graph has no Hamiltonian cycles, and, accordingly, the corresponding program has no answer sets. But the set

```
    vertex(v0), vertex(v1), edge(v0,v0),
    edge(v1,v1), in(v0,v0), in(v1,v1),
    reachable(v0), reachable(v1)
```

is a model of the program's completion.

Acknowledgements

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Appendix: Definitions

The notion of an answer set is defined first for programs whose rules do not contain negation as failure. Let Π be such a program, and let X be a consistent set of literals. We say that X is *closed* under Π if, for every rule

$$Head \leftarrow Body$$

in Π, $Head \in X$ whenever $Body \subseteq X$. (For a constraint, this condition means that the body is not contained in X.) We say that X is an *answer set* for Π if X is minimal among the sets closed under Π w.r.t. set inclusion. It is clear that a program without negation as failure can have at most one answer set.

To extend this definition to arbitrary programs, take any program Π, and let X be a consistent set of literals. The *reduct* Π^X of Π relative to X is the set of rules

$$Head \leftarrow L_1, \ldots, L_m$$

for all rules (3) in Π such that $L_{m+1}, \ldots, L_n \notin X$. Thus Π^X is a program without negation as failure. We say that X is an *answer set* for Π if X is an answer set for Π^X.

A set X of literals is *closed under* Π if, for every rule (3) in Π, $Head \in X$ whenever $L_1, \ldots, L_m \in X$ and $L_{m+1}, \ldots, L_n \notin X$. We say that X is *supported by* Π if, for every $L \in X$, there is a rule (3) in Π such that $Head = L$, $L_1, \ldots, L_m \in X$ and $L_{m+1}, \ldots, L_n \notin X$.

Let Π be a finite program without classical negation. If H is an atom or the symbol $\perp$, by $Comp(\Pi, H)$ we denote the formula

$$H \equiv \bigvee (A_1 \wedge \cdots \wedge A_m \wedge \neg A_{m+1} \wedge \cdots \wedge \neg A_n)$$

where the disjunction extends over all rules

$$H \leftarrow A_1, \ldots, A_m, not\ A_{m+1}, \ldots, not\ A_n$$

in Π with the head H. The *completion* of Π is set of formulas $Comp(\Pi, H)$ for all H.