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Olusegun Michael Otunuga
Marshall University, otunuga@marshall.edu

Gangaram S. Ladde

Nathan G. Ladde

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Local Lagged Adapted Generalized Method Of Moments And Applications[1]

OLUSEGUN M. OTUNUGA*
Department of Mathematics and Statistics, Marshall University, Huntington, West Virginia, 25755, USA.
GANGARAM S. LADDE
Department of Mathematics and Statistics, University of South Florida, Tampa, Florida, 33620, USA.
NATHAN G. LADDE
CSM, 5775 Glenridge Dr NE, Atlanta, Georgia, 30328, USA.

Abstract

In this work, an attempt is made for developing the local lagged adapted generalized method of moments (LLGMM). This proposed method is composed of: (1) development of the stochastic model for continuous-time dynamic process, (2) development of the discrete-time interconnected dynamic model for statistic process, (3) utilization of Euler-type discretized scheme for nonlinear and non-stationary system of stochastic differential equations, (4) development of generalized method of moment/observation equations by employing lagged adaptive expectation process, (5) introduction of the conceptual and computational parameter estimation problem, (6) formulation of the conceptual and computational state estimation scheme and (7) definition of the conditional mean square ϵ-best sub optimal procedure. The development of LLGMM is motivated by parameter and state estimation problems in continuous-time nonlinear and non-stationary stochastic dynamic model validation problems in biological, chemical, engineering, financial, medical, physical and social sciences. The byproducts of LLGMM are the balance between model specification and model prescription of continuous-time dynamic process and the development of discrete-time interconnected dynamic model of local sample mean and variance statistic process (DTIDMLSMVSP). DTIDMLSMVSP is the generalization of statistic (sample mean and variance) drawn from the static dynamic population problems. Moreover, it is also an alternative approach to the GARCH (1,1) model and its many related variant models (e.g., EGARCH model, GJR GARCH model). It provides an iterative scheme for updating statistic coefficients in a system of generalized method of moment/observation equation. Furthermore, application of the LLGMM method to stochastic differential dynamic models for energy commodity price, U. S. Treasury Bill Yield Interest Rate and U. S.-U.K. Foreign Exchange Rate exhibits its unique role and scope.

Keywords: Conceptual computational/theoretical parameter estimation scheme; Sample mean/ variance dynamical model; Local Lagged adapted GMM (LLGMM); Local moving sample mean/variance; Reaction/response time delay.

*Corresponding Author, [1]U.S. Patent Pending

1. Introduction

Recently, several models have been developed to investigate the volatility process described by stochastic differential equations [30, 49] and stochastic difference equations [16]. It is well-recognized that volatility is predictable in many asset markets [4]. Moreover, it is observed that the volatility predictability varies significantly. Engle [16] developed a class of discrete-time models where the variance depends on the past history of the commodity/service. Bollerslev [4] generalized models in [16] to the GARCH(p,q).

The estimate for the variance of general statistic from a stationary sequence can be obtained using the concept of moving average [7]. Employing the batched mean, the grand mean of the individual batch mean and introducing ASAP3 [29], it is shown that ASAP3 fits AR(1) time series model to the batch mean and it provides a better technique for determining points and confidence-interval estimators. The Kalman Filtering approach is another technique for estimation scheme. It is widely known and well recognized [15, 28, 41] that the Kalman filtering approach for the system parameter and state estimation problems is based on the continuous time coupled system of state dynamic and observation systems. Using the batched mean and the first order iterative process for $\bar{X}_n$ [48], a first order iterative process [48] is developed to estimate the population variance from a given time series data set.

For the past 40 years, researchers [3, 9, 11, 15, 17, 18, 19, 29, 31, 32, 35, 36, 38, 39, 40, 41] have given a lot of attention to estimating continuous-time dynamic models from discrete time data sets. The Generalized Method of Moments (GMM) developed by Hansen [17] and its extensions [11, 18, 19] have played a significant role in the literature related to the parameter and state estimation problems in linear and nonlinear stochastic dynamic processes. Under the continuous-time dynamic and discrete time data collection processes, the GMM and its extensions/generalizations are comprised of these components, namely: 1). Stochastic differential equations of It$\hat{o}$-Doob type, 2). Euler-type discretization scheme/using econometric specification, 3). the general moment function, 4). minimizing functional or objective criterion function [17].

Most of the existing parameter and state estimation techniques except for the Kalman filtering are centered around the usage of either overall data sets [11, 18, 19], batched data sets [7], or local data sets [39] drawn on an interval of finite length T. This leads to an overall parameter estimate on the interval of length T.

In this paper, an innovative method, the ”Local Lagged Adapted Generalized Method of Moments” (LLGMM) is developed in a systematic and unified way. It is based on a foundation of correctly utilized mathematics: (a) the It$\hat{o}$-Doob Stochastic Calculus, (b) the formation of continuous-time differential equations for suitable functions of dynamic state with respect to original SDE (using It$\hat{o}$-Doob differential formula), (c) constructing corresponding

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Euler-type discretization schemes, (d) developing general discrete-time interconnected dynamic model of local sample mean and variance statistic processes (DTIDMLSMVSP), (e) the fundamental properties of solution process of system of stochastic differential equations, for example: existence, uniqueness, continuous dependence of parameters. The LLGMM approach is composed of seven interconnected components: (1). development of the stochastic mathematical model of continuous time dynamic process [23, 24], (2). development of the discrete-time interconnected dynamic model for statistic process, (3). utilizing the Euler-type discretized scheme [21] for nonlinear and non-stationary system of stochastic differential equations (1), (4). employing lagged adaptive expectation process [33] for developing generalized method of moment/observation equations, (5). introduction of the conceptual and computational parameter estimation problem, (6). formulation of the conceptual and computational state simulation scheme and (7). definition of the mean square ϵ-sub optimal procedure.

A brief outline for the motivation and rationale for the development of the multi-component LLGMM is given below. The LLGMM is motivated by parameter and state estimation problems of continuous time nonlinear stochastic dynamic model of energy commodity markets [30]. In fact, it is also applicable to apparently different dynamic processes that are conceptually similar dynamic processes in biological, chemical, engineering, financial, medical and physical and social sciences. Moreover, one of the goals of the parameter and state estimation problems is for model validation rather than model misspecification [11]. For the continuous-time dynamic model validation, we need to utilize the existing real world data set. Of course, the real world time varying data is drawn/recorded at discrete-time on a time interval of finite length. In view of this, instead of using existing econometric specification/Euler-type numerical scheme, we construct the stochastic numerical approximation scheme [21] using continuous time stochastic differential equations. In almost all real world dynamic modeling problems [23, 24, 25, 33, 34], future states of continuous time dynamic processes are influenced by the past state history (that is, history) and response/reaction time delay processes influencing present states [25, 30, 33]. In fact, the discrete-time dynamic models depend on the past state of a system [22, 25]. The influence of state history, the concept of lagged adaptive expectation process [33] and the idea of a moving average [20] lead to the development of the general discrete-time interconnected dynamic model of local sample mean and variance statistic processes (DTIDMLSMVSP). A few by-products of the discrete-time sample mean and variance statistic process are: (a) to initiate ideas for the usage of discrete-time interconnected dynamic approach parallel to the continuous-time dynamic process, (b) shorten the computation time and (c) to significantly reduce the state error estimates. Utilizing the Euler-type [21] stochastic discretization of the continuous time stochastic differential equations/moment/observation, the discrete-time interconnected dynamic approach parallel to the continuous-time dynamic process and the given real world time series data and the method of moments [8], systems of local moment/observation equations are constructed. Using discrete-time interconnected dynamic model for statistic process and the lagged adaptive expectation process [33] for developing generalized method of moment equations, the notions of data coordination, theoretical iterative and simulation schedule processes, conceptual and computational parameter estimation scheme, conceptual and computation state simulation and mean-square optimal procedure are introduced. In fact, our approach is more suitable and robust for forecasting problems than the existing methods. It also provides upper and lower bounds for the forecasted state of the system. Moreover, its computational aspect is a nested "two scale hierarchic" quadratic mean-square optimization process whereas the existing GMM and its extensions are "single-shot".

The organization of this paper is as follows:

In Section 2, using the role of time-delay processes, the concept of lagged adaptive expectation process [33], moving average [20], local finite sequence, local mean and variance, discrete-time dynamic sample mean and variance statistic processes, local conditional sequence and local sample mean and variance, we develop a general discrete time interconnected dynamic model for local sample mean and variance statistic processes (DTIDMLSMVSP). Moreover, DTIDMLSMVSP is the generalization of statistic of random sample drawn from the "static" population. In Section 3, we construct a local observation system from a nonlinear stochastic functional differential equations. This is based on the It$\hat{o}$-Doob stochastic differential formula and Euler-type numerical scheme in the context of the original stochastic systems of differential equations and the given data. In addition, using the method of moments [8] in the context of lagged adaptive expectation process [33], we briefly outline a procedure to estimate the state parameters, locally. Using the local lagged adaptive process and the discrete-time interconnected dynamic model for statistic process, the idea of time series data collection schedule synchronization with both numerical and simulation time schedules induces a finite chain of concepts in Section 4, namely: (a) local admissible set of lagged sample/data/ observation size, (b) local class of admissible lagged-adapted finite sequence of conditional sample/data, (c) local admissible sequence of parameter estimates and corresponding admissible sequence of simulated values, (d) ϵ-best sub-optimal admissible subset of set of m_k-size local conditional samples at time t_k in (a), (e) ϵ-sub-optimal lagged-adapted finite sequence of conditional sample/data and (f) finally, the ϵ-best sub optimal parameter estimates and simulated value at time t_k for $k = 1, 2, ..., N$ in a systematic way. In addition, the local lagged adaptive process and DTIDMLSMVSP generate a finite chain of discrete-time admissible sets/sub-data and corresponding chain described by simulation algorithm. Furthermore, a conceptual Matlab code and its implementation scheme are designed.

2. Derivation of Discrete Time Dynamic Model for sample mean and variance Processes.

In this section, we introduce an innovative component of an innovative GMM-based approach that plays a role not only in state and parameter estimation problems in continuous time nonlinear stochastic dynamic model of the energy commodity market [30], but also plays the same role in models developed in the areas of economics/finance, in particular, to the US Treasury Bill Interest Rate and the U.S.-U.K. Foreign Exchange rate models. The existing GMM-based parameter and state estimation techniques for testing/selecting continuous-time dynamic models [10, 11, 18, 19] are centered around discretization and model misspecifications errors in the context of usage of entire time-series data, algebraic manipulations and econometric specification for formation of orthogonality condition parameter vectors (OCPV). The existing approaches lead to an overall/single-shot state and parameter estimates and requires the ergodic stationary condition for convergence. Furthermore, the existing GMM-based single-shot approaches are not flexible to correctly validate the features of continuous-time dynamic models that are influenced by the state parameter and hereditary processes. In fact, in many real-life problems, the past and present dynamic states influence the future state dynamic. In the formulation of one of the components of the LLGMM approach, we incorporate the "past state history" via local lagged adaptive process [33].

Moreover, based on one of the goals of applied mathematical and statistical research, we develop a tool or method that is applicable or useful for apparently different yet conceptually similar processes in biological, chemical, engineering, energy commodity markets, financial, medical and physical and social sciences. In the framework of the stated goal, employing the hereditary influence of a systems [25, 30, 33], the concept of lagged adaptive expectation process [33] and the idea of moving average [20], we develop a very general discrete-time interconnected dynamic model of local sample mean and variance statistic processes (DTIDMLSMVSP) with respect to an arbitrary continuous-time stochastic dynamic process. The development of idea and discrete-time interconnected dynamic model of statistic for mean and variance processes is motivated by the state and parameter estimation problems of any continuous time non-linear stochastic dynamic model. Moreover, the idea of DTIDMLSMVSP was primarily based on the sample mean and sample variance ideas as statistic for a random sample drawn from a static population in the descriptive statistics [8]. The role and scope of DTIDMLSMVSP, in particular, the problems of long-term price forecasting and interval estimation problems with a high degree of confidence are also addressed. For this purpose, we need to introduce a few definitions and notations.

Let τ and γ be finite constant time delays such that $0 < \gamma \leq \tau$. Here, τ characterizes the influence of the past performance history of state of dynamic process and γ describes the reaction or response time delay. In general, these time delays are unknown random variables. These types of delay play a role in developing mathematical models of continuous time [25] and discrete time [22, 33] dynamic processes. Based upon the practical nature of data collection process, it is essential to either transform these time delays into positive integers or to design the data collection schedule in relation to the aforementioned delays. For this purpose, we describe the discrete version of time delays of τ and γ as

$$r = \left[\left|\frac{\tau}{\Delta t_i}\right|\right] + 1, \text{ and } q = \left[\left|\frac{\gamma}{\Delta t_i}\right|\right] + 1, \tag{2.1}$$

respectively. Moreover, for the sake of simplicity, we assume that $0 < \gamma < 1$ (q=1).

Definition 2.1. Let x be a continuous time stochastic dynamic process defined on an interval $[-\tau, T]$ into $\Re$, for some $T > 0$. For $t \in [-\tau, T]$, let $\mathcal{F}_t$ be an increasing sub-sigma algebra of a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ for which $x(t)$ is $\mathcal{F}_t$ measurable. Let P be a partition of $[-\tau, T]$ defined by

$$P := \{t_i = -\tau + (r + i)\Delta t\}, \quad \text{for} \quad i \in I_{-r}(N), \tag{2.2}$$

where $\Delta t = \frac{\tau+T}{N}$, and $I_i(k)$ is defined by $I_i(k) = \{j \in \mathbb{Z} \,|i \le j \le k\}$.

Let $\{x(t_i)\}_{i=-r}^{N}$ be a finite sequence corresponding to the stochastic dynamic process x and partition P in Definition 2.1. We further note that $x(t_i)$ is $\mathcal{F}_{t_i}$ measurable for $i \in I_{-r}(N)$. We recall the definition of forward time shift operator F [6] :

$$F^i x(t_k) = x(t_{k+i}). \tag{2.3}$$

In addition, let us denote $x(t_i)$ by x_i for $i \in I_{-r}(N)$.

Definition 2.2. For $q = 1$ and $r \ge 1$, each $k \in I_0(N)$ and each $m_k \in I_2(r + k - 1)$, a partition P_k of closed interval $[t_{k-m_k}, t_{k-1}]$ is called local at time t_k and it is defined by

$$P_k := t_{k-m_k} < t_{k-m_k+1} < ... < t_{k-1}. \tag{2.4}$$

Moreover, P_k is referred as the m_k-point sub-partition of the partition P in (2.2) of the closed sub-interval $[t_{k-m_k}, t_{k-1}]$ of $[-\tau, T]$.

Definition 2.3. For each $k \in I_0(N)$ and each $m_k \in I_2(r + k - 1)$, a local finite sequence at a time t_k of the size m_k is restriction of $\{x(t_i)\}_{i=-r}^{N}$ to P_k in (2.4) [2], and it is defined by

$$S_{m_k,k} := \{F^i x_{k-1}\}_{i=-m_k+1}^{0}. \tag{2.5}$$

As m_k varies from 2 to $k + r - 1$, the corresponding local sequence $S_{m_k,k}$ at t_k varies from $\{x_i\}_{i=k-2}^{k-1}$ to $\{x_i\}_{i=-r+1}^{k-1}$. As a result of this, the sequence defined in (2.5) is also called a m_k-local moving sequence. Furthermore, the average corresponding to the local sequence $S_{m_k,k}$ in (2.5) is defined by

$$\bar{S}_{m_k,k} := \frac{1}{m_k} \sum_{i=-m_k+1}^{0} F^i x_{k-1}. \tag{2.6}$$

The average/mean defined in (2.6) is also called the m_k-local average/mean. Moreover, the m_k-local variance corresponding to the local sequence $S_{m_k,k}$ in (2.5) is defined by

$$s^2_{m_k,k} := \begin{cases} \frac{1}{m_k} \sum\limits_{i=-m_k+1}^{0} \left(F^i x_{k-1} - \frac{1}{m_k} \sum\limits_{j=-m_k+1}^{0} F^j x_{k-1} \right)^2, & \text{for small } m_k \\ \frac{1}{m_k-1} \sum\limits_{i=-m_k+1}^{0} \left(F^i x_{k-1} - \frac{1}{m_k} \sum\limits_{j=-m_k+1}^{0} F^j x_{k-1} \right)^2, & \text{for large } m_k \end{cases} \tag{2.7}$$

Definition 2.4. For each fixed $k \in I_0(N)$, and any $m_k \in I_2(k + r - 1)$, the sequence $\{\bar{S}_{i,k}\}_{i=k-m_k}^{k-1}$ is called a m_k- local moving average/mean process at t_k. Moreover, the sequence $\{s^2_{i,k}\}_{i=k-m_k}^{k-1}$ is called a m_k- local moving variance process at t_k.

Definition 2.5. Let $\{x(t_i)\}_{i=-r}^{N}$ be a random sample of continuous time stochastic dynamic process collected at partition P in (2.2). The local sample average/mean in (2.6) and local sample variance in (2.7) are called discrete time dynamic processes of sample mean and sample variance statistics.

Definition 2.6. Let $\{x(t_i)\}_{i=-r}^{N}$ be a random sample of continuous time stochastic dynamic process collected at partition P in (2.2). The m_k-local moving average and variance defined in (2.6) and (2.7) are called the m_k-local moving sample average/mean and local moving sample variance at time t_k, respectively. Moreover, m_k-local sample average and m_k-local sample variance are referred to as local sample mean and local sample variance statistics for the local mean and variance of the continuous time stochastic dynamic process at time t_k, respectively. Moreover, $\bar{S}_{m_k}$ and $s^2_{m_k}$ are called sample statistic time series processes.

Definition 2.7. Let $\{\mathbb{E}[x(t_i)|\mathcal{F}_{t_{i-1}}]\}_{i=-r+1}^{N}$ be a conditional random sample of continuous time stochastic dynamic process with respect to sub-σ algebra $\mathcal{F}_{t_i}$, $t_i \in P$ in (2.2). The m_k-local conditional moving average and variance defined in the context of (2.6) and (2.7) are called the m_k-local conditional moving sample average/mean and local conditional moving sample variance, respectively.

The concept of sample statistic time-series/ process extends the concept of random sample statistic [8] for static dynamic populations in a natural and unified way. In the following, employing Definition 2.7 , we introduce n discrete-time interconnected dynamic model of local sample mean and variance statistic processes (DTIDMLSMVSP), namely, conditional sample mean and variance. This discrete-time algorithm/model would play an important role in state and parameter estimation problems for nonlinear and non-stationary continuous-time stochastic differential and difference equations. Moreover, it provides feedback for both continuous-time dynamic model and corresponding discrete-time statistic dynamic model for modifications and updates under the influence of exogenous and endogenous varying forces or conditions in a systematic and unified way. Moreover, it is obvious that the discrete-time algorithm eases the updates in the time-series statistic. Now we are ready to state and prove a change in $\bar{S}_{m_k,k}$ and $s^2_{m_k,k}$ with respect to change in time t_k. This fundamental result is motivated by Exercise 5.15 in [8].

Lemma 2.1. ***DISCRETE TIME INTERCONNECTED DYNAMIC MODEL OF LOCAL SAMPLE MEAN AND VARIANCE PROCESSES (DTIDMLSMVSP).*** *Let $\{\mathbb{E}[x(t_i)|\mathcal{F}_{t_{i-1}}]\}_{i=-r+1}^{N}$ be a conditional random sample of continuous time stochastic dynamic process with respect to sub-σ algebra $\mathcal{F}_{t_i}$, t_i belong to partition P in (2.2). Let $\bar{S}_{m_k,k}$ and $s^2_{m_k,k}$ be m_k-local conditional sample average and local conditional sample variance at t_k for each $k \in I_0(N)$. Then, a discrete time interconnected dynamic model of local conditional sample mean and sample variance statistic is described by*

$$\begin{cases} \bar{S}_{m_{k-p+1},k-p+1} = \frac{m_{k-p}}{m_{k-p+1}}\bar{S}_{m_{k-p},k-p} + \eta_{m_{k-p},k-p}, \quad \bar{S}_{m_0,0} = \bar{S}_0 \\ s^2_{m_k,k} = \begin{cases} \frac{m_k-1}{m_k}\left[\sum_{i=1}^{p}\left[\frac{m_{k-i}}{\prod_{j=0}^{i-1} m_{k-j}}\right]s^2_{m_{k-i},k-i} + \frac{m_{k-p}}{\prod_{j=0}^{p-1} m_{k-j}}\bar{S}^2_{m_{k-p},k-p}\right] + \varepsilon_{m_{k-1},k-1}, & \textit{for small } m_k, m_{k-1} \leq m_k \\ \sum_{i=1}^{p}\left[\frac{m_{k-i}-1}{\prod_{j=0}^{i-1} m_{k-j}}\right]s^2_{m_{k-i},k-i} + \frac{m_{k-p}}{\prod_{j=0}^{p-1} m_{k-j}}\bar{S}^2_{m_{k-p},k-p} + \epsilon_{m_{k-1},k-1}, & \textit{for large } m_k,\ m_{k-1} \leq m_k \\ s^2_{m_i,i} = s^2_i, i \in I_{-p}(0), & \textit{initial conditions} \end{cases} \end{cases} \tag{2.8}$$

where

$$\begin{cases} \eta_{m_{k-p},k-p} = \frac{1}{m_{k-p+1}}\left[\sum_{i=-m_{k-p+1}+1}^{-m_{k-p}+1} F^i x_{k-p} - F^{-m_{k-p}+1}x_{k-p} - F^{-m_{k-p}}x_{k-p} + F^0 x_{k-p}\right], \\ \varepsilon_{m_{k-1},k-1} = \frac{m_k-1}{m_k}\left[\sum_{i=1}^{p}\frac{\left(F^{-i+1}x_{k-1}\right)^2}{\prod_{j=0}^{i-1} m_{k-j}} - \sum_{i=1}^{p}\frac{\left(F^{-i+1-m_{k-i}}x_{k-1}\right)^2}{\prod_{j=0}^{i-1} m_{k-j}} - \sum_{i=1}^{p}\frac{\left(F^{-i+2-m_{k-i}}x_{k-1}\right)^2}{\prod_{j=0}^{i-1} m_{k-j}}\right] \\ \quad + \frac{m_k-1}{m_k}\left[\sum_{i=1}^{p}\left[\frac{\sum_{l=-i+2-m_{k-i+1}}^{-i+2-m_{k-i}}(F^l x_{k-1})^2}{\prod_{j=0}^{i-1} m_{k-j}}\right] + \sum_{i=1}^{p}\left[\frac{\sum_{\substack{l,s=-i+2-m_{k-i+1} \\ l\neq s}}^{-i+1} F^l x_{k-1}F^s x_{k-1}}{\prod_{j=0}^{i-1} m_{k-j}}\right]\right] \\ \quad - \frac{1}{m_k}\sum_{\substack{l,s=-m_k+1 \\ l\neq s}}^{0} F^l x_{k-1}F^s x_{k-1}, \\ \epsilon_{m_{k-1},k-1} = \sum_{i=1}^{p}\frac{\left(F^{-i+1}x_{k-1}\right)^2}{\prod_{j=0}^{i-1} m_{k-j}} - \sum_{i=1}^{p}\frac{\left(F^{-i+1-m_{k-i}}x_{k-1}\right)^2}{\prod_{j=0}^{i-1} m_{k-j}} - \sum_{i=1}^{p}\frac{\left(F^{-i+2-m_{k-i}}x_{k-1}\right)^2}{\prod_{j=0}^{i-1} m_{k-j}} + \sum_{i=1}^{p}\left[\frac{\sum_{l=-i+2-m_{k-i+1}}^{-i+2-m_{k-i}}(F^l x_{k-1})^2}{\prod_{j=0}^{i-1} m_{k-j}}\right] \\ \quad + \sum_{i=1}^{p}\left[\frac{\sum_{\substack{l,s=-i+2-m_{k-i+1} \\ l\neq s}}^{-i+1} F^l x_{k-1}F^s x_{k-1}}{\prod_{j=0}^{i-1} m_{k-j}}\right] - \frac{1}{m_k-1}\sum_{\substack{l,s=-m_k+1 \\ l\neq s}}^{0} F^l x_{k-1}F^s x_{k-1} \end{cases} \tag{2.9}$$

PROOF. The proof of Lemma 2.1 for small m_k, $m_{k-1} \le m_k$, is given in *Appendix A*. The case for small m_k, $m_k \le m_{k-1}$ is also described in *Appendix B*. The proof for large m_k, $m_{k-1} \le m_k$, is given in *Appendix C*. ■

Remark 2.1. The interconnected dynamic statistic system (2.8) can be re-written as the one-step Gauss-Sidel dynamic system [26] of iterative process described by

$$\mathbf{X}(k;p) = \mathbf{A}(k, \mathbf{X}(k-1;p);p)\mathbf{X}(k-1;p) + \mathbf{e}(k;p), \tag{2.10}$$

where $\mathbf{X}(k;p) = \begin{pmatrix} \mathbf{X}_1(k;p) \\ \mathbf{X}_2(k;p) \end{pmatrix}$, $\mathbf{X}_1(k;p) = \bar{S}_{m_{k-p+1},k-p+1}$, $\mathbf{X}_2(k) = \begin{pmatrix} s^2_{m_{k-p+1},k-p+1} \\ s^2_{m_{k-p+2},k-p+2} \\ \vdots \\ s^2_{m_{k-1},k-1} \\ s^2_{m_k,k} \end{pmatrix}$,

$\mathbf{A}(k, \mathbf{X}(k-1;p);p) = \begin{pmatrix} \mathbf{A}_{11}(k;p) & \mathbf{A}_{12}(k;p) \\ \mathbf{A}_{21}(k, \mathbf{X}(k-1;p);p) & \mathbf{A}_{22}(k;p) \end{pmatrix}$, $\mathbf{A}_{11}(k;p) = \frac{m_{k-p}}{m_k - p + 1}$, $\mathbf{A}_{12}(k;p) = \begin{pmatrix} 0 & 0 & \dots & 0 \end{pmatrix}$,

$$\mathbf{A}_{21}(k;p) = \begin{cases} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{(m_k-1)m_{k-p}}{m_k \prod_{j=0}^{p-1} m_{k-j}} \bar{S}_{m_{k-p},k-p} \end{pmatrix}, & \text{for small } m_k \\ \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{m_{k-p}}{\prod_{j=0}^{p-1} m_{k-j}} \bar{S}_{m_{k-p},k-p} \end{pmatrix}, & \text{for large } m_k, \end{cases},$$

$$\mathbf{A}_{22}(k;p)=\begin{cases}\begin{pmatrix}0 & 1 & 0 & 0 & \dots & 0\\ 0 & 0 & 1 & 0 & \dots & 0\\ \vdots & 0 & 0 & 0 & \ddots & \vdots\\ 0 & \dots & 0 & 0 & 0 & 1\\ \frac{(m_k-1)m_{k-p}}{m_k\prod_{j=0}^{p-1}m_{k-j}} & \frac{(m_k-1)m_{k-p+1}}{m_k\prod_{j=0}^{p-2}m_{k-j}} & \dots & \frac{(m_k-1)m_{k-p+i-1}}{m_k\prod_{j=0}^{p-i}m_{k-j}} & \dots & \frac{(m_k-1)m_{k-1}}{m_k^2}\end{pmatrix}, \text{ for small } m_k\\ \begin{pmatrix}0 & 1 & 0 & 0 & \dots & 0\\ 0 & 0 & 1 & 0 & \dots & 0\\ \vdots & 0 & 0 & 0 & \ddots & \vdots\\ 0 & \dots & 0 & 0 & 0 & 1\\ \frac{m_{k-p}-1}{\prod_{j=0}^{p-1}m_{k-j}} & \frac{m_{k-p+1}-1}{\prod_{j=0}^{p-2}m_{k-j}} & \dots & \frac{m_{k-p+i-1}-1}{\prod_{j=0}^{p-i}m_{k-j}} & \dots & \frac{m_{k-1}-1}{m_k}\end{pmatrix}, \text{ for large } m_k\end{cases}$$

$$\mathbf{e}(k;p)=\begin{pmatrix}\mathbf{e}_1(k;p)\\ \mathbf{e}_2(k;p)\end{pmatrix}, \mathbf{e}_1(k;p)=\eta_{m_{k-p},k-p}, \mathbf{e}_2(k;p)=\begin{pmatrix}0\\ 0\\ \vdots\\ \epsilon^*_{m_{k-1},k-1}\end{pmatrix}, \epsilon^*_{m_{k-1},k-1}=\begin{cases}\varepsilon_{m_{k-1},k-1}, & for\ small\ m_k\\ \epsilon_{m_{k-1},k-1}, & for\ large\ m_k\end{cases}$$

Remark 2.2. For each $k \in I_0(N)$, $p = 2$ and small m_k, the inter-connected system (2.8) reduces to the following special case:

$$\mathbf{X}(k;2)=\mathbf{A}(k,\mathbb{X}(k-1;2);2)\mathbf{X}(k-1;2)+\mathbf{e}(k;2), \tag{2.11}$$

where $\mathbf{X}(k;2)$, $\mathbf{A}(k;2)$ and $\mathbf{e}(k;2)$ are defined by $\mathbf{X}(k;2)=\begin{pmatrix}\mathbf{X}_1(k;2)\\ \mathbf{X}_2(k;2)\end{pmatrix}$, $\mathbf{X}_1(k;2)=\bar{S}_{m_{k-1},k-1}$, $\mathbf{X}_2(k;2)=\begin{pmatrix}s^2_{m_{k-1},k-1}\\ s^2_{m_k,k}\end{pmatrix}$,

$\mathbf{A}(k;2)=\begin{pmatrix}\mathbf{A}_{11}(k;2) & \mathbf{A}_{12}(k;2)\\ \mathbf{A}_{21}(k;2) & \mathbf{A}_{22}(k;2)\end{pmatrix}$, $\mathbf{A}_{11}(k;2)=\frac{m_{k-2}}{m_k-1}$, $\mathbf{A}_{12}(k;2)=\begin{pmatrix}0 & 0\end{pmatrix}$, $\mathbf{A}_{21}(k;2)=\begin{pmatrix}0\\ \frac{(m_k-1)m_{k-2}}{m_k^2m_{k-1}}\bar{S}_{m_{k-2},k-2}\end{pmatrix}$,

$\mathbf{A}_{22}(k;2)=\begin{pmatrix}0 & 1\\ \frac{(m_k-1)m_{k-2}}{m_k^2m_{k-1}} & \frac{(m_k-1)m_{k-1}}{m_k^2}\end{pmatrix}$, $\mathbf{e}(k;2)=\begin{pmatrix}\mathbf{e}_1(k;2)\\ \mathbf{e}_2(k;2)\end{pmatrix}$; $\mathbf{e}_1(k;2)=\eta_{m_{k-2},k-2}$, $\mathbf{e}_2(k;2)=\begin{pmatrix}0\\ \varepsilon_{m_{k-1},k-1}\end{pmatrix}$ and

$$
\begin{cases}
\eta_{m_{k-2},k-2} = \frac{1}{m_k}\left[\sum_{i=-m_{k-1}+1}^{-m_{k-2}+1} F^i x_{k-2} - F^{-m_{k-2}+1}x_{k-2} - F^{-m_{k-2}}x_{k-2} + F^0 x_{k-2}\right], \\
\varepsilon_{m_{k-1},k-1} = \frac{m_k-1}{m_k}\left[\frac{\left(F^0 x_{k-1}\right)^2-\left(F^{-m_{k-1}}x_{k-1}\right)^2-\left(F^{1-m_{k-1}}x_{k-1}\right)^2}{m_k} + \frac{\left(F^{-1}x_{k-1}\right)^2-\left(F^{-1-m_{k-2}}x_{k-1}\right)^2-\left(F^{-m_{k-2}}x_{k-1}\right)^2}{m_k m_{k-1}}\right] \\
\qquad + \frac{m_k-1}{m_k}\left[\frac{\sum_{i=-m_{k-1}}^{-m_{k-2}}(F^i x_{k-1})^2}{m_k m_{k-1}} + \frac{\sum_{\substack{i,j=-m_{k-1}\\ i\neq j}}^{-1} F^i x_{k-1}F^j x_{k-1}}{m_k m_{k-1}} + \frac{\sum_{i=1-m_k}^{1-m_{k-1}}(F^i x_{k-1})^2}{m_k}\right] - \frac{\sum_{\substack{i,j=1-m_k\\ i\neq j}}^{0} F^i x_{k-1}F^j x_{k-1}}{m_k^2}.
\end{cases}
$$

Remark 2.3. Define $\varphi_1 = \frac{m_k-1}{m_k}\frac{m_{k-1}}{m_k}$, $\varphi_2 = \frac{m_k-1}{m_k}\frac{m_{k-2}}{m_k m_{k-1}}$, and $\varphi_3 = \frac{m_{k-2}}{m_{k-1}}$. For small m_k, $m_{k-1} \leq m_k$, $\forall k$, we have $\varphi_1 < 1$, $\varphi_2 < 1$, and $\varphi_3 \leq 1$. From $0 < \varphi_i, i = 1,2,3$, and the fact that $\varphi_1 + \varphi_2 = \frac{m_k-1}{m_k^2}\left[m_{k-1} + \frac{m_{k-2}}{m_{k-1}}\right] \leq \frac{m_k-1}{m_k^2}\left[m_{k-1}+1\right] \leq \frac{m_k^2-1}{m_k^2} < 1$, the stability of the trivial solution ($\mathbf{X}(k;2) = 0$) of the homogeneous system corresponding to (2.10) follows. Moreover, under the above stated conditions, the convergence of solutions of (2.10) also follows.

Remark 2.4. From Remark 2.2, we note that the local sample variance statistics at time t_k depends on the state of the m_{k-1} and m_{k-2}-local sample variance statistics at time t_{k-1} and t_{k-2}, respectively, and the m_{k-2}-local sample mean statistics at time t_{k-2}.

Remark 2.5. The role and scope of the DTIDMLSMVSP are summarized below:

- The " discrete-time interconnected dynamic model for statistic process" (DTIDMLSMVSP) " is the second component of the LLGMM approach.
- The "discrete-time interconnected dynamic model for statistic process" (DTIDMLSMVSP) (Lemma 2.1) is also valid for a transformation of data.
- It is generalization of "statistic" of random sample drawn from "static" population problems.
- Moreover, "Lemma-2.1" provides iterative scheme for updating statistic coefficients in the local systems of moment/observation equations in the LLGMM approach.
- This indeed accelerates the speed of computation .
- The DTIDMLSMVSP does not require any type of stationary condition.
- The DTIDMLSMVSP plays a very significant role in the local discretization and model validation errors.
- The presented approach is more suitable for forecasting problems.
- These features will be further emphasized in the subsequent sections.

Remark 2.6. The further usefulness of the discrete time interconnected dynamic model of local sample mean and local sample variance statistic process (DTIDMLSMVSP) arises in the estimation of volatility process of a stochastic differential or difference equations. This model provides an alternative approach to the GARCH(p,q) model [4, 5]. We shall later compare the m_k-local sample variance statistics with the GARCH(p,q) model and show that the m_k-local sample variance statistics gives a better forecast than the GARCH(p,q) model.

3. Theoretical Parametric Estimation Procedure

In this section, we formulate a foundation based on a mathematically rigorous theoretical state and parameter estimation procedure for a very general continuous-time nonlinear and non-stationary stochastic dynamic model described by a systems stochastic differential equations [24]. This work is not only motivated by the continuous-time dynamic model validation problem [30] in the context of real data energy commodities, but also motivated by any continuous-time nonlinear and non-stationary stochastic dynamic model validation problems in biological, chemical, engineering, financial, medical, physical and social sciences. This is because of the fact that the development of existing Orthogonality Condition Based GMM (OCBGMM) procedure [9, 10, 18, 19] is primarily composed of five components: (1) testing/selecting continuous-time stochastic models for a particular dynamic process in finance that is described by stochastic differential equation, (2) using either Euler-type discretization scheme or a discrete-time econometric specification regarding the stochastic differential equation specified in (1), (3) formation of orthogonality condition parameter vector (OCPV) using algebraic manipulation, (4) using (2), (3) and the entire time series data set, finding a system of moment equations for the OCBGMM and (5) single-shot parameter and state estimates using positive-definite quadratic form. The existing OCBGMM lacks the usage of Itô-Doob calculus, properties of stochastic differential equations and its connectivity with the usage of econometric specification based discretization scheme, orthogonality conditional vector and the quadratic form. In this section, we make an attempt to eliminate the drawbacks, operational limitations and the lack of connectivity and limited scope of the existing OCBGMM. This is achieved by utilizing (i) historical role played by hereditary process in dynamic modeling [23, 24, 25, 33],(ii) Itô-Doob calculus [21, 22, 24], (iii), fundamental properties of stochastic system of differential equations, (iv) the lagged adaptive process [33] and (v) the discrete-time interconnected dynamics of local sample mean and variances statistic processes model in Section 2 (Lemma 2.1), (vi) developing the Euler-type numerical schemes [21] for both stochastic differential equations generated from the original stochastic systems of differential equations and the original stochastic systems of differential equations, (vii) systems of moments/observation equations and (viii) forming a local observation/measurements systems in the context of real world data.

We consider a general system of stochastic differential equations under the influence of hereditary effects in both the drift and diffusion coefficients described by

$$d\mathbf{y} = \mathbf{f}(t, \mathbf{y}_t)dt + \boldsymbol{\sigma}(t, \mathbf{y}_t)d\mathbf{W}(t), \mathbf{y}_{t_0} = \boldsymbol{\varphi}_0, \tag{3.1}$$

where $\mathbf{y}_t(\theta) = \mathbf{y}(t+\theta)$, $\theta \in [-\tau, 0]$, $\mathbf{f}$, $\boldsymbol{\sigma} : [0, T] \times C \to \mathfrak{R}^q$ are Lipschitz continuous bounded functionals; C is the Banach space of continuous functions defined on $[-\tau, 0]$ into $\mathfrak{R}^q$ equipped with the supremum norm; $W(t)$ is standard Wiener process defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathbb{P})$; $\boldsymbol{\varphi}_0 \in C$, and $\mathbf{y}_0(t_0 + \theta)$ is $(\mathcal{F})_{t_0}$ measurables; the filtration function $(\mathcal{F})_{t\geq 0}$ is right-continuous, and each $\mathcal{F}_t$ with $t \geq t_0$ contains all $\mathbb{P}$-null events in $\mathcal{F}$; the solution process $\mathbf{y}(t_0, \boldsymbol{\varphi}_0)(t)$ of (3.1) is adapted and non-anticipating with respect to $(\mathcal{F})_{t\geq 0}$.

*3.1. Transformation of System of Stochastic Differential Equations (*3.1*)*

Let $V \in C[[-\tau, \infty] \times \mathfrak{R}^q, \mathfrak{R}^m]$, and its partial derivatives V_t, $\frac{\partial V}{\partial \mathbf{y}}$, $\frac{\partial^2 V}{\partial \mathbf{y}^2}$ exist and are continuous. We apply Itô-Doob stochastic differential formula [24] to V , and we obtain

$$dV(t, \mathbf{y}) = LV(t, \mathbf{y}, \mathbf{y}_t)dt + V_{\mathbf{y}}(t, \mathbf{y})\boldsymbol{\sigma}(t, \mathbf{y}_t)d\mathbf{W}(t) \tag{3.2}$$

where the L operator is defined by

$$\begin{cases} LV(t, \mathbf{y}, \mathbf{y}_t) &= V_t(t, \mathbf{y}) + V_{\mathbf{y}}(t, \mathbf{y})\mathbf{f}(t, \mathbf{y}_t) + \frac{1}{2}tr(V_{\mathbf{yy}}(t, y)b(t, \mathbf{y}_t)) \\ b(t, \mathbf{y}_t) &= \boldsymbol{\sigma}(t, \mathbf{y}_t)\boldsymbol{\sigma}^T(t, \mathbf{y}_t). \end{cases} \tag{3.3}$$

*3.2. Euler-type Discretization Scheme for (*3.1*) and (*3.2*)*

For (3.1) and (3.2), we present the Euler-type discretization scheme [21]:

$$\begin{cases} \Delta \mathbf{y}_i &= \mathbf{f}(t_{i-1}, \mathbf{y}_{t_{i-1}})\Delta t_i + \boldsymbol{\sigma}(t_{i-1}, \mathbf{y}_{t_{i-1}})\Delta \mathbf{W}(t_i), \quad i \in I_1(N) \\ \Delta V(t_i, \mathbf{y}(t_i)) &= LV(t_{i-1}, \mathbf{y}(t_i), \mathbf{y}_{t_{i-1}})\Delta t_i + V_{\mathbf{y}}(t_{i-1}, \mathbf{y}(t_{i-1}))\boldsymbol{\sigma}(t_{i-1}, \mathbf{y}_{t_{i-1}})\Delta W(t_i) \end{cases} \tag{3.4}$$

Define $\mathcal{F}_{t_{i-1}} \equiv \mathcal{F}_{i-1}$ as the filtration process up to time t_{i-1}.

*3.3. Formation of Generalized Moment Equations From (*3.4*)*

With regard to the continuous time dynamic system (3.1) and its transformed system (3.2), the more general moments of $\Delta \mathbf{y}(t_i)$ are as follows:

$$\begin{cases} E\left[\Delta \mathbf{y}(t_i)|\mathcal{F}_{i-1}\right] & = \mathbf{f}(t_{i-1}, \mathbf{y}_{t_{i-1}})\Delta t_i, \\ E\left[\left(\Delta \mathbf{y}(t_i) - E\left[\Delta \mathbf{y}(t_i)|\mathcal{F}_{i-1}\right]\right)\left(\Delta \mathbf{y}(t_i) - E\left[\Delta \mathbf{y}(t_i)|\mathcal{F}_{i-1}\right]\right)^T |\mathcal{F}_{i-1}\right] & = \left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)(t_{i-1}, \mathbf{y}_{t_{i-1}})\Delta t_i, \\ E\left[\Delta V(t_i, \mathbf{y}(t_i))|\mathcal{F}_{i-1}\right] & = LV(t_{i-1}, \mathbf{y}(t_i), \mathbf{y}_{t_{i-1}})\Delta t_i \\ E\left[\left(\Delta V(t_i, \mathbf{y}(t_i)) - E\left[\Delta V(t_i, \mathbf{y}(t_i))|\mathcal{F}_{i-1}\right]\right)\left(\Delta V(t_i, y(t_i)) - E\left[\Delta V(t_i, y(t_i))|\mathcal{F}_{i-1}\right]\right)^T |\mathcal{F}_{i-1}\right] & = B(t_{i-1}, \mathbf{y}(t_{i-1}), \mathbf{y}_{t_{i-1}}) \end{cases} \tag{3.5}$$

where $B(t_{i-1}, \mathbf{y}(t_{i-1}), \mathbf{y}_{t_{i-1}}) = V_{\mathbf{y}}(t_{i-1}, \mathbf{y}(t_{i-1}))b(t_{i-1}, \mathbf{y}_{t_{i-1}})V_{\mathbf{y}}(t_{i-1}, \mathbf{y}(t_{i-1}))^T\Delta t_i$, and T stands for the transpose of the matrix.

3.4. Basis for Local Lagged Adaptive Discrete-time Expectation Process

From (3.4) and (3.5), we have

$$\begin{cases} \Delta \mathbf{y}_i & = E\left[\Delta \mathbf{y}(t_i)|\mathcal{F}_{i-1}\right] + \boldsymbol{\sigma}(t_{i-1}, \mathbf{y}_{t_{i-1}})\Delta \mathbf{W}(t_i), \quad i \in I_1(N) \\ \Delta V(t_i, \mathbf{y}(t_i)) & = E\left[\Delta V(t_i, \mathbf{y}(t_i))|\mathcal{F}_{i-1}\right] + V_{\mathbf{y}}(t_{i-1}, \mathbf{y}(t_{i-1}))\boldsymbol{\sigma}(t_{i-1}, \mathbf{y}_{t_{i-1}})\Delta \mathbf{W}(t_i) \end{cases} \tag{3.6}$$

This provides the basis for the development of the concept of lagged adaptive expectation process [33] with respect to continuous time stochastic dynamic systems (3.1) and (3.2). This indeed leads to a formulation of m_k-local generalized method of moments at t_k.

Remark 3.1. (**Block Orthogonality Condition Vector for** (3.1) **and** (3.2))

From (3.6), we note that one can define a block vector of orthogonality condition [10] as

$$H(t_{i-1}, \mathbf{y}(t_i), \mathbf{y}(t_{i-1})) = \begin{pmatrix} \Delta \mathbf{y}(t_i) - \mathbf{f}(t_{i-1}, \mathbf{y}(t_{i-1}))\Delta t_i \\ \Delta V(t_{i-1}, \mathbf{y}(t_i)) - LV(t_{i-1}, \mathbf{y}(t_{i-1}, \mathbf{y}_{t_{i-1}}))\Delta t_i \end{pmatrix} \tag{3.7}$$

Moreover, unlike the orthogonality condition vector defined in the literature [8, 10, 42], the definition of the block vector of orthogonality condition (3.7) is based on the discretization scheme [21] associated with nonlinear and non-stationary continuous-time stochastic system of differential equations (3.1) and (3.2) and the ItÃ´-Doob stochastic differential calculus [21, 24].

Example 1:

For $V(t, y)$ in (3.2) is defined by $V(t, y) = \|y\|_p^p = \sum_{j=1}^{n} |y^j|^p$. In this case, we have

$$dV = \left[p \sum_{j=1}^{n} |y^j|^{p-1} sgn(y^j) f(t, y_t^j) + \frac{p(p-1)}{2} |y^j|^{p-2} \sigma(t, y_t^j) \right] \Delta t + p \sum_{j=1}^{n} |y^j|^{p-1} sgn(y^j) \sigma(t, y_t^j) \Delta W_i^j. \tag{3.8}$$

Hence, the discretized form of (3.8) is given by

$$\Delta V_i = \left[p \sum_{j=1}^{n} |y_{i-1}^j|^{p-1} sgn(y_{i-1}^j) f(t_{i-1}, y_{t_{i-1}}^j) + \frac{p(p-1)}{2} |y_{i-1}^j|^{p-2} \sigma(t_{i-1}, y_{t_{i-1}}^j) \right] \Delta t + p \sum_{j=1}^{n} |y_{i-1}^j|^{p-1} sgn(y_{i-1}^j) \sigma(t_{i-1}, y_{t_{i-1}}^j) \Delta W_i^j. \tag{3.9}$$

In this special case, (3.6) reduces to

$$\begin{cases} \Delta \mathbf{y}_i &= E\left[\Delta \mathbf{y}(t_i) | \mathcal{F}_{i-1}\right] + \boldsymbol{\sigma}(t_{i-1}, \mathbf{y}_{t_{i-1}}) \Delta \mathbf{W}_i, \quad i \in I_1(N) \\ \Delta \left(\sum_{j=1}^{n} |y_i^j|^p \right) &= E\left[\Delta \left(\sum_{j=1}^{n} |y_i^j|^p \right) | \mathcal{F}_{i-1} \right] + p \sum_{j=1}^{n} |y_{i-1}^j|^{p-1} sgn(y_{i-1}^j) \sigma(t_{i-1}, y_{t_{i-1}}^j) \Delta W_i^j. \end{cases} \tag{3.10}$$

Example 2:

We consider a multivariate AR(1) model as another example to exhibit the parameter and state estimation problem. The AR(1) model is of the following type

$$\boldsymbol{x}_t = \boldsymbol{a}_{t-1} \boldsymbol{x}_{t-1} + \boldsymbol{\sigma}_{t-1} \boldsymbol{e}_t, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \quad \text{for } t = 0, 1, 2, ..., t, ..., N, \tag{3.11}$$

where $\boldsymbol{x}_t, \boldsymbol{x}_0 \in \Re^n$, $\boldsymbol{e}_t \in \Re^m$ is $\mathcal{F}_t$ measurable normalized discrete-time Gaussian process; $\boldsymbol{a}_{t-1}$ and $\boldsymbol{\sigma}_{t-1}$ are $n \times n$ and $n \times m$ discrete-time varying matrix functions, respectively. Here

$$\begin{pmatrix} E\left[\boldsymbol{x}_t | \mathcal{F}_{t-1}\right] \\ E\left[\boldsymbol{x}_t \boldsymbol{x_t}^T | \mathcal{F}_{t-1}\right] \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}_{t-1} \boldsymbol{x}_{t-1} \\ \boldsymbol{a}_{t-1} \boldsymbol{x}_{t-1} (\boldsymbol{a}_{t-1} \boldsymbol{x}_{t-1})^T + \boldsymbol{\sigma}_{t-1} (\boldsymbol{\sigma}_{t-1})^T \end{pmatrix}. \tag{3.12}$$

In this case, the block orthogonality condition vector [10] is based on multivariate stochastic system of difference equation and difference calculus for (3.11) and (3.12):

$$H(t_{i-1}, \boldsymbol{x}_t, \boldsymbol{x}_{t-1}, \boldsymbol{a}_{t-1}, \boldsymbol{\sigma}_{t-1}) = \begin{pmatrix} \boldsymbol{x}_t - \boldsymbol{a}_{t-1} \boldsymbol{x}_{t-1} \\ \Delta V(\boldsymbol{x}_t) - LV(t, \boldsymbol{x}_{t-1}) \Delta t \end{pmatrix} \tag{3.13}$$

where Δ and L are difference and L operators with respect to $V = \boldsymbol{x}\boldsymbol{x}^T$ for $x \in \Re^n$, and are defined by

$$\begin{cases} \Delta V(\boldsymbol{x}_t) = V(\boldsymbol{x}_t) - V(\boldsymbol{x}_{t-1}), \quad \text{for } t = 1, 2, ..., t, ..., N \\ LV(t, \boldsymbol{x}_{t-1}) = \boldsymbol{a}_{t-1}\boldsymbol{x}_{t-1}\left((2 + \boldsymbol{a}_{t-1})\,\boldsymbol{x}_{t-1}\right)^T + \boldsymbol{\sigma}_{t-1}\boldsymbol{\sigma}_{t-1}^T, \end{cases} \tag{3.14}$$

and differential of V with respect to multivariate difference system (3.11) parallel to continuous-time version (3.2) is as:

$$\Delta V(\boldsymbol{x}_t) = \boldsymbol{a}_{t-1}\boldsymbol{x}_{t-1}\left((2 + \boldsymbol{a}_{t-1})\,\boldsymbol{x}_{t-1}\right)^T + \boldsymbol{\sigma}_{t-1}\boldsymbol{\sigma}_{t-1}^T + 2(1 + \boldsymbol{a}_{t-1}\boldsymbol{x}_{t-1})(\boldsymbol{\sigma}_{t-1}\boldsymbol{e}_t)^T \tag{3.15}$$

From the above discussion, it is obvious that the orthogonality condition parameter vector in (3.13) is constructed with respect to multivariate stochastic system of difference equations and elementary difference calculus.

Remark 3.2. From the transformation of system of stochastic differential equations (3.2) in Sub-section 3.1, the construction of Euler-type Discretization Scheme for (3.1) and (3.2) in Sub-section 3.2, the Formation of Generalized Moment Equations from (3.4) in Sub-section 3.3 and the Basis for Local Lagged Adaptive Discrete-time Expectation Process in Sub-section 3.4 are in the frame-work of correct mathematical reasoning, logical and interconnected/interactive within the context of the continuous-time dynamic system (3.1). Moreover, a continuous-time state dynamic process (described by systems of stochastic differential equations (3.1)) moves forward in time. The theoretical parameter estimation procedure in this section adapts to and incorporates the continuous-time changes in the state and parameters of the system and moves into a discrete-time theoretical numerical schemes in (3.4) as a model validation of (3.1). It further successively moves in the local moment equations within the context of local lagged adaptive, local discrete-time statistic and computational processes in a natural, systematic and coherent manner. On the other hand, the existing OCBGMM approach is "single-shot with a global approach" and it is highly dependent on the second component of the OCBGMM, that is, the "usage of either Euler-type discretization scheme or a discrete-time econometric specification regarding the stochastic differential equation". We refer to OCBGMM as the single-shot or global approach with formation of a single moment equation in a quadratic form.

In the following, we state a result that exhibits the existence of solution of system of non linear algebraic equations. For ease of reference, we shall state the Implicit function theorem without proof.

Theorem 3.1. ***Implicit Function Theorem****[2] Let $\boldsymbol{F} = \{F_1, F_2, ..., F_q\}$ be a vector-valued function defined on an open set $S \in \Re^{q+k}$ with values in $\Re^q$. Suppose $\boldsymbol{F} \in C'$ on S. Let $(\boldsymbol{u}_0; \boldsymbol{v}_0)$ be a point in S for which $\boldsymbol{F}(\boldsymbol{u}_0; \boldsymbol{v}_0) = 0$ and for which the $q \times q$ determinant det $\left[D_j\boldsymbol{F}_i(\boldsymbol{u}_0; \boldsymbol{v}_0)\right] \neq 0$. Then there exists a $k-$ dimensional open set $\boldsymbol{T}_0$ containing $\boldsymbol{v}_0$*

and unique vector-valued function $\boldsymbol{g}$, defined on $\boldsymbol{T}_0$ and having values in $\Re^q$, such that $\boldsymbol{g} \in C'$ on $\boldsymbol{T}_0$, $\boldsymbol{g}(\boldsymbol{v}_0) = \boldsymbol{u}_0$, and $\boldsymbol{F}(\boldsymbol{g}(\boldsymbol{v}); \boldsymbol{v}) = 0$ for every $\boldsymbol{v} \in \boldsymbol{T}_0$.

3.5. Illustration 1: Dynamic Model for Energy Commodity Price

We consider the stochastic dynamic model of energy commodities [30] described by the following nonlinear stochastic differential equation

$$dy = a(t)y(\mu(t) - y)dt + \sigma(t, y_t)ydW(t), y_{t_0} = \boldsymbol{\varphi}_0, \tag{3.16}$$

where $y_t(\theta) = y(t+\theta)$; $\theta \in [-\tau, 0]$, $\mu, a : [t_0, T] \to \Re$ are continuous functions; the initial process $\boldsymbol{\varphi}_0 = \{y(t_0+\theta)\}_{\theta\in[-\tau,0]}$ is $\mathcal{F}_{t_0}$-measurable and independent of $\{W(t), t \in [0, T]\}$; $W(t)$ is a standard Wiener process defined in (3.1); $\sigma : [0, T] \times C \to \Re^+$ is a Lipschitz continuous and bounded functional; C is the Banach space of continuous functions defined on $[-\tau, 0]$ into $\Re$ equipped with the supremum norm.

The solution $y(t)$ of (3.16) satisfy

$$y(t) - y(t_0) = \int_{t_0}^{t} a(s)y(s)(\mu(s) - y(s))ds + \int_{t_0}^{t} \sigma(s, y_s)y(s)dW(s),$$

and

$$\mathbb{E}\left[y(t) - y(t_0)|\mathcal{F}_{s<t}\right] = \int_{t_0}^{t} a(s)y(s)(\mu(s) - y(s))ds.$$

Transformation of Stochastic Differential Equation (3.16): We pick a Lyapunov function $V(t, y) = \ln(y)$ in (3.2) for (3.16). Using Itô-differential formula [24], we have

$$d\left(\ln(y)\right) = \left[a(t)(\mu(t) - y) - \frac{1}{2}\sigma^2(t, y_t)\right]dt + \sigma(t, y_t)dW. \tag{3.17}$$

The Euler-type Discretization Schemes for (3.16) **and** (3.17): By setting $\Delta t_i = t_i - t_{i-1}$, $\Delta y_i = y_i - y_{i-1}$, the combined Euler discretized scheme for (3.16) and (3.17) is

$$\begin{cases} \Delta y_i &= a_{i-1}y_{i-1}(\mu_{i-1} - y_{i-1})\Delta t_i + \sigma(t_{i-1}, y_{t_{i-1}})y_{i-1}\Delta W(t_i), \quad y_{t_0} = \boldsymbol{\varphi}_0, \\ \Delta\left(\ln(y_i)\right) &= \left[a_{i-1}(\mu_{i-1} - y_{i-1}) - \frac{1}{2}\sigma^2(t_{i-1}, y_{t_{i-1}})\right]\Delta t_i + \sigma(t_{i-1}, y_{t_{i-1}})\Delta W(t_i), y_{t_0} = \boldsymbol{\varphi}_0. \end{cases} \tag{3.18}$$

where $\boldsymbol{\varphi}_0 = \{y_i\}_{i=-r}^{0}$ is a given finite sequence of $\mathcal{F}_0-$ measurable random variables, and it is independent of $\{\Delta W(t_i\}_{i=0}^{N}$.

Generalized Moment Equations: Applying conditional expectation to (3.18) with respect to $\mathcal{F}_{t_{i-1}} \equiv \mathcal{F}_{i-1}$, we obtain

$$\begin{array}{lcl} \mathbb{E}\left[\Delta y_i|\mathcal{F}_{i-1}\right] & = & a_{i-1}y_{i-1}(\mu_{i-1}-y_{i-1})\Delta t \\ \mathbb{E}\left[\Delta\left(\ln(y_i)\right)|\mathcal{F}_{i-1}\right] & = & \left[a_{i-1}(\mu_{i-1}-y_{i-1})-\frac{1}{2}\sigma^2(t_{i-1},y_{t_{i-1}})\right]\Delta t \\ \mathbb{E}\left[\left(\Delta\left(\ln(y_i)\right)-\mathbb{E}\left[\Delta\left(\ln(y_i)\right)|\mathcal{F}_{i-1}\right]\right)^2|\mathcal{F}_{i-1}\right] & = & \sigma^2(t_{i-1},y_{t_{i-1}})\Delta t. \end{array} \tag{3.19}$$

Basis for Lagged Adaptive Discrete-time Expectation Process: From (3.19), (3.18) reduces to

$$\begin{cases} \Delta y_i & = \mathbb{E}\left[\Delta y_i|\mathcal{F}_{i-1}\right]+\sigma(t_{i-1},y_{t_{i-1}})y_{i-1}\Delta W(t_i) \\ \Delta\left(\ln(y_i)\right) & = \mathbb{E}\left[\Delta\left(\ln(y_i)\right)|\mathcal{F}_{i-1}\right]+\sigma(t_{i-1},y_{t_{i-1}})\Delta W(t_i). \end{cases} \tag{3.20}$$

(3.20) provides the basis for the development of the concept of lagged adaptive expectation process [33] with respect to continuous time stochastic dynamic systems (3.16) and (3.17).

Remark 3.3. (Orthogonality Condition Vector for (3.16) **and** (3.17)

Following Remark 3.1 and using (3.18), (3.19) and (3.20), we further remark that the orthogonality condition vector [10] with respect to continuous-time stochastic dynamic model (3.16) is represented by:

$$H(t_{i-1},y(t_i),y(t_{i-1}))=\begin{pmatrix} \Delta y(t_i)-a(t_{i-1})y(t_{i-1})(\mu(t_{i-1})-y(t_{i-1}))\Delta t_i \\ \Delta\ln(y(t_i))-L\ln(y(t_{i-1}),y_{t_{i-1}})\Delta t_i \\ (\Delta\ln(y(t_i))-L\ln(y(t_{i-1}),y_{t_{i-1}})\Delta t_i)^2-\sigma^2(t_{i-1},y_{t_{i-1}})\Delta t_i \end{pmatrix} \tag{3.21}$$

where $\mathrm{L}\ln(y(t_{i-1}),y_{t_{i-1}})\Delta t_i = \left(a(t_{i-1})(\mu(t_{i-1})-y(t_{i-1}))-\frac{1}{2}\sigma^2(t_{i-1},y_{t_{i-1}})\right)\Delta t_i$. Moreover, unlike the orthogonality condition vector defined in the literature [8, 10, 38], this orthogonality condition vector is based on the discretization scheme (3.18) associated with nonlinear continuous-time stochastic differential equations (3.16) and (3.17) and the Itô-Doob stochastic differential calculus [21, 24]

Local Observation System of Algebraic Equations: For $k \in I_0(N)$, applying the lagged adaptive expectation process [33], from Definitions 2.3 – 2.7, and using the discretized form of (2.8) and (3.20), we formulate a local observation/measurement process at t_k as a algebraic functions of m_k-local functions of restriction of the overall finite sample sequence $\{y_i\}_{i=-r}^{N}$ to a subpartition P_k in Definition 2.2. Let $a_{t_{k*}}$, $\mu_{t_{k*}}$ be estimates of a_t and μ_t, respectively, at each time t. We have

$$\begin{cases} \frac{1}{m_k}\sum\limits_{i=k-m_k}^{k-1}\mathbb{E}\left[\Delta y_i|\mathcal{F}_{i-1}\right] & = & a_{t_{k*}}\left[\frac{\mu_{t_{k*}}}{m_k}\sum\limits_{i=k-m_k}^{k-1} y_{i-1}-\frac{1}{m_k}\sum\limits_{i=k-m_k}^{k-1} y_{i-1}^2\right]\Delta t, \\ \frac{1}{m_k}\sum\limits_{i=k-m_k}^{k-1}\mathbb{E}\left[\Delta\left(\ln(y_i)\right)|\mathcal{F}_{i-1}\right] & = & a_{t_{k*}}\left[\mu_{t_{k*}}-\frac{1}{m_k}\sum\limits_{i=k-m_k}^{k-1} y_{i-1}\right]\Delta t-\frac{1}{2m_k}\sum\limits_{i=k-m_k}^{k-1}\mathbb{E}\left[\left(\Delta\left(\ln(y_i)\right)-\mathbb{E}\left[\Delta\left(\ln(y_i)\right)|\mathcal{F}_{i-1}\right]\right)^2|\mathcal{F}_{i-1}\right], \\ \hat{\sigma}^2_{m_k,k} & = & \begin{cases} \frac{1}{m_k\Delta t}\sum\limits_{i=k-m_k}^{k-1}\mathbb{E}\left[\left(\Delta\left(\ln(y_i)\right)-\mathbb{E}\left[\Delta\left(\ln(y_i)\right)|\mathcal{F}_{i-1}\right]\right)^2|\mathcal{F}_{i-1}\right] & \text{if } m_k \text{ is small} \\ \frac{1}{(m_k-1)\Delta t}\sum\limits_{i=k-m_k}^{k-1}\mathbb{E}\left[\left(\Delta\left(\ln(y_i)\right)-\mathbb{E}\left[\Delta\left(\ln(y_i)\right)|\mathcal{F}_{i-1}\right]\right)^2|\mathcal{F}_{i-1}\right] & \text{if } m_k \text{ is large.} \end{cases} \end{cases} \tag{3.22}$$

From the third equation in (3.22), it follows that the average volatility square $\hat{\sigma}^2_{m_k,k}$ is given by

$$\hat{\sigma}^2_{m_k,k}=\frac{s^2_{m_k,k}}{\Delta t}, \tag{3.23}$$

where $s^2_{m_k,k}$ is the local sample variance statistics for volatility at t_k in the context of $x(t_i)=\Delta\left(\ln(y_i)\right)$. We define

$$\begin{aligned} F_1\left(\mathbb{E}\left[\Delta y_i|\mathcal{F}_{i-1}\right],\mathbb{E}\left[\Delta(\ln y_i)|\mathcal{F}_{i-1}\right];a_{t_{k*}},\mu_{t_{k*}}\right) &= \frac{\sum\limits_{i=k-m_k}^{k-1}\mathbb{E}\left[\Delta y_i|\mathcal{F}_{i-1}\right]}{m_k}-a_{t_{k*}}\left[\frac{\mu_{t_{k*}}\sum\limits_{i=k-m_k}^{k-1} y_{i-1}}{m_k}-\frac{\sum\limits_{i=k-m_k}^{k-1} y_{i-1}^2}{m_k}\right]\Delta t \\ F_2\left(\mathbb{E}\left[\Delta y_i|\mathcal{F}_{i-1}\right],\mathbb{E}\left[\Delta(\ln y_i)|\mathcal{F}_{i-1}\right];a_{t_{k*}},\mu_{t_{k*}}\right) &= \frac{1}{m_k}\sum\limits_{i=k-m_k}^{k-1}\mathbb{E}\left[\Delta(\ln y_i)|\mathcal{F}_{i-1}\right]-a_{t_{k*}}\left[\mu_{t_{k*}}-\frac{1}{m_k}\sum\limits_{i=k-m_k}^{k-1} y_{i-1}\right]\Delta t+\frac{s^2_{m_k,k}}{2}. \end{aligned} \tag{3.24}$$

Then we have

$$\begin{cases} F_1\left(\mathbb{E}\left[\Delta y_i|\mathcal{F}_{i-1}\right],\mathbb{E}\left[\Delta(\ln y_i)|\mathcal{F}_{i-1}\right];a_{t_{k*}},\mu_{t_{k*}}\right) & = & 0, \\ F_2\left(\mathbb{E}\left[\Delta y_i|\mathcal{F}_{i-1}\right],\mathbb{E}\left[\Delta(\ln y_i)|\mathcal{F}_{i-1}\right];a_{t_{k*}},\mu_{t_{k*}}\right) & = & 0. \end{cases} \tag{3.25}$$

Let $F=\{F_1,F_2\}$. The determinant of the Jacobian matrix of F is given by

$$JF(a_{t_{k*}},\mu_{t_{k*}})=-\frac{a_{t_{k*}}}{m_k}\left[\sum_{i=k-m_k}^{k-1} y_{i-1}^2-\frac{1}{m_k}\left(\sum_{i=k-m_k}^{k-1} y_{i-1}\right)^2\right](\Delta t)^2=-a_{t_{k*}}\,var(y(t_{i-1})_{i=k-m_k}^{k-1})(\Delta t)^2\neq 0, \tag{3.26}$$

provided that $a\neq 0$ or the sequence $\{x(t_{i-1})\}_{i=-r+1}^{N}$ is neither zero nor a constant sequence. This fulfils the hypothesis of Theorem 3.1.

Thus, by the application of Theorem 3.1 (Implicit Function Theorem), we conclude that for every non-constant m_k-local sequence $\{x(t_i)\}_{i=k-m_k}^{k-1}$, there exists a unique solution of system of algebraic equations (3.25), $\hat{a}_{m_k,k}$ and $\hat{\mu}_{m_k,k}$ as a point estimates of a and μ, respectively.

We also note that the estimated values $\hat{a}_{m_k,k}\equiv a_{t_{k*}}$, $\hat{\mu}_{m_k,k}\equiv\mu_{t_{k*}}$, of a and μ, respectively, change at each time t_k. For instance, at time $t_0=0$ and the given $\mathcal{F}_{-1}$ measurable discrete-time process $y_{-r+1},y_{-r+2},...,y_{-1}$, (3.22) reduces to

$$
\begin{cases}
\frac{1}{m_0}\sum\limits_{i=-m_0}^{0}\Delta y_i & = \hat{a}_{m_0,0}\left[\frac{\hat{\mu}_{m_0,0}}{m_0}\sum\limits_{i=-m_0}^{0} y_{i-1}-\frac{1}{m_0}\sum\limits_{i=-m_0}^{0} y_{i-1}^2\right]\Delta t,\\
\frac{1}{m_0}\sum\limits_{i=-m_0}^{0}\Delta(\ln y_i) & = \hat{a}_{m_0,0}\left[\hat{\mu}_{m_0,0}-\frac{1}{m_0}\sum\limits_{i=-m_0}^{0} y_{i-1}\right]\Delta t-\frac{s_{m_0,0}^2}{2},\\
\hat{\sigma}_{m_0,0}^2 & = \frac{s_{m_0,0}^2}{\Delta t}.
\end{cases}
\tag{3.27}
$$

The initial solution of algebraic equations (3.27) at time t_0 is given by

$$
\begin{cases}
\hat{a}_{m_0,0} & = \dfrac{\left(\frac{1}{m_0}\sum\limits_{i=-m_0}^{0}\Delta(\ln y_i)+\frac{s_{m_0,0}^2}{2}\right)\left(\frac{1}{m_0}\sum\limits_{i=-m_0}^{0} y_{i-1}\right)-\frac{1}{m_0}\sum\limits_{i=-m_0}^{0}\Delta y_i}{\frac{1}{m_0}\left[\sum\limits_{i=-m_0}^{0} y_{i-1}^2-\frac{1}{m_0}\left(\sum\limits_{i=-m_0}^{0} y_{i-1}\right)^2\right]\Delta t}\\
\hat{\mu}_{m_0,0} & = \dfrac{\frac{1}{m_0\Delta t}\sum\limits_{i=-m_0}^{0}\Delta(\ln y_i)+\frac{s_{m_0,0}^2}{2\Delta t}+\frac{\hat{a}_{m_0,0}}{m_0}\left(\sum\limits_{i=-m_0}^{0} y_{i-1}\right)}{\hat{a}_{m_0,0}}\\
\hat{\sigma}_{m_0,0}^2 & = \frac{s_{m_0,0}^2}{\Delta t}.
\end{cases}
\tag{3.28}
$$

At time $t_1 = 1$ and the given $\mathcal{F}_0$ measurable discrete-time process $y_{-r}, y_{-r+1}, ..., y_{-1}, y_0$, we have

$$
\begin{cases}
\hat{a}_{m_1,1} & = \dfrac{\left(\frac{1}{m_1}\sum\limits_{i=1-m_1}^{0}\Delta(\ln y_i)+\frac{s_{m_1,1}^2}{2}\right)\left(\frac{1}{m_1}\sum\limits_{i=1-m_1}^{0} y_{i-1}\right)-\frac{1}{m_1}\sum\limits_{i=1-m_1}^{0}\Delta y_i}{\frac{1}{m_1}\left[\sum\limits_{i=1-m_1}^{0} y_{i-1}^2-\frac{1}{m_1}\left(\sum\limits_{i=1-m_1}^{0} y_{i-1}\right)^2\right]\Delta t}\\
\hat{\mu}_{m_1,1} & = \dfrac{\frac{1}{m_1\Delta t}\sum\limits_{i=1-m_1}^{0}\Delta(\ln y_i)+\frac{s_{m_1,1}^2}{2\Delta t}+\frac{\hat{a}_{m_1,1}}{m_1}\left(\sum\limits_{i=1-m_1}^{0} y_{i-1}\right)}{\hat{a}_{m_1,1}}\\
\hat{\sigma}_{m_1,1}^2 & = \frac{s_{m_1,1}^2}{\Delta t}.
\end{cases}
\tag{3.29}
$$

Repeating the above procedure, from (3.22) and applying the principle of mathematical induction [23], we have

$$
\begin{cases}
\hat{a}_{m_k,k} & = \dfrac{\left(\frac{1}{m_k}\sum\limits_{i=k-m_k}^{k-1}\Delta(\ln y_i)+\frac{s_{m_k,k}^2}{2}\right)\left(\frac{1}{m_k}\sum\limits_{i=k-m_k}^{k-1} y_{i-1}\right)-\frac{1}{m_k}\sum\limits_{i=k-m_k}^{k-1}\Delta y_i}{\frac{1}{m_k}\left[\sum\limits_{i=k-m_k}^{k-1} y_{i-1}^2-\frac{1}{m_k}\left(\sum\limits_{i=k-m_k}^{k-1} y_{i-1}\right)^2\right]\Delta t}\\
\hat{\mu}_{m_k,k} & = \dfrac{\frac{1}{m_k\Delta t}\sum\limits_{i=k-m_k}^{k-1}\Delta(\ln y_i)+\frac{s_{m_k,k}^2}{2\Delta t}+\frac{\hat{a}_{m_k,k}}{m_k}\left(\sum\limits_{i=k-m_k}^{k-1} y_{i-1}\right)}{\hat{a}_{m_k,k}},\\
\hat{\sigma}_{m_k,k}^2 & = \frac{s_{m_k,k}^2}{\Delta t}.
\end{cases}
\tag{3.30}
$$

Remark 3.4. We note that without loss in generality, the discrete-time data set $\{y_{-r+i} : i \in I_1(r-1)\}$ is assumed to be close to the true values of the solution process of the continuous-time dynamic process. In fact, this assumption is feasible in view of the uniqueness and continuous dependence of solution process of stochastic functional or ordinary differential equation with respect to the initial data [24].

Remark 3.5. If the sample $\{y_i\}_{i=k-m_k-1}^{k-1}$ is a constant sequence, then it follows from (3.30) and the fact that $\Delta(\ln y_i) = 0$ and $s_{m_k,k}^2 = 0$, that $\hat{\mu}_{m_k,k} \to \frac{1}{m_k}\sum\limits_{i=k-m_k}^{k-1} y_{i-1}$. Hence, it follows from (3.22) that $\hat{a}_{m_k,k} = 0$.

Remark 3.6. As we stated before, estimated parameters a, μ, and σ^2 depend upon the time at which data point is drawn. This is what we expected because of the fact that nonlinearity of the dynamic model together with environmental stochastic perturbations generate non stationary solution process. Using this locally estimated parameters of the continuous-time dynamic system, we can find the average of these local parameters over the entire size of data set as follows:

$$\begin{cases} \bar{a} &= \frac{1}{N}\sum_{i=0}^{N} a_{\hat{m}_i,i}, \\ \bar{\mu} &= \frac{1}{N}\sum_{i=0}^{N} \mu_{\hat{m}_i,i} \\ \overline{\sigma^2} &= \frac{1}{N}\sum_{i=0}^{N} \sigma^2_{\hat{m}_i,i}. \end{cases} \tag{3.31}$$

$\bar{a}$, $\bar{\mu}$, and $\overline{\sigma^2}$ are referred to as aggregated parameter estimates of a, μ, and σ^2 over the given entire finite interval of time, respectively.

Remark 3.7. The "discrete-time interconnected dynamic model for statistic process" (DTIDMLSMVSP) (Lemma 2.1) and its transformation of data are utilized in (3.22), (3.23), (3.24), (3.30), and (3.31) for updating statistic coefficients of equations in (3.19). This indeed accelerates the computation process. Furthermore, DTIDMLSMVSP plays a very significant role in the local discretization and model validation errors.

3.6. Illustration 2: Dynamic Model for U.S. Treasury Bill Interest Rate and the U.S.-U.K. Foreign Exchange Rate

We also apply the above presented scheme for estimating parameters of a continuous-time model for U.S. Treasury Bill Interest Rate [43] and U.S.-U.K. Foreign Exchange Rate [44] processes. By employing dynamic modeling process [23, 24], a continuous time dynamic model of interest rate process under random environmental perturbations can be described by

$$dy = (\beta y + \mu y^{\delta})dt + \sigma y^{\gamma} dW(t), y(t_0) = y_0, \tag{3.32}$$

where $\beta, \mu, \delta, \sigma, \gamma \in \Re$; $y(t, t_0, y_0)$ is adapted, non-anticipating solution process with respect to $\mathcal{F}_t$; the initial process y_0 is $\mathcal{F}_{t_0}$-measurable and independent of $\{W(t), t \in [t_0, T]\}$; $W(t)$ is a standard Wiener process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathbb{P})$.

Transformation of Stochastic Differential Equation (3.32): For (3.32), we consider the Lyapunov functions

$V_1(t,y) = \frac{1}{2}y^2$, and $V_2(t,y) = \frac{1}{3}y^3$ as in (3.2). The Itô differentials of V_i, for $i = 1,2$, are given by

$$\begin{cases} dV_1 &= \left[y(\beta y + \mu y^{\delta}) + \frac{1}{2}\sigma^2 y^{2\gamma}\right]dt + \sigma y^{\gamma+1}dW \\ dV_2 &= \left[y^2(\beta y + \mu y^{\delta}) + \sigma^2 y^{2\gamma+1}\right]dt + \sigma y^{\gamma+2}dW. \end{cases} \tag{3.33}$$

The Euler-type Numerical Schemes for (3.32) **and** (3.33): Following the approach in Section 3 and illustration 3.5, the Euler discretized scheme ($\Delta t = 1$) for (3.32) is defined by

$$\begin{cases} \Delta y_i &= (\beta y_{i-1} + \mu y_{i-1}^{\delta}) + \sigma y_{i-1}^{\gamma}\Delta W(t_i) \\ \frac{1}{2}\Delta(y_i^2) &= y_{i-1}(\beta y_{i-1} + \mu y_{i-1}^{\delta}) + \frac{1}{2}\sigma^2 y_{i-1}^{2\gamma} + \sigma y_{i-1}^{\gamma+1}\Delta W_i \\ \frac{1}{3}\Delta(y_i^3) &= y_{i-1}^2(\beta y_{i-1} + \mu y_{i-1}^{\delta}) + \sigma^2 y_{i-1}^{2\gamma+1} + \sigma y_{i-1}^{\gamma+2}\Delta W_i. \end{cases} \tag{3.34}$$

Generalized Moment Equations: Applying conditional expectation to (3.34) with respect to $\mathcal{F}_{i-1}$, we obtain

$$\begin{aligned} \mathbb{E}\left[\Delta y_i|\mathcal{F}_{i-1}\right] &= \beta y_{i-1} + \mu y_{i-1}^{\delta} \\ \frac{1}{2}\mathbb{E}\left[\Delta(y_i^2)|\mathcal{F}_{i-1}\right] &= \beta y_{i-1}^2 + \mu y_{i-1}^{\delta+1} + \frac{1}{2}\sigma^2 y_{i-1}^{2\gamma} \\ \frac{1}{3}\mathbb{E}\left[\Delta(y_i^3)|\mathcal{F}_{i-1}\right] &= \beta y_{i-1}^3 + \mu y_{i-1}^{\delta+2} + \sigma^2 y_{i-1}^{2\gamma+1} \\ \mathbb{E}\left[(\Delta y_i - \mathbb{E}\left[\Delta y_i|\mathcal{F}_{i-1}\right])^2|\mathcal{F}_{i-1}\right] &= \sigma^2 y_{i-1}^{2\gamma}, \\ \frac{1}{4}\mathbb{E}\left[\left(\Delta(y_i^2) - \mathbb{E}\left[\Delta(y_i^2)\right]\right)^2|\mathcal{F}_{i-1}\right] &= \sigma^2 y_{i-1}^{2\gamma+2}. \end{aligned} \tag{3.35}$$

Basis for Lagged Adaptive Discrete-time Expectation Process: From (3.35), (3.34) reduces to

$$\begin{cases} \Delta y_i &= \mathbb{E}\left[\Delta y_i|\mathcal{F}_{i-1}\right] + \sigma y_{i-1}^{\gamma}\Delta W(t_i) \\ \frac{1}{2}\Delta(y_i^2) &= \frac{1}{2}\mathbb{E}\left[\Delta(y_i^2)|\mathcal{F}_{i-1}\right] + \sigma y_{i-1}^{\gamma+1}\Delta W_i \\ \frac{1}{3}\Delta(y_i^3) &= \frac{1}{3}\mathbb{E}\left[\Delta(y_i^3)|\mathcal{F}_{i-1}\right] + \sigma y_{i-1}^{\gamma+2}\Delta W_i. \end{cases} \tag{3.36}$$

Remark 3.8. Orthogonality Condition Vector for (3.32) **and** (3.33)**:** Again, imitating Remarks 3.1, 3.2 and 3.3 and in the context of (3.32), (3.33), (3.34), (3.35) and (3.36), the orthogonality condition vector [10] with respect to continuous-time stochastic dynamic model (3.32) is as:

$$H(t_{i-1}, y(t_i), y(t_{i-1})) = \begin{pmatrix} \Delta y(t_i) - (\beta y(t_{i-1}) + \mu y^{\delta}(t_{i-1}))\Delta t_i \\ \frac{1}{2}\Delta(y^2(t_i)) - L(y^2(t_{i-1}))\Delta t_i \\ \frac{1}{3}\Delta(y^3(t_i)) - L(y^3(t_{i-1}))\Delta t_i \\ \left(\Delta y(t_i) - (\beta y(t_{i-1}) + \mu y^{\delta}(t_{i-1}))\Delta t_i\right)^2 - \sigma^2 y^{2\gamma}(t_{i-1})\Delta t_i \\ \left(\frac{1}{2}\Delta(y^2(t_i)) - L(y^2(t_{i-1}))\Delta t_i\right)^2 - \sigma^2 y^{2\gamma+2}(t_{i-1})\Delta t_i \end{pmatrix} \tag{3.37}$$

where $L(y^2(t_{i-1}))\Delta t_i = \left(y(t_{i-1})\left(\beta y(t_{i-1}) + \mu y^{\delta}(t_{i-1})\right) + \frac{1}{2}\sigma^2 y^{2\gamma}(t_{i-1})\right)\Delta t_i$ and $L(y^3(t_{i-1}))\Delta t_i = \left(y^2(t_{i-1})\left(\beta y(t_{i-1}) + \mu y^{\delta}(t_{i-1})\right) + \sigma^2 y^{2\gamma+1}(t_{i-1})\right)\Delta t_i$. Moreover, unlike the orthogonality condition vector defined in the literature [8, 10, 38], this orthogonality condition vector is based on the discretization scheme (3.34) associated with nonlinear continuous-time stochastic differential equations (3.32) and (3.33).

Local Observation System of Algebraic Equations: Following the argument used in (3.22), for $k \in I_0(N)$, applying the lagged adaptive expectation process [33], from Definitions 2.3 − 2.7, and using (2.8) and (3.35), we formulate a local observation/measurement process at t_k as a algebraic functions of m_k-local functions of restriction of the overall finite sample sequence $\{y_i\}_{i=-r}^{N}$ to subpartition P_k in Definition 2.2:

$$\begin{cases} \frac{1}{m_k}\sum\limits_{i=k-m_k}^{k-1} \mathbb{E}\left[\Delta y_i|\mathcal{F}_{i-1}\right] & = \beta\frac{\sum\limits_{i=k-m_k}^{k-1} y_{i-1}}{m_k} + \mu\frac{\sum\limits_{i=k-m_k}^{k-1} y_{i-1}^{\delta}}{m_k} \\ \frac{1}{2m_k}\sum\limits_{i=k-m_k}^{k-1}\left[\mathbb{E}\left[\Delta(y_i^2)|\mathcal{F}_{i-1}\right] - \mathbb{E}\left[\left(\Delta y_i - \mathbb{E}\left[\Delta y_i|\mathcal{F}_{i-1}\right]\right)^2|\mathcal{F}_{i-1}\right]\right] & = \beta\frac{\sum\limits_{i=k-m_k}^{k-1} y_{i-1}^2}{m_k} + \mu\frac{\sum\limits_{i=k-m_k}^{k-1} y_{i-1}^{\delta+1}}{m_k} \\ \frac{1}{m_k}\sum\limits_{i=k-m_k}^{k-1}\left[\frac{1}{3}\mathbb{E}\left[\Delta(y_i^3)|\mathcal{F}_{i-1}\right] - \sigma^2 y_{i-1}^{2\gamma+1}\right] & = \beta\frac{\sum\limits_{i=k-m_k}^{k-1} y_{i-1}^3}{m_k} + \mu\frac{\sum\limits_{i=k-m_k}^{k-1} y_{i-1}^{\delta+2}}{m_k} \\ \frac{1}{m_k}\sum\limits_{i=k-m_k}^{k-1}\mathbb{E}\left[\left(\Delta y_i - \mathbb{E}\left[\Delta y_i|\mathcal{F}_{i-1}\right]\right)^2|\mathcal{F}_{i-1}\right] & = \sigma^2\frac{\sum\limits_{i=k-m_k}^{k-1} y_{i-1}^{2\gamma}}{m_k}, \\ \frac{1}{4m_k}\sum\limits_{i=k-m_k}^{k-1}\mathbb{E}\left[\left(\Delta(y_i^2) - \mathbb{E}\left[\Delta(y_i^2)\right]\right)^2|\mathcal{F}_{i-1}\right] & = \sigma^2\frac{\sum\limits_{i=k-m_k}^{k-1} y_{i-1}^{2\gamma+2}}{m_k}. \end{cases} \tag{3.38}$$

Following the approach discussed in Section 3.5, the solution of $\sigma_{m_k,k}$ is given by

$$\sigma_{m_k,k} = \left[\frac{s_{m_k,k}^2}{\frac{1}{m_k}\sum\limits_{i=k-m_k}^{k-1} y_{i-1}^{2\gamma_{m_k,k}}}\right]^{1/2} \tag{3.39}$$

and $\gamma_{m_k,k}$ satisfies the following nonlinear algebraic equation

$$s_{m_k,k}^2 \sum_{i=k-m_k}^{k-1} y_{i-1}^{2\gamma_{m_k,k}+2} - \frac{1}{4}\mathbf{s}_{m_k,k}^2 \sum_{i=k-m_k}^{k-1} y_{i-1}^{2\gamma_{m_k,k}} = 0, \tag{3.40}$$

where $s_{m_k,k}^2$, and $\mathbf{s}_{m_k,k}^2$ denotes the local moving variance of Δy_i and $\Delta(y_i^2)$ respectively.

To solve for the parameters β, μ and δ, we define the conditional moment functions

$$F_j \equiv F_j\left(\mathbb{E}\left[\Delta y_i|\mathcal{F}_{i-1}\right], \mathbb{E}\left[\Delta(y_i)^2|\mathcal{F}_{i-1}\right], \mathbb{E}\left[\Delta(y_i)^3|\mathcal{F}_{i-1}\right]\right), \quad j = 1, 2, 3$$

as

$$
\begin{aligned}
F_1 &= \frac{1}{m_k}\sum_{i=k-m_k}^{k-1}\mathbb{E}\left[\Delta y_i|\mathcal{F}_{i-1}\right]-\beta\frac{\sum_{i=k-m_k}^{k-1} y_{i-1}}{m_k}-\mu\frac{\sum_{i=k-m_k}^{k-1} y_{i-1}^{\delta}}{m_k}\\
F_2 &= \frac{1}{2m_k}\sum_{i=k-m_k}^{k-1}\left[\mathbb{E}\left[\Delta(y_i^2)|\mathcal{F}_{i-1}\right]-\mathbb{E}\left[\left(\Delta y_i-\mathbb{E}\left[\Delta y_i|\mathcal{F}_{i-1}\right]\right)^2|\mathcal{F}_{i-1}\right]\right]-\beta\frac{\sum_{i=k-m_k}^{k-1} y_{i-1}^2}{m_k}-\mu\frac{\sum_{i=k-m_k}^{k-1} y_{i-1}^{\delta+1}}{m_k} \qquad (3.41)\\
F_3 &= \frac{1}{m_k}\sum_{i=k-m_k}^{k-1}\left[\frac{1}{3}\mathbb{E}\left[\Delta(y_i^3)|\mathcal{F}_{i-1}\right]-\sigma^2 y_{i-1}^{2\gamma+1}\right]-\beta\frac{\sum_{i=k-m_k}^{k-1} y_{i-1}^3}{m_k}-\mu\frac{\sum_{i=k-m_k}^{k-1} y_{i-1}^{\delta+2}}{m_k}.
\end{aligned}
$$

Using (3.38), we have

$$
\begin{cases}
F_1 = 0\\
F_2 = 0\\
F_3 = 0
\end{cases} \qquad (3.42)
$$

Let $F=\{F_1,F_2,F_3\}$. The determinant of the Jacobian matrix of F is given by

$$
JF(\beta,\mu,\delta) = -\frac{1}{m_k^3}det\begin{pmatrix}
\sum_{i=k-m_k}^{k-1} y_{i-1} & \sum_{i=k-m_k}^{k-1} y_{i-1}^{\delta} & \sum_{i=k-m_k}^{k-1}(\ln y_{i-1})y_{i-1}^{\delta}\\
\sum_{i=k-m_k}^{k-1} y_{i-1}^2 & \sum_{i=k-m_k}^{k-1} y_{i-1}^{\delta+1} & \sum_{i=k-m_k}^{k-1}(\ln y_{i-1})y_{i-1}^{\delta+1}\\
\sum_{i=k-m_k}^{k-1} y_{i-1}^3 & \sum_{i=k-m_k}^{k-1} y_{i-1}^{\delta+2} & \sum_{i=k-m_k}^{k-1}(\ln y_{i-1})y_{i-1}^{\delta+2}
\end{pmatrix} \neq 0 \qquad (3.43)
$$

provided $\delta \neq 1$ and the sequence $\{y(t_{i-1})\}_{i=k-m_k}^{k-1}$ is neither zero nor a constant sequence. Thus, by the application of Theorem 3.1 (Implicit Function Theorem), we conclude that for every non-constant m_k-local sequence $\{y(t_i)\}_{i=k-m_k}^{k-1}$, $\delta \neq 1$, there exist a solution of system of algebraic equations (3.42) $\hat{\beta}_{m_k,k}$, $\hat{\mu}_{m_k,k-1}$, $\hat{\delta}_{m_k,k}$ as a point estimates of β and μ, and δ respectively.

The solution of equation (3.42) is given by

$$
\begin{cases}
\hat{\mu}_{m_k,k} = \dfrac{\frac{1}{m_k}\sum_{i=k-m_k}^{k-1}\Delta y_i \sum_{i=k-m_k}^{k-1} y_{i-1}^2-\frac{1}{2}\left[\frac{1}{m_k}\sum_{i=k-m_k}^{k-1}\Delta(y_i^2)-s_{m_k,k}^2\right]\sum_{i=k-m_k}^{k-1} y_{i-1}}{\frac{1}{m_k}\left[\sum_{i=k-m_k}^{k-1} y_{i-1}^{\delta_{m_k,k}}\sum_{i=k-m_k}^{k-1} y_{i-1}^2-\sum_{i=k-m_k}^{k-1} y_{i-1}^{1+\delta_{m_k,k}}\sum_{i=k-m_k}^{k-1} y_{i-1}\right]}\\
\hat{\beta}_{m_k,k} = \dfrac{\sum_{i=k-m_k}^{k-1}\Delta y_i-\hat{\mu}_{m_k,k}\sum_{i=k-m_k}^{k-1} y_{i-1}^{\delta_{m_k,k}}}{\sum_{i=k-m_k}^{k-1} y_{i-1}},
\end{cases} \qquad (3.44)
$$

where $\delta_{m_k,k}$ satisfies the third equation in (3.38) described by

$$
\frac{1}{3m_k}\sum_{i=k-m_k}^{k-1}\Delta(y_i^3)-\frac{\sigma_{m_k,k}^2}{m_k}\sum_{i=k-m_k}^{k-1} y_{i-1}^{2\gamma_{m_k,k}+1}-\beta\frac{\sum_{i=k-m_k}^{k-1} y_{i-1}^3}{m_k}-\mu\frac{\sum_{i=k-m_k}^{k-1} y_{i-1}^{\delta+2}}{m_k}=0 \qquad (3.45)
$$

We further note that the parameters of continuous-time dynamic process described by(3.32) are time-varying functions. This justifies the modifications/correctness needed for the development of continuous-time models of dynamic processes.

Remark 3.9. The presented Illustrations exhibit the important features described in Remark 3.2 of the theoretical parameter estimation procedure. The illustrations further clearly differentiate the It$\hat{o}$-Doob differential formula [24] based formation of orthogonality condition vectors in Remarks 3.3 and 3.8 and the algebraic manipulation and discretized scheme using the econometric specification based orthogonality condition vectors in [9, 11, 17].

Remark 3.10. The "discrete-time interconnected dynamic model for statistic process" (DTIDMLSMVSP) (Lemma 2.1) and its transformation of data are utilized in (3.38), (3.39), (3.40), (3.44) and (3.45) for updating statistic coefficient of equations in (3.35). This indeed accelerates the computation process. Furthermore, DTIDMLSMVSP plays a very significant role in the local discretization and model validation errors.

4. Computational Algorithm

In this section, we outline computational, data organizational and simulation schemes. We introduce the ideas of iterative data process and data simulation process time schedules in relation with the real time data observation/collection schedule. For the computational estimation of continuous time stochastic dynamic system state and parameters, it is essential to determine an admissible set of local conditional sample average and sample variance, in particular, the size of local conditional sample in the context of a partition of time interval $[-\tau, T]$. Moreover, the discrete time dynamic model of conditional sample mean and sample variance statistic processes in Section 2 and the theoretical parameter estimation scheme in Section 3 coupled with the lagged adaptive expectation process motivate to outline a computational scheme in a systematic and coherent manner. A brief conceptual computational scheme and simulation process summary is described below:

4.1. Coordination of data observation, Iterative process, and Simulation schedules:

Without loss of generality, we assume that the real data observation/collection partition schedule P is defined in (2.2). Now, we present definitions of iterative process and simulation time schedules.

Definition 4.1. The iterative process time schedule in relation with the real data collection schedule is defined by

$$IP = \{F^{-r}t_i : \text{ for } t_i \in P\}, \tag{4.1}$$

where $F^{-r}t_i = t_{i-r}$, and F^{-r} is a forward shift operator [6].

The simulation time is based on the order p of the time series model of m_k-local conditional sample mean and variance processes in Lemma 2.1 in Section 2.

Definition 4.2. The simulation process time schedule in relation with the real data observation schedule is defined by

$$SP = \begin{cases} \{F^r t_i : \text{ for } t_i \in P\}, \text{ if } p \le r \\ \{F^p t_i : \text{ for } t_i \in P\}, \text{ if } p > r. \end{cases} \tag{4.2}$$

Remark 4.1. We note that the initial times of iterative and simulation processes are equal to the real data times t_r and t_p, respectively. Moreover, iterative and simulation process time in (4.1) and (4.2), respectively, justify Remark 3.4. In short, t_i is the scheduled time clock for the collection of the ith observation of the state of the system under investigation. The iterative process and simulation process times are t_{i+r} and t_{i+p}, respectively.

4.2. Conceptual Computational Parameter Estimation Scheme

For the conceptual computational dynamic system parameter estimation, we need to introduce a few concepts of local admissible sample/data observation size, m_k-local admissible conditional finite sequence at $t_k \in SP$, local finite sequence of parameter estimates at t_k.

Definition 4.3. For each $k \in I_0(N)$, we define local admissible sample/data observation size m_k at t_k as $m_k \in OS_k$, where

$$OS_k = \begin{cases} I_2(r + k - 1), \text{ if } p \le r, \\ I_2(p + k - 1), \text{ if } p > r, \end{cases} \tag{4.3}$$

Moreover, OS_k is referred as the local admissible set of lagged sample/data observation size at t_k.

Definition 4.4. For each admissible $m_k \in OS_k$ in Definition 4.3, a m_k-local admissible lagged-adapted finite restriction sequence of conditional sample/data observation at t_k to subpartition P_k of P in Definition 2.3 is defined by $\{\mathbb{E}[y_i|\mathcal{F}_{i-1}]\}_{i=k-m_k}^{k-1}$. Moreover, a m_k- class of admissible lagged-adapted finite sequences of conditional sample/data observation of size m_k at t_k is defined by

$$\mathcal{AS}_k = \{\{\mathbb{E}[y_i|\mathcal{F}_{i-1}]\}_{i=k-m_k}^{k-1} : m_k \in OS_k\} = \{\{\mathbb{E}[y_i|\mathcal{F}_{i-1}]\}_{i=k-m_k}^{k-1}\}_{m_k \in OS_k}. \tag{4.4}$$

Without loss of generality, in the case of energy commodity model, for each $m_k \in OS_k$, we find corresponding m_k- local admissible adapted finite sequence of conditional sample/data observation at t_k, $\{\mathbb{E}[y_i|\mathcal{F}_{i-1}]\}_{i=k-m_k}^{k-1}$. Using this sequence and (3.30), we compute $\hat{a}_{m_k,k}$, $\hat{\mu}_{m_k,k}$ and $\hat{\sigma}^2_{m_k,k}$. This leads to a local admissible finite sequence of parameter estimates at t_k defined on OS_k as follows: $\left\{(\hat{a}_{m_k,k}, \hat{\mu}_{m_k,k}, \hat{\sigma}^2_{m_k,k})\right\}_{m_k \in OS_k} = \{(\hat{a}_{m_k,k}, \hat{\mu}_{m_k,k}, \hat{\sigma}^2_{m_k,k})\}_{m_k\in 2}^{r+k-1}$ or $\{(\hat{a}_{m_k,k}, \hat{\mu}_{m_k,k}, \hat{\sigma}^2_{m_k,k})\}_{m_k\in 2}^{p+k-1}$. It is denoted by

$$(\mathcal{A}_k, \mathbb{M}_k, \mathcal{S}_k) = \left\{(\hat{a}_{m_k,k}, \hat{\mu}_{m_k,k}, \hat{\sigma}^2_{m_k,k})\right\}_{m_k \in OS_k} \tag{4.5}$$

4.3. Conceptual Computation of State Simulation Scheme

For the development of a conceptual computational scheme, we need to employ the method of induction. The presented simulation scheme is based on the idea of lagged adaptive expectation process [33]. An autocorrelation function (ACF) analysis [6, 8] performed on $s^2_{m_k,k}$ suggests that the discrete time interconnected dynamic model of local conditional sample mean and sample variance statistic in (2.8) is of order $p = 2$. In view of this, we need to identify the initial data. We begin with a given initial data y_{t_0}, $\{\hat{s}^2_{m_0,0}\}_{m_0 \in OS_0}$, $\{\hat{s}^2_{m_{-1},-1}\}_{m_{-1} \in OS_{-1}}$, and $\{\bar{S}^2_{m_{-1},-1}\}_{m_{-1} \in OS_{-1}}$.

Let $y^s_{m_k,k}$ be a simulated value of $\mathbb{E}[y_k|\mathcal{F}_{k-1}]$ at time t_k corresponding to a local admissible lagged-adapted finite sequences of conditional sample/data observation of size m_k at t_k $\{\mathbb{E}[y_i|\mathcal{F}_{i-1}]\}_{i=k-m_k}^{k-1} \in \mathcal{AS}_k$ in (4.4). This simulated value is derived from the discretized Euler scheme (3.18) by

$$y^s_{m_k,k} = y^s_{m_{k-1},k-1} + \hat{a}_{m_{k-1},k-1}(\hat{\mu}_{m_{k-1},k-1} - y^s_{m_{k-1},k-1})y^s_{m_{k-1},k-1}\Delta t + \hat{\sigma}_{m_{k-1},k-1}y^s_{m_{k-1},k-1}\Delta W_{m_k,k}. \tag{4.6}$$

Let

$$\{y^s_{m_k,k}\}_{m_k \in OS_k} \tag{4.7}$$

be a m_k- local admissible sequence of simulated values corresponding to m_k-class $\mathcal{AS}_k$ of local admissible lagged-adapted finite sequences of conditional sample/data observation of size m_k at t_k in (4.4).

4.4. Mean-Square Sub-Optimal Procedure

To find the best estimate of $\mathbb{E}[y_k|\mathcal{F}_{k-1}]$ at time t_k from a m_k -local admissible finite sequence $\{y^s_{m_k,k}\}_{m_k \in OS_k}$ of a simulated value of $\{\mathbb{E}[y_i|\mathcal{F}_{i-1}]\}$, we need to compute a local admissible finite sequence of quadratic mean square error corresponding to $\{y^s_{m_k,k}\}_{m_k \in OS_k}$. The quadratic mean square error is defined below.

Definition 4.5. The quadratic mean square error of $\mathbb{E}[y_k|\mathcal{F}_{k-1}]$ relative to each member of the term of local admissible sequence $\{y^s_{m_k,k}\}_{m_k \in OS_k}$ of simulated values is defined by

$$\Xi_{m_k,k,y_k} = \left(\mathbb{E}[y_k|\mathcal{F}_{k-1}] - y^s_{m_k,k}\right)^2. \tag{4.8}$$

For any arbitrary small positive number ϵ and for each time t_k, to find the best estimate from the m_k-local admissible sequence $\{y^s_{m_k,k}\}_{m_k \in OS_k}$ of simulated values, we determine the following ϵ-sub-optimal admissible subset of set of m_k-size local admissible lagged sample size m_k at t_k (OS_k) as:

$$\mathcal{M}_k = \{m_k : \Xi_{m_k,k,y_k} < \epsilon \ \text{ for } \ m_k \in OS_k\}. \tag{4.9}$$

There are three different cases that determine the ϵ-best sub-optimal sample size $\hat{m}_k$ at time t_k.

Case 1: If $m_k \in \mathcal{M}_k$ gives the minimum, then m_k is recorded as $\hat{m}_k$.

Case 2: If more than one value of $m_k \in \mathcal{M}_k$, then the largest of such m_k's is recorded as $\hat{m}_k$.

Case 3: If condition (4.9) is not met at time t_k, (that is, $\mathcal{M}_k = \emptyset$), then the value of m_k where the minimum $\min_{m_k} \Xi_{m_k,k,y_k}$ is attained, is recorded as $\hat{m}_k$. The $\epsilon-$ best sub-optimal estimates of the parameters $\hat{a}_{m_k,k}$, $\hat{\mu}_{m_k,k}$ and $\hat{\sigma}^2_{m_k,k}$ at the ϵ-best sub-optimal sample size $\hat{m}_k$ are also recorded as $a_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$ and $\sigma^2_{\hat{m}_k,k}$, respectively.

Finally, the simulated value $y^s_{m_k,k}$ at time t_k with $\hat{m}_k$ is now recorded as the ϵ-best sub-optimal state estimate for $\mathbb{E}[y_k|\mathcal{F}_{k-1}]$ at time t_k. This ϵ-best sub-optimal simulated value of $\mathbb{E}[y_k|\mathcal{F}_{k-1}]$ at time t_k is denoted by $y^s_{\hat{m}_k,k}$. Similar reasoning can be provided for the estimates of the parameters of the U.S. Treasury Bill Yield Interest Rate and U.S.-U.K. Foreign Exchange Rate model.

Remark 4.2. In additions to comparative statements in Sections 2 together with Remarks 3.1, 3.2, 3.3, 3.7, 3.8, 3.9, and 3.10, we further augment a few more Conceptual Computational Comparison between the LLGMM and the existing OCBGMM as follows.

a: The LLGMM approach is focused on parameter and state estimation problems at each data collection/observation time t_k using the local lagged adaptive expectation process. In fact, LLGMM is discrete-time dynamic process. On the other hand, the OCBGMM is centered on the state and parameter estimates using the entire data that is to the left of the final data collection time $T_N = T$. Implied weakness in forecasting, as seen in the next section, is explicitly shown with the OCBGMM approach and the ensuing results.

b: We note that Remark 3.2 exhibits the interactions/interdependence between the first three components of LL-GMM,"(1) development of the stochastic model for continuous-time dynamic process, (2) development of the discrete-time interconnected dynamic model for statistic process, (3) utilization of the Euler-type discretized scheme for nonlinear and non-stationary system of stochastic differential equations" and their interactions. On the other hand, the OCBGMM is partially connected.

c: From the development of the computational algorithm section, We remark that the interdependence/ interconnectedness of the four remaining components of the LLGMM, "(4) employing lagged adaptive expectation process for developing generalized method of moment equations, (5) introducing conceptual computational parameter estimation problem, (6) formulating conceptual computational state estimation scheme and (7) defining conditional mean square ϵ-sub optimal procedure" is clearly illustrated. Moreover, the above stated components as well as data are directly connected with the original continuous-time SDE. On the other hand, the OCBGMM is composed of single size, single sequence, single estimates, single simulated value and single error. Hence, the OCBGMM is the "single shot approach". Moreover, the OCBGMM is highly dependent on its second component rather than the first component. See, Section 3.

d: The LLGMM is a discrete-time dynamic system composed of seven interactive interdependent components. On the other hand, the OCBGMM is static dynamic process of five almost isolated components.

e: Furthermore, the LLGMM is a "two scale hierarchic" quadratic mean-square optimization process", but the optimization process of OCBGMM is "single-shot"

f: Moreover, the LLGMM performs in discrete-time but operates like the original continuous-time dynamic process. The performance of the LLGMM approach is more superior than OCBGMM and IRGMM approaches.

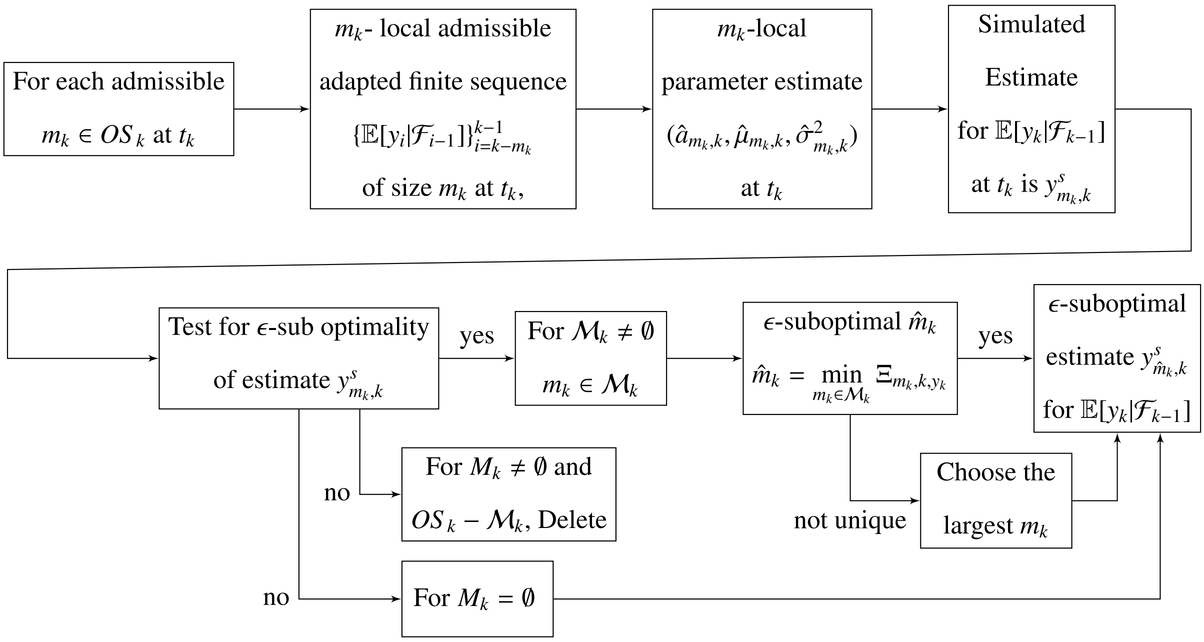
g: The LLGMM does not require a large size data set. In addition, as k increases, it generates a larger size of lagged adapted data set thereby further stabilizing the state and parameter estimation procedure with finite size data set, on the other hand the OCBGMM does not have this flexibility.

h: In fact, the local adaptive process component of LLGMM generates conceptual finite chain of discrete-time admissible sets/sub-data. See "Flowchart-1: LLGMM Conceptual Computational Algorithm". The OCBGMM does not possess this feature.

i: Item (h) indeed generates a finite computational chain that is described by "Flowchart-2: LLGMM Simulation Algorithm". The OCBGMM does not possess this feature.

Remark 4.3. We note that the choice of $p = 2$ was determined based on the statistical procedure known as the Autocorrelation Function Analysis [6, 8].

A detailed flowchart of the conceptual algorithm is as follows:



Flowchart 1: LLGMM Conceptual Computational Algorithm.

Moreover, a detailed simulation algorithm is presented in *Appendix D*

APPENDIX

Appendix A. Proof

Proof of Lemma 2.1 for small m_k, $m_{k-1} \leq m_k$,

PROOF.

$$
\begin{aligned}
\bar{S}_{m_k,k} &= \frac{1}{m_k}\sum_{i=1-m_k}^{0} F^i x_{k-1} \\
&= \frac{1}{m_k}\left[\sum_{i=1-m_k}^{-1-m_{k-1}} F^i x_{k-1} + \sum_{i=-m_{k-1}}^{-1} F^i x_{k-1} + F^0 x_{k-1}\right] \\
&= \frac{m_{k-1}}{m_k}\bar{S}_{m_{k-1},k-1} + \frac{1}{m_k}\left[\sum_{i=1-m_k}^{1-m_{k-1}} F^i x_{k-1} - F^{1-m_{k-1}} x_{k-1} - F^{-m_{k-1}} x_{k-1} + F^0 x_{k-1}\right]
\end{aligned}
$$

$$
\begin{aligned}
s^2_{m_k,k} &= \frac{1}{m_k}\left[\sum_{i=-m_k+1}^{0}\left(F^i x_{k-1}\right)^2 - \frac{1}{m_k}\left(\sum_{j=-m_k+1}^{0} F^j x_{k-1}\right)^2\right] \\
&= \frac{1}{m_k}\left[\sum_{i=-m_k+1}^{-m_{k-1}-1}\left(F^i x_{k-1}\right)^2 + \sum_{i=-m_{k-1}}^{-1}\left(F^i x_{k-1}\right)^2 + (F^0 x_{k-1})^2 - \frac{1}{m_k}\left(\sum_{j=-m_k+1}^{0} F^j x_{k-1}\right)^2\right]
\end{aligned}
$$

$$= \frac{1}{m_k}\left[\sum_{i=-m_{k-1}}^{-1}\left(F^i x_{k-1}\right)^2 - \frac{1}{m_{k-1}}\left(\sum_{i=-m_{k-1}}^{-1} F^i x_{k-1}\right)^2 + \frac{1}{m_{k-1}}\left(\sum_{i=-m_{k-1}}^{-1} F^i x_{k-1}\right)^2\right]$$
$$+\frac{1}{m_k}\left[\left(F^0 x_{k-1}\right)^2 - (F^{-m_{k-1}} x_{k-1})^2 - \left(F^{-m_{k-1}+1} x_{k-1}\right)^2 - \frac{1}{m_k}\left(\sum_{i=-m_k+1}^{0} F^i x_{k-1}\right)^2 + \sum_{i=-m_k+1}^{-m_{k-1}+1}\left(F^i x_{k-1}\right)^2\right]$$
$$= \frac{m_{k-1}}{m_k} s^2_{m_{k-1},k-1} + \frac{m_{k-1}}{m_k}\bar{S}^2_{m_{k-1},k-1} - \bar{S}^2_{m_k,k} + \frac{\left(F^0 x_{k-1}\right)^2 - (F^{-m_{k-1}} x_{k-1})^2 - \left(F^{-m_{k-1}+1} x_{k-1}\right)^2}{m_k}$$
$$+\frac{\sum_{i=-m_k+1}^{-m_{k-1}+1}\left(F^i x_{k-1}\right)^2}{m_k}.$$

Hence,

$$s^2_{m,k} = \frac{m_{k-1}}{m_k} s^2_{m_{k-1},k-1} + \frac{m_{k-1}}{m_k}\bar{S}^2_{m_{k-1},k-1} - \bar{S}^2_{m_k,k} + \frac{\left(F^0 x_{k-1}\right)^2 - \left(F^{-m_{k-1}} x_{k-1}\right)^2 - \left(F^{-m_{k-1}+1} x_{k-1}\right)^2}{m_k} + \frac{\sum_{i=-m_k+1}^{-m_{k-1}+1}\left(F^i x_{k-1}\right)^2}{m_k}. \tag{A.1}$$

Next, we find an expression connecting $\bar{S}^2_{m_k,k}, \bar{S}^2_{m_k-1,k-1}$ and $s^2_{m_k-1,k-1}$. By definition and simplification,

$$\begin{aligned} m_k^2 \bar{S}^2_{m_k,k} &= \left[\sum_{i=-m_k+1}^{0} F^i x_{k-1}\right]^2 = \sum_{i=-m_k+1}^{0}\left(F^i x_{k-1}\right)^2 + \sum_{\substack{l,s=-m_k+1 \\ l\neq s}}^{0} F^l x_{k-1} F^s x_{k-1} \\ &= (m_{k-1}) s^2_{m_{k-1},k-1} + m_{k-1}\bar{S}^2_{m_{k-1},k-1} + (F^0 x_{k-1})^2 - (F^{-m_{k-1}} x_{k-1})^2 - (F^{-m_{k-1}+1} x_{k-1})^2 \\ &\quad + \sum_{i=-m_k+1}^{-m_{k-1}+1} (F^i x_{k-1})^2 + \sum_{\substack{l,s=-m_k+1 \\ l\neq s}}^{0} F^l x_{k-1} F^s x_{k-1} \end{aligned} \tag{A.2}$$

Substituting $(A.2)$ into $(A.1)$, we have

$$s^2_{m,k} = \frac{m_k-1}{m_k}\left[\frac{m_{k-1}}{m_k} s^2_{m_{k-1},k-1} + \frac{m_{k-1}}{m_k}\bar{S}^2_{m_{k-1},k-1} + \frac{\left(F^0 x_{k-1}\right)^2 - \left(F^{-m_{k-1}} x_{k-1}\right)^2 - \left(F^{-m_{k-1}+1} x_{k-1}\right)^2}{m_k} + \frac{\sum_{i=-m_k+1}^{-m_{k-1}+1}\left(F^i x_{k-1}\right)^2}{m_k}\right] - \frac{\sum_{\substack{l,s=-m_k+1 \\ l\neq s}}^{0} F^l x_{k-1} F^s x_{k-1}}{m_k(m_k-1)}. \tag{A.3}$$

Likewise, using equation $(A.2)$,

$$\begin{aligned} m_{k-1}^2 \bar{S}^2_{m_{k-1},k-1} &= (m_{k-2}) s^2_{m_{k-2},k-2} + m_{k-2}\bar{S}^2_{m_{k-2},k-2} + (F^{-1} x_{k-1})^2 - (F^{-m_{k-2}-1} x_{k-1})^2 - (F^{-m_{k-2}} x_{k-1})^2 \\ &\quad + \sum_{i=-m_{k-1}}^{-m_{k-2}} (F^i x_{k-1})^2 + \sum_{\substack{l,s=-m_{k-1} \\ l\neq s}}^{-1} F^l x_{k-1} F^s x_{k-1}. \end{aligned}$$

Also,

$$\begin{aligned} m_{k-2}^2 \bar{S}^2_{m_{k-2},k-2} &= (m_{k-3}) s^2_{m_{k-3},k-3} + m_{k-3}\bar{S}^2_{m_{k-3},k-3} + (F^{-2} x_{k-1})^2 - (F^{-m_{k-3}-2} x_{k-1})^2 - (F^{-m_{k-3}-1} x_{k-1})^2 \\ &\quad + \sum_{i=-m_{k-2}-1}^{-m_{k-3}-1} (F^i x_{k-1})^2 + \sum_{\substack{l,s=-m_{k-2}-1 \\ l\neq s}}^{-2} F^l x_{k-1} F^s x_{k-1}. \end{aligned}$$

Continuing in this sense and substituting $\bar{S}^2_{m_{k-i},k-i}$, $i = 2, ..., p-1$ into $\bar{S}^2_{m_{k-1},k-1}$, we have

$$
\begin{aligned}
(m_{k-1})\bar{S}^2_{m_k-1,k-1} \quad = \quad & \sum_{i=2}^{p}\left[\frac{m_{k-i}}{\prod_{j=1}^{i-1} m_{k-j}}\right] s^2_{m_{k-i},k-i} + \frac{m_{k-p}}{\prod_{j=1}^{p-1} m_{k-j}} \bar{S}^2_{m_{k-p},k-p} + \sum_{i=2}^{p} \frac{\left(F^{-i+1}x_{k-1}\right)^2}{\prod_{j=1}^{i-1} m_{k-j}} - \sum_{i=2}^{p} \frac{\left(F^{-i+1-m_{k-i}}x_{k-1}\right)^2}{\prod_{j=1}^{i-1} m_{k-j}} \\
& - \sum_{i=2}^{p} \frac{\left(F^{-i+2-m_{k-i}}x_{k-1}\right)^2}{\prod_{j=1}^{i-1} m_{k-j}} + \sum_{i=2}^{p}\left[\frac{\sum_{l=-i+2-m_{k-i+1}}^{-i+2-m_{k-i}} (F^l x_{k-1})^2}{\prod_{j=1}^{i-1} m_{k-j}}\right] + \sum_{i=2}^{p}\left[\frac{\sum_{\substack{l,s=-i+2-m_{k-i+1}\\ l\neq s}}^{-i+1} F^l x_{k-1} F^s x_{k-1}}{\prod_{j=1}^{i-1} m_{k-j}}\right]
\end{aligned}
\tag{A.4}
$$

Finally, the result follows by substituting (A.4) into (A.3). ■

Appendix B. Proof

Proof of Lemma 2.1 for small m_k, $m_k \leq m_{k-1}$,

PROOF. Following the same steps, if $m_k \leq m_{k-1}$,

$$
\begin{cases}
s^2_{m_k,k} = \frac{m_k-1}{m_k}\left[\sum_{i=1}^{p}\left[\frac{m_{k-i}}{\prod_{j=0}^{i-1} m_{k-j}}\right] s^2_{m_{k-i},k-i} + \frac{m_{k-p}}{\prod_{j=0}^{p-1} m_{k-j}} \bar{S}^2_{m_{k-p},k-p}\right] + \varpi_{m_{k-1},k-1},\ m_k \leq m_{k-1} \\
\varpi_{m_{k-1},k-1} = \frac{m_k-1}{m_k}\left[\sum_{i=1}^{p} \frac{\left(F^{-i+1}x_{k-1}\right)^2}{\prod_{j=0}^{i-1} m_{k-j}} - \sum_{i=1}^{p}\left[\frac{\sum_{l=-i+1-m_{k-i}}^{-i+1-m_{k-i+1}} (F^l x_{k-1})^2}{\prod_{j=0}^{i-1} m_{k-j}}\right] + \sum_{i=1}^{p}\left[\frac{\sum_{\substack{l,s=-i+2-m_{k-i+1}\\ l\neq s}}^{-i+1} F^l x_{k-1} F^s x_{k-1}}{\prod_{j=0}^{i-1} m_{k-j}}\right]\right] \\
\quad - \frac{1}{m_k} \sum_{\substack{l,s=-m_k+1\\ l\neq s}}^{0} F^l x_{k-1} F^s x_{k-1},
\end{cases}
$$

■

Appendix C. Proof

Proof of Lemma 2.1 for large m_k

PROOF.

$$
\begin{aligned}
s^2_{m_k,k} &= \frac{1}{m_k-1}\left[\sum_{i=-m_k+1}^{0}\left(F^i x_{k-1}\right)^2 - \frac{1}{m_k}\left(\sum_{j=-m_k+1}^{0} F^j x_{k-1}\right)^2\right] \\
&= \frac{1}{m_k-1}\left[\sum_{i=-m_{k-1}}^{-1}\left(F^i x_{k-1}\right)^2 - \frac{1}{m_{k-1}}\left(\sum_{i=-m_{k-1}}^{-1} F^i x_{k-1}\right)^2 + \frac{1}{m_{k-1}}\left(\sum_{i=-m_{k-1}}^{-1} F^i x_{k-1}\right)^2\right] \\
&\quad + \frac{1}{m_k-1}\left[\left(F^0 x_{k-1}\right)^2 - (F^{-m_{k-1}} x_{k-1})^2 - \left(F^{-m_{k-1}+1} x_{k-1}\right)^2 - \frac{1}{m_k}\left(\sum_{i=-m_k+1}^{0} F^i x_{k-1}\right)^2\right] \\
&\quad + \frac{1}{m_k-1}\left[\left(\sum_{i=-m_k+1}^{-m_{k-1}+1} F^i x_{k-1}\right)^2\right]
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{m_{k-1}-1}{m_k-1} s^2_{m_{k-1},k-1} + \frac{m_{k-1}}{m_k-1} \bar{S}^{\,2}_{m_{k-1},k-1} - \frac{m_k}{m_k-1} \bar{S}^{\,2}_{m_k,k} + \frac{\left(F^0 x_{k-1}\right)^2 - (F^{-m_{k-1}} x_{k-1})^2 - \left(F^{-m_{k-1}+1} x_{k-1}\right)^2}{m_k-1} \\
& + \frac{\sum_{i=-m_k+1}^{-m_{k-1}+1} \left(F^i x_{k-1}\right)^2}{m_k-1}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
s^2_{m,k} \quad = \quad & \tfrac{m_{k-1}-1}{m_k-1} s^2_{m_{k-1},k-1} + \tfrac{m_{k-1}}{m_k-1} \bar{S}^{\,2}_{m_{k-1},k-1} - \tfrac{m_k}{m_k-1} \bar{S}^{\,2}_{m_k,k} + \tfrac{\left(F^0 x_{k-1}\right)^2 - \left(F^{-m_{k-1}} x_{k-1}\right)^2 - \left(F^{-m_{k-1}+1} x_{k-1}\right)^2}{m_k-1} \\
& + \tfrac{\sum_{i=-m_k+1}^{-m_{k-1}+1} \left(F^i x_{k-1}\right)^2}{m_k-1}.
\end{aligned}
\tag{C.1}
$$

Next, we find an expression connecting $\bar{S}^{\,2}_{m_k,k}, \bar{S}^{\,2}_{m_k-1,k-1}$ and $s^2_{m_k-1,k-1}$. By definition and simplification,

$$
\begin{aligned}
m_k^2 \bar{S}^{\,2}_{m_k,k} \quad &= \quad \left[\sum_{i=-m_k+1}^{0} F^i x_{k-1}\right]^2 = \sum_{i=-m_k+1}^{0} \left(F^i x_{k-1}\right)^2 + \sum_{\substack{l,s=-m_k+1 \\ l\neq s}}^{0} F^l x_{k-1} F^s x_{k-1} \\
&= \quad (m_{k-1}-1) s^2_{m_{k-1},k-1} + m_{k-1} \bar{S}^{\,2}_{m_{k-1},k-1} + (F^0 x_{k-1})^2 - (F^{-m_{k-1}} x_{k-1})^2 - (F^{-m_{k-1}+1} x_{k-1})^2 \\
& \qquad + \sum_{i=-m_k+1}^{-m_{k-1}+1} (F^i x_{k-1})^2 + \sum_{\substack{l,s=-m_k+1 \\ l\neq s}}^{0} F^l x_{k-1} F^s x_{k-1}
\end{aligned}
\tag{C.2}
$$

Substituting $(C.2)$ into $(C.1)$, we have

$$
\begin{aligned}
s^2_{m,k} \quad = \quad & \tfrac{m_{k-1}-1}{m_k} s^2_{m_{k-1},k-1} + \tfrac{m_{k-1}}{m_k} \bar{S}^{\,2}_{m_{k-1},k-1} + \tfrac{\left(F^0 x_{k-1}\right)^2 - \left(F^{-m_{k-1}} x_{k-1}\right)^2 - \left(F^{-m_{k-1}+1} x_{k-1}\right)^2}{m_k} + \tfrac{\sum_{i=-m_k+1}^{-m_{k-1}+1} \left(F^i x_{k-1}\right)^2}{m_k} \\
& - \tfrac{\sum_{\substack{l,s=-m_k+1 \\ l\neq s}}^{0} F^l x_{k-1} F^s x_{k-1}}{m_k(m_k-1)}.
\end{aligned}
\tag{C.3}
$$

Likewise,

$$
\begin{aligned}
m_{k-1}^2 \bar{S}^{\,2}_{m_{k-1},k-1} \quad = \quad & (m_{k-2}-1) s^2_{m_{k-2},k-2} + m_{k-2} \bar{S}^{\,2}_{m_{k-2},k-2} + (F^{-1} x_{k-1})^2 - (F^{-m_{k-2}-1} x_{k-1})^2 - (F^{-m_{k-2}} x_{k-1})^2 \\
& + \sum_{i=-m_{k-1}}^{-m_{k-2}} (F^i x_{k-1})^2 + \sum_{\substack{l,s=-m_{k-1} \\ l\neq s}}^{-1} F^l x_{k-1} F^s x_{k-1},
\end{aligned}
$$

$$
\begin{aligned}
m_{k-2}^2 \bar{S}^{\,2}_{m_{k-2},k-2} \quad = \quad & (m_{k-3}-1) s^2_{m_{k-3},k-3} + m_{k-3} \bar{S}^{\,2}_{m_{k-3},k-3} + (F^{-2} x_{k-1})^2 - (F^{-m_{k-3}-2} x_{k-1})^2 - (F^{-m_{k-3}-1} x_{k-1})^2 \\
& + \sum_{i=-m_{k-2}-1}^{-m_{k-3}-1} (F^i x_{k-1})^2 + \sum_{\substack{l,s=-m_{k-2} \\ l\neq s}}^{-2} F^l x_{k-1} F^s x_{k-1}.
\end{aligned}
$$

Continuing in this sense and substituting $\bar{S}_{m_{k-i},k-i}$, $i = 2, ..., p-1$ into $\bar{S}_{m_{k-1},k-1}$, we have

$$
\begin{aligned}
(m_{k-1}) \bar{S}^{\,2}_{m_{k-1},k-1} \quad = \quad & \sum_{i=2}^{p} \left[\frac{m_{k-i}-1}{\prod_{j=1}^{i-1} m_{k-j}}\right] s^2_{m_{k-i},k-i} + \frac{m_{k-p}}{\prod_{j=1}^{p-1} m_{k-j}} \bar{S}^{\,2}_{m_{k-p},k-p} + \sum_{i=2}^{p} \frac{\left(F^{-i+1} x_{k-1}\right)^2}{\prod_{j=1}^{i-1} m_{k-j}} - \sum_{i=2}^{p} \frac{\left(F^{-i+1-m_{k-i}} x_{k-1}\right)^2}{\prod_{j=1}^{i-1} m_{k-j}} \\
& - \sum_{i=2}^{p} \frac{\left(F^{-i+2-m_{k-i}} x_{k-1}\right)^2}{\prod_{j=1}^{i-1} m_{k-j}} + \sum_{i=2}^{p} \left[\frac{\sum_{l=-i+2-m_{k-i+1}}^{-i+2-m_{k-i}} F^l x_{k-1}}{\prod_{j=1}^{i-1} m_{k-j}}\right] + \sum_{i=2}^{p} \left[\frac{\sum_{\substack{l,s=-i+2-m_{k-i+1} \\ l\neq s}}^{-i+1} F^l x_{k-1} F^s x_{k-1}}{\prod_{j=1}^{i-1} m_{k-j}}\right]
\end{aligned}
\tag{C.4}
$$

Finally, the result follows by substituting $(C.4)$ into $(C.3)$. ■

Appendix D. Algorithm and Flowchart For Simulation

The simulated estimate $y^s_{m_k,k}$ for the energy commodity model follows the Euler scheme

$$y^s_{m_k,k} = y^s_{m_{k-1},k-1} + \hat{a}_{m_{k-1},k-1}(\hat{\mu}_{m_{k-1},k-1} - y^s_{m_{k-1},k-1})y^s_{m_{k-1},k-1}\Delta t + \hat{\sigma}_{m_{k-1},k-1}y^s_{m_{k-1},k-1}\Delta W_{m_k,k}. \quad \text{(D.1)}$$

Algorithm 1 Simulation scheme

Given initials r, ϵ, $\{\hat{s}^2_{m_0,0}\}_{m_0\in OS_0}$, $\{\hat{s}^2_{m_{-1},-1}\}_{m_{-1}\in OS_{-1}}$, $\{\bar{S}^2_{m_{-1},-1}\}_{m_{-1}\in OS_{-1}}$, $\{y^s_{m_0,0}\}_{m_0\in OS_0}$,
for $k = 1$ to N **do**,
  for $m_{k-1} = 2$ to $r + k - 2$ **do**,
    Compute $\hat{a}_{m_{k-1},k-1}$, $\hat{\mu}_{m_{k-1},k-1}$,
    for $m_{k-2} = 2$ to $r + k - 3$ **do**,
      Compute $\bar{S}^2_{m_{k-1},k-1}$, $\hat{s}^2_{m_k,k}$,$y^s_{m_k,k}$, Ξ_{m_k,k,y_k}
    end for
  end for
end for
if $\Xi_{m_k,k,y_k} < \epsilon$ **then**,
Save $\hat{m}_k$, $\hat{m}_{k-1}$, $\hat{m}_{k-2}$
else
  Find $\hat{m}_k$ that minimizes Ξ_{m_k,k,y_k}.
end if
Compute $a_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$,$s^2_{\hat{m}_k,k}$,$y^s_{\hat{m}_k,k}$.

Similar algorithm can be generated for the interest rate model.

Remark Appendix D.1. We give the first iterate for the energy commodity model.

Given initials r, ϵ, $\{s^2_{m_0,0}\}_{m_0\in OS_0}$, $\{s^2_{m_{-1},-1}\}_{m_{-1}\in OS_{-1}}$, $\{\bar{S}^2_{m_{-1},-1}\}_{m_{-1}\in OS_{-1}}$, $\{y^s_{m_0,0}\}_{m_0\in OS_0}$,

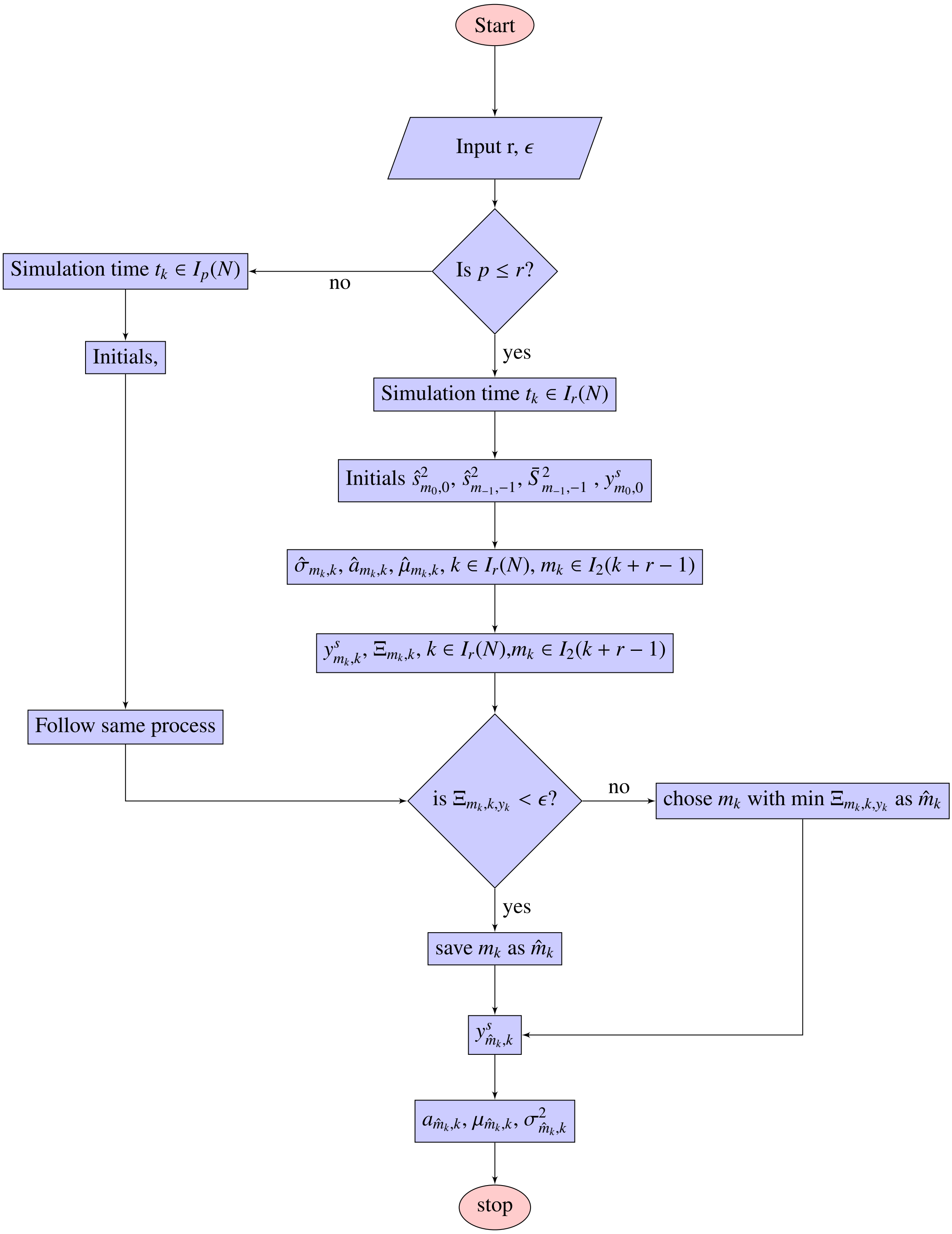
Compute $a_{m_0,0}$, $\mu_{m_0,0}$.

For k=1:

Compute $y^s_{m_1,1}$ using $(D.1)$. If $\Xi_{m_1,1,y_1} < \epsilon$, save $\hat{m}_1, \hat{m}_0, \hat{m}_{-1}$, else, find values of m_1 that minimizes $\Xi_{m_1,1,y_1}$.

Compute $a_{\hat{m}_0,0}$, $\mu_{\hat{m}_0,0}$,$s^2_{\hat{m}_1,1}$,$y^s_{\hat{m}_1,1}$.

Next, we give a flowchart similar to the algorithm above.



Flowchart 2: LLGMM Simulation Algorithm.

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